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Obtaining rigorous bounds for topological entropy for discrete time dynamical systems

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1. Introduction

In this work we develop a method for finding rigorous bounds for topological entropy of discrete time dynamical systems based on construction of symbolic dynamics embedded within the considered nonlinear map. In order to prove the existence of symbolic dynamics the concept of topological covering [5] is used. The covering relations between the topological rectangles $N_1, N_2, \ldots, N_p$ are rigorously proved using interval arithmetic. Once the existence of covering relations is ensured, the transition matrix $A = (a_{ij})$ for the subshift on p symbols is formed, such that the map is semiconjugate with the subshift on p symbols, with the transition matrix A. Topological entropy of a subshift of a finite type with transition matrix A is used to obtain a lower bound for the topological entropy for the map f.

In this work we address the problem how to find sets N_i, which lead to complex symbolic dynamics and estimates for topological entropy close to the true entropy of the system. For this task algorithms for finding nonwandering part of a given set are used. The method starts with generation of the nonwandering part of a chosen region. The set obtained is covered by quadrangles, which are adjusted by hand to fulfill the assumptions of the theorem on existence of symbolic dynamics.

As an example we consider simple two–dimensional chaotic maps, namely the Hénon map and the Ikeda map. For the Hénon map we find the symbolic dynamics on 29 symbols with topological entropy larger than 0.43. For the Ikeda map we find the symbolic dynamics on 18 symbols with the topological entropy larger than 0.485. The rigorous bounds for topological entropy found are best known to date.

2. Existence of Symbolic Dynamics

In this section we describe a topological method, which can be used to prove the existence of symbolic dynamics. The method is based on the concept of covering [5]. For simplicity we consider two–dimensional systems only. For the description of covering relations in higher dimension see [5]. Let us assume that f is a continuous two–dimensional map. Let us choose p pairwise disjoint quadrangles $N_1, N_2, \ldots, N_p$. For each N_i we choose two opposite edges and call them "horizontal". The two others are called "vertical". We say that N_i f–covers N_j and we use the notation $N_i \overset{f}{\Rightarrow} N_j$ if

(i) the image of N_i under f has empty intersection with the horizontal edges of N_j,

(ii) the images of vertical edges of N_i has empty intersection with N_j and they are located geometrically on the opposite sides of N_j.

In the paper we will use the extension of the above definitions by allowing that the edges of topological quadrangles are broken lines instead of segments.

The existence of topological coverings can be rigorously checked by means of computer assisted proofs using interval arithmetic. To prove that a certain covering relation $N_i \overset{f}{\Rightarrow} N_j$ holds, the edges of N_i are covered by boxes of a specified size. Next, images of these boxes under f are found and the conditions (i) and (ii) are checked.

Once the existence of covering relations is proved, we have the existence of symbolic dynamics, as stated by the following theorem (compare [3]).

Theorem 1 *Let $N_1, N_2, \ldots, N_p$ be pairwise disjoint quadrangles. Let $A = (a_{i,j})_{i,j=1}^{p}$ be a square matrix, where*

$$a_{i,j} = \begin{cases} 1 & \text{if } N_i \overset{f}{\Rightarrow} N_j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then f is semiconjugate with the subshift on p symbols, with the transition matrix A.

2.1. Symbolic dynamics and topological entropy

From the fact that f is semiconjugate with a subshift of a finite type, we can make conclusions on the topological entropy of f.

Topological entropy $\mathrm{H}(f)$ characterizes "mixing" of points by the map f. Topological entropy can be defined using the notion of separated sets. A set $E \subset X$ is called (n, ε)–*separated* if for every two different points $x, y \in E$, there exists $0 \le j < n$ such that the distance between $f^j(x)$ and $f^j(y)$ is greater than ε.

Topological entropy of f is defined by

$$\mathrm{H}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon), \tag{2}$$

where $s_n(\varepsilon)$ is the cardinality of a maximum (n, ε)–separated set.

Topological entropy of a subshift of finite type with transition matrix A equals to the logarithm of the dominant eigenvalue λ_1 of A, i.e., λ_1 is such that $\lambda_1 \ge |\lambda_j|$ for all eigenvalues of A (see [4][Theorem 1.9, p. 340]). The topological entropy of a map semiconjugate to a subshift is not less than the topological entropy of this subshift. For each symbolic sequence there exist a trajectory realizing the sequence, which serves as an (n, ε)–separated set for every ε smaller that the minimum distance between the sets N_i. Thus, we have the following result.

