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# STABILNOŚĆ ZACHOWAŃ SYNCHRONICZNYCH W PĘTLI SPRZĘŻONYCH OBWODÓW CHAOTYCZNYCH

# STABILITY OF SYNCHRONIZED MOTIONS IN A RING OF COUPLED CHAOTIC CIRCUITS

## STRESZCZENIE

Niniejsza praca poświęcona jest badaniu zachowań dynamicznych w układzie sprzężonych ze sobą oscylatorów chaotycznych połączonych w zamknięty łańcuch. Badano zjawiska synchronizacji drgań pomiędzy poszczególnymi obwodami w łańcuchu. W zależności od parametrów sprzężeń obserwowano rozmaite typy synchronizacji "w fazie" i "w odwróconej fazie". Przedstawione wnioski oparte są na wszechstronnych badaniach numerycznych.

## 1. INTRODUCTION

Lattice models, exhibiting various types of collective behavior can be found in studies of neural networks and simulations of a variety of phenomena observed in real systems in physics, solid state electronics, chemical reactions, biology and medicine [1, 2, 3, 7]. Existence of many interesting dynamic phenomena in arrays of discretely coupled bi-stable or oscillatory systems has been confirmed both in laboratory and numerical experiments [6].

The dynamics of individual cells and the coupling between them predefines the overall system behavior. Among other types of collective dynamics one can observe various kinds of spatial, temporal or spatio-temporal ordered structures referred to as self-organization [1] or "pattern formation". "Organized" behavior is usually linked with coherent (synchronized) behavior of a number of cells in the network. Organized spatio-temporal behavior includes also propagation of waves including solitons and autowaves, target waves, spiral waves and traveling wavefronts [6].

In our previous works we studied cooperative behavior in one- and two-dimensional arrays of Chua's circuits with double resistive coupling between the cells [4, 5]. In the present study we investigate steady-state behavior observed in a ring (one–dimensional array with connected edges) of coupled Chua's chaotic circuits. Comparing with our previous studies coupling has been modified by addition of a self-coupling term to each cell thus enabling simultaneous development of synchronized chaotic motion in all cells. Using computer experiments we have confirmed the existence of a very large number of stable final states depending on the connection strength and initial conditions applied in the individual cells. Existence of various synchronized states is studied experimentally.

## 2. DYNAMICS OF THE NETWORK

Let us consider a one–dimensional lattice of simple third–order electronic oscillators (Chua's circuits). The oscillators are coupled bi-directionally by means of two resistors cross-connected between the capacitors $C_1$ and $C_2$ of the neighboring cells. Every cell is connected with two nearest neighbors. The dynamics of the one–dimensional lattice composed of $n$ cells can be described by the following set of ordinary differential equations:

$$\begin{aligned} C_2\dot{x}_i &= -y_i + (G-2G_1)(z_i - x_i) + \\ &\quad + G_1(z_{i-1} - x_i) + G_1(z_{i+1} - x_i), \\ L\dot{y}_i &= x_i, \\ C_1\dot{z}_i &= (G-2G_1)(x_i - z_i) - f(z_i) + \\ &\quad + G_1(x_{i-1} - z_i) + G_1(x_{i+1} - z_i), \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, n$ and we use the following boundary conditions: $x_{n+1} = x_1$, $z_{n+1} = z_1$, $x_0 = x_n$, $z_0 = z_n$ (the first and the last cells are also connected,