Theorem 2 *The topological entropy of the map f is not smaller than the logarithm of the dominant eigenvalue of the matrix A, defined be equation (1)*

$$\mathrm{H}(f) \ge \log \lambda_1. \tag{3}$$

2.2. Finding sets on which symbolic dynamics exists

There is no fully automatic method for finding sets N_i on which complicated symbolic dynamics is defined. In previous work the sets N_i were found by "trial and error" using some information on the positions of low-period cycles and their stable and unstable directions [5, 1, 3].

In order to locate the sets N_i we use the technique based on construction of nonwandering part of a given set. A point x is called *nonwandering* for f if for any neighborhood U of x there exists $n > 0$ such that $f^n(U) \cap U \ne \emptyset$. For a given set A we define the *nonwandering part* of A as the set of nonwandering points of the map $f|\mathrm{Inv}(A)$. An enclosure for the invariant or nonwandering part of a given set can be easily found using the methods presented in [2].

The method of construction of sets N_i consists of several steps. First, we choose the set containing the interesting dynamics. If we know the trapping region for the system, we may choose this set. In the opposite case we choose a set containing the numerically observed attractor. In the next step, we find the nonwandering part of this set. Usually this set is connected and it does not help us much in defining the sets N_i. To break this set into several pieces we remove part of this set and find the invariant part of what is left. In many cases the result is a small number of connected components, which after minor modification can serve as the rectangles N_i.

It is possible to refine the enclosure of the nonwandering part by dividing up the boxes constituting the enclosure. In this way one may obtain a more detailed enclosure of the nonwandering part and define symbolic dynamics on more sets.

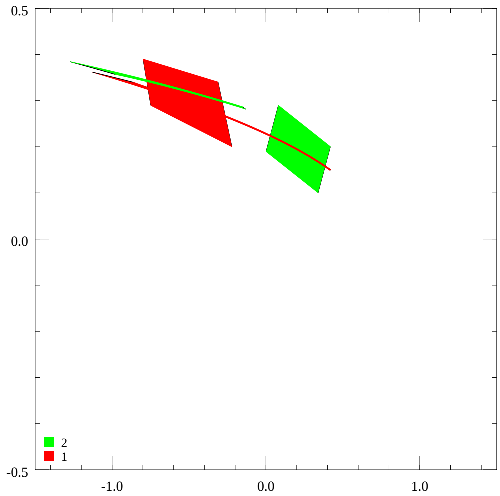

Figure 1: Symbolic dynamics on 2 symbols for the second iterate of the Hénon map

3. Hénon Map

As a first example let us consider the Hénon map

$$h(x, y) = (1 + y - ax^2, bx), \tag{4}$$

with standard parameter values $a = 1.4$, $b = 0.3$.

In [1] we have proved the existence of symbolic dynamics on 2 symbols for the second iteration of the Hénon map. The corresponding quadrangles are shown in Fig. 1.

The covering relations between these sets correspond to the following transition matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

and we obtain the lower bound for the topological entropy for the Hénon map

$$\mathrm{H}(h) \ge \frac{1}{2} \log \frac{\sqrt{5}+1}{2} > 0.2406. \tag{6}$$

The factor $\frac{1}{2}$ comes from the fact that covering relations involve the second iterate of the map. In the above example the sets N_i were found by "trial and error".

To prove the existence of a more complex symbolic dynamics, we first find the nonwandering part of the set $[-2, 2] \times [-2, 2]$ containing the numerically observed attractor. Then we remove boxes for which $x < -1$, and find the invariant part of the remaining set. The result of this procedure is shown in Fig. 2. This set is then used as an initial guess for the position of rectangles, on which the symbolic

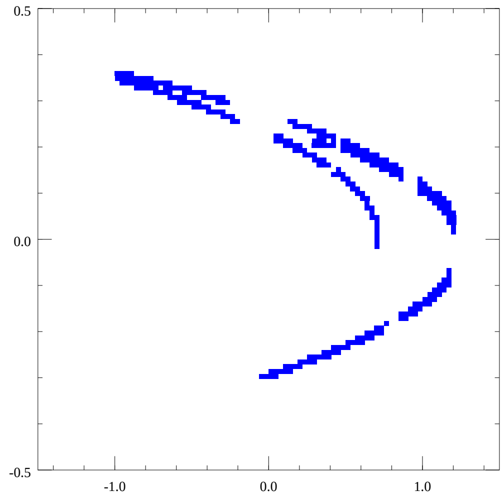

Figure 2: Enclosure of the nonwandering part of $[-1, 2] \times [-2, 2]$

dynamics is defined. Since the nonwandering part is composed of 8 connected subsets, we choose 8 quadrangles (see Fig. 3(a)). There are only four covering relations between these sets. The transition matrix is almost empty and hence there is no interesting symbolic dynamics on these sets. We modify the position of the rectangles by hand, so that a large number of covering relations hold. The improved sets and their images under the Hénon map are shown in Fig. 3(b).

Finally, we check rigorously the existence of covering relations between the chosen sets. The coverings correspond to the symbolic dynamics on eight symbols with the following transition matrix:

$$A = \begin{pmatrix} & 1 & 1 & & & & & \\ & & & & & & 1 & 1 \\ 1 & & & & & & & \\ 1 & & & & & & & \\ & & & & 1 & 1 & & \\ & & & 1 & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \end{pmatrix}. \tag{7}$$

It follows that the symbolic dynamics with the transition matrix (7) is embedded in h and that the topological entropy of the Hénon map is bounded by $\mathrm{H}(h) > 0.382$. This is better than the best estimate known to date ($\mathrm{H}(h) > 0.338$, see [3]).

We have performed several other attempts to find complex symbolic dynamics for the Hénon map. The largest bound for the topological entropy $\mathrm{H}(h) > 0.430$ was obtained for the sets shown in Fig. 3(c). This bound is close to the non-rigorous estimation of topological entropy based on the number of low-period cycles $\mathrm{H}(h) \approx 0.465$ (see [2]).

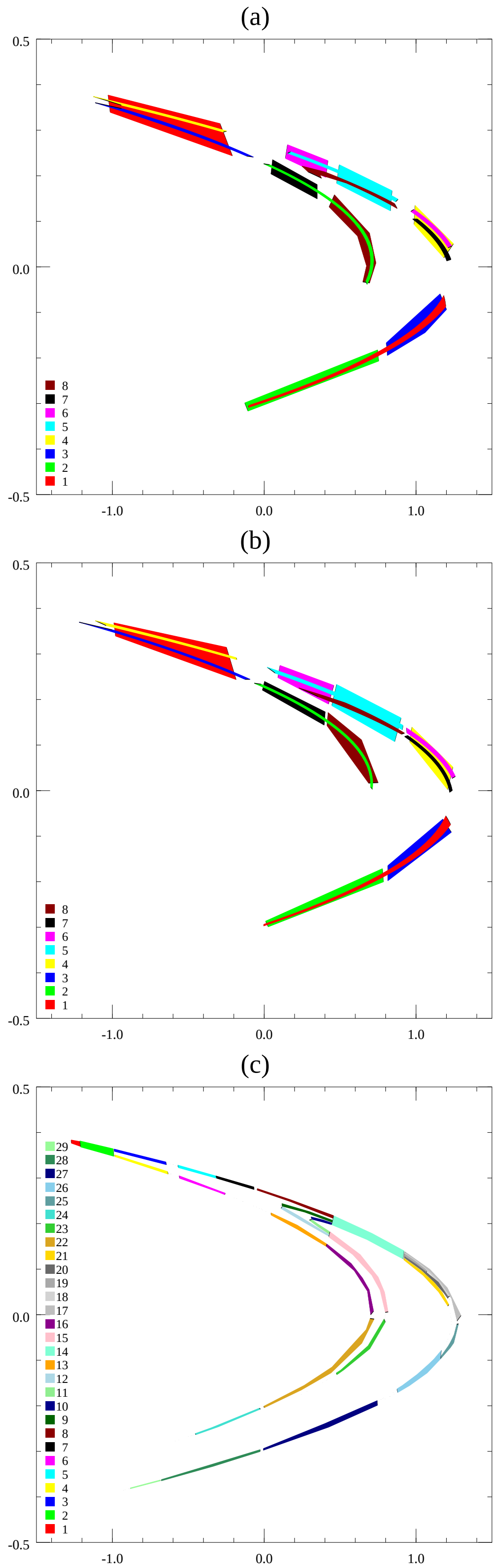

Figure 3: (a) Symbolic dynamics on 8 symbols, initial quadrangles, (b) Symbolic dynamics on 8 symbols, improved quadrangles, (c) Symbolic dynamics on 29 symbols

4. Ikeda Map

As a second example let us consider the Ikeda map

$$f(x, y) = (p + B(x \cos t - y \sin t), B(x \sin t + y \cos t)), \quad (8)$$

where $t = t(x, y) = \kappa - \alpha/(1 + x^2 + y^2)$, $p = 1$, $B = 0.9$, $\kappa = 0.4$ and $\alpha = 6$.