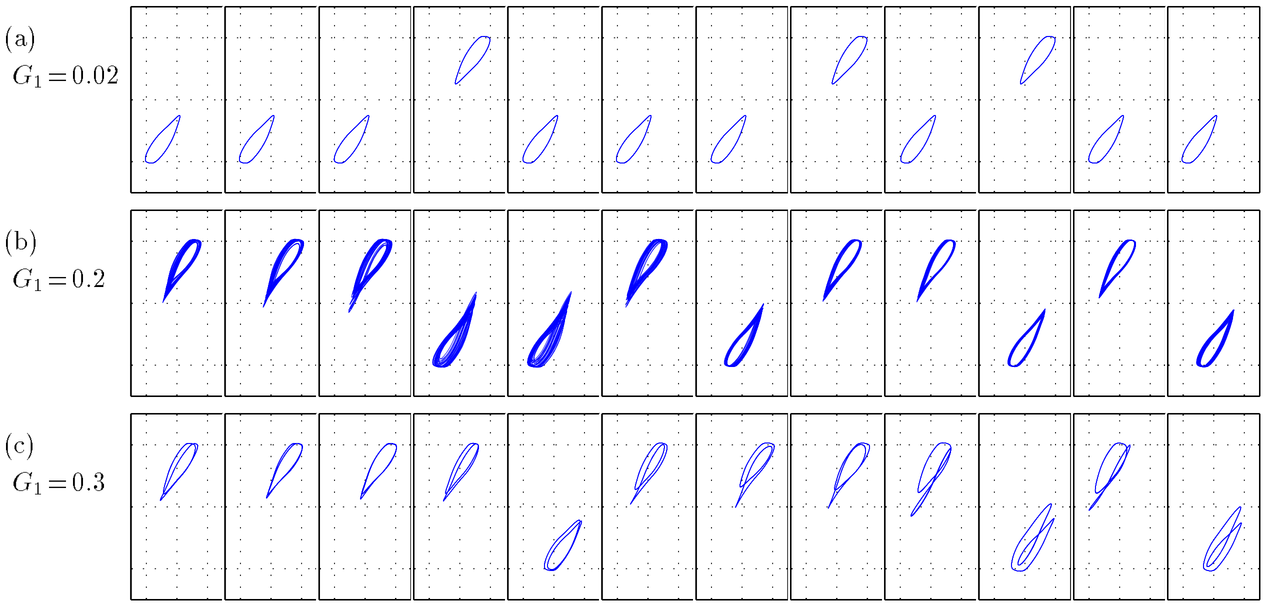


**Figure 1:** Steady state behavior of the lattice composed of $n = 12$ chaotic cells for different coupling conductance $G_1$. The projection of attractor onto the $y_i, z_i$ plane for cells $i = 1, \ldots, 12$ is shown, $y_i, z_i \in [-3, 3]$

hence the lattice forms a ring). $f$ is a five–segment piecewise linear function

$$f(z) = m_2 z + \frac{1}{2}(m_1 - m_2)(|z + B_{p_2}| - |z - B_{p_2}|) + \frac{1}{2}(m_0 - m_1)(|z + B_{p_1}| - |z - B_{p_1}|). \quad \mathbf{(2)}$$

In simulation experiments we used typical parameter values for which an isolated Chua's circuit generates chaotic oscillations — the double scroll attractor ($C_1 = 1/9F$, $C_2 = 1F$, $L = 1/7H$, $G = 0.7S$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$). For the integration of the system the fourth-order Runge-Kutta method was used with the time step $\tau = 0.1$.

The setup described above is slightly different from the one used in our previous experiments. We have modified the value of the resistor connecting capacitors $C_1$ and $C_2$ is a single cell. This ensures the existence of synchronized chaotic solution. If we apply identical initial conditions to every cell in the array ($x_i(0) = x(0)$, $y_i(0) = y(0)$, $z_i(0) = z(0)$ for $i = 1, \ldots, n$) then all the cells oscillate synchronously and the equations describing the array can be written as

$$\begin{aligned} C_2\dot{x} &= -y + G(z - x), \\ L\dot{y} &= x, \\ C_1\dot{z} &= G(x - z) - f(z), \end{aligned} \quad \mathbf{(3)}$$

where $x_i = x$, $y_i = y$ and $z_i = z$ for $i = 1, \ldots, n$. Hence in the case of equal initial conditions the network as a whole behaves as a single uncoupled cell. For the dynamical system (3) and the parameter values we consider there exist two attractors — the "double–scroll" chaotic attractor and a periodic orbit of large amplitude. One of our aims is the investigation of stability of these synchronous solutions.

## 3. STABILITY OF THE SYNCHRONIZED MOTION

In this section we will outline the results of our experiments on stability of synchronous solutions and on the search for other steady–state solutions.

In order to test the stability of a particular solution one can perturb this solution by a random additive signal with a small amplitude and observe the steady–state behavior of the system. If the system converges to the solution under consideration we claim that the solution is stable. We have performed such an experiment for both of the synchronous behaviors, corresponding to the "double–scroll" chaotic attractor and large amplitude periodic oscillations of a single circuit.

Additionally we tested the existence of other steady–state behaviors by using different initial conditions.

### Stability of chaotic synchronous solution

In the first series of experiments we have perturbed the chaotic synchronous solution by a random additive signal of amplitude 0.0001. Then we have integrated the system with different values of coupling coefficients.