We have found several examples of symbolic dynamics embedded withing the Ikeda map. For the symbolic dynamics on four symbols (see Fig. 4(a)) the transition matrix is

$$A = \begin{pmatrix} & & & 1 \\ 1 & & & \\ 1 & 1 & & \\ & & 1 & \end{pmatrix}, \quad (9)$$

and the bound for topological entropy is $\mathrm{H}(f) > 0.199$. The symbolic dynamics on seven symbols (Fig. 4(b)) gives $\mathrm{H}(f) > 0.401$. The largest bound $\mathrm{H}(f) > 0.485$ is obtained for the symbolic dynamics on 18 symbols (Fig. 4(c)).

5. Conclusions

In this work a method for choosing sets in the state space, which can be used for the construction of complex symbolic dynamics has been proposed. Lower bounds for topological entropy has been found for the Hénon map: $\mathrm{H}(h) > 0.430$ and for the Ikeda map: $\mathrm{H}(f) > 0.485$.

Acknowledgments

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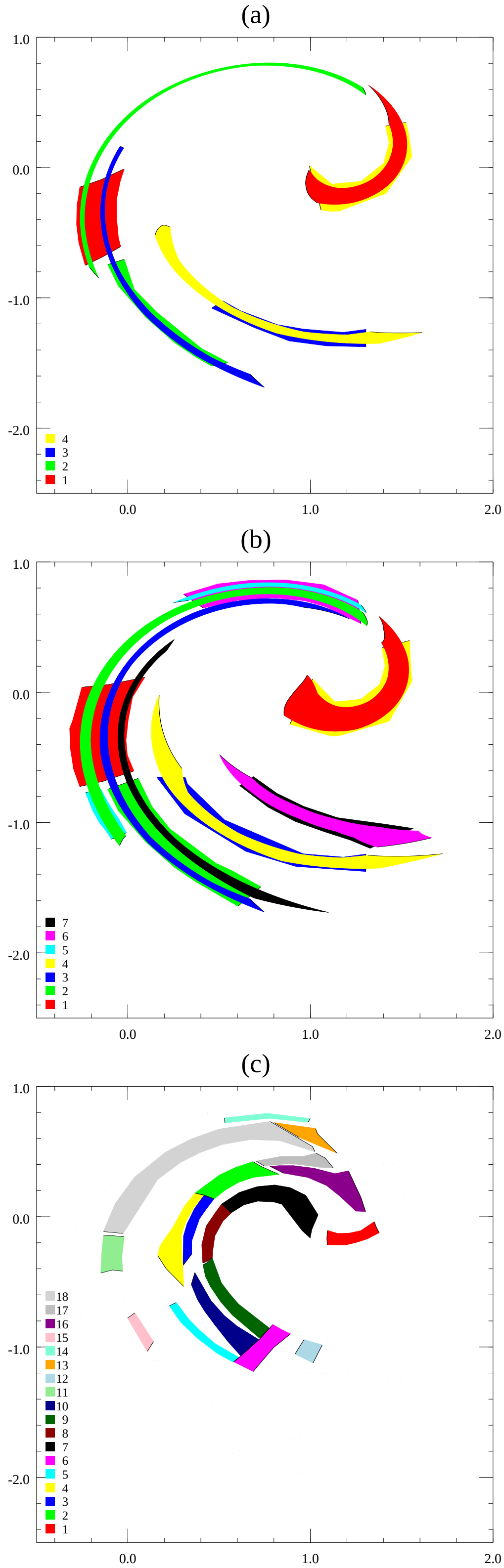

Figure 4: (a) Symbolic dynamics on 4 symbols, (b) symbolic dynamics on 7 symbols, (c) symbolic dynamics on 18 symbols