For zero coupling after short time all the cells were uncorrelated. One could not recognize that they were initialized at very close initial conditions. This phenomenon is well–known in the theory of chaotic system as the "sensitive dependence on initial conditions".

For $G_1 = 0.01$ cells behave chaotically. Observing the trajectory it is difficult to see that the cells are coupled. Trajectory of every cell forms the double–scroll attractor. But there is a slight difference in comparison with the case of uncoupled cells. With non–zero coupling the switchings between different scrolls are less frequent. We have computed the number of switchings

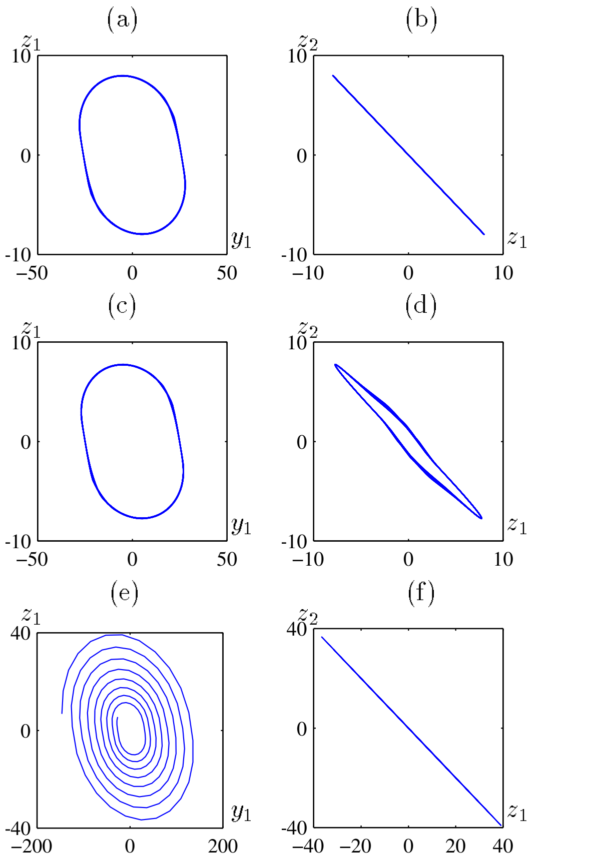


**Figure 2:** Behavior of the network for large coupling, (a)–(b) $G_1 = 0.4$, $n = 30$, (c)–(d) $G_1 = 0.4$, $n = 31$, (e)–(f) $G_1 = 0.45$, $n = 30$

for all cells which occur during a fixed time interval for both cases. For the uncoupled cells there were more than 200 switchings, while for the coupling $G_1 = 0.01$ the number of switchings was 24 during the same time ($t \in [0, 1000]$).

For $G_1 = 0.02$ trajectory of every cell in the steady–state is periodic and enclosed in one of the scrolls. The steady–state behavior of the network is shown in Fig. 1(a).

For $G_1 \in [0.03, 0.09]$ every cell forms the double–scroll attractor, with infrequent switchings between different scrolls.

For $G_1 \in [0.1, 0.25]$ the steady–state is chaotic. Trajectory of every cell is similar to the Rössler–type attractor. An example for $G_1 = 0.2$ is shown in Fig. 1(b). If we start the network from random initial conditions then some of the cells end up in the upper half-space while the others in the lower half-space. We have observed different patters depending on initial conditions. We believe that in a steady–state there are no switchings between different half-spaces. The transient can be very long, especially if we consider larger networks. In one of the experiments with the network composed of 30 cells we have recorded the last switching after 700000 iterations. But then for another 3000000 iterations there were no other switchings.

For $G_1 = 0.3$ we have observed another interesting phenomena. For this coupling the steady state behavior is periodic but this time the trajectory of every cell forms a double–loop periodic attractor. Its shape is different for every cell (compare Fig. 1(c).

For very large coupling ($G_1 = 0.4$) the array as a whole operates is a periodic mode. The trajectory of a single cell is shown in Fig 2(a). Every cell displays identical periodic oscillations of large amplitude. The amplitude of oscillations grows very quickly with $G_1$. For $G_1 = 0.36$ the amplitude is 10, while for $G_1 = 0.41$ the amplitude is 40.

If we further increase the value of the coupling conductance ($G_1 > 0.44$) the network becomes unstable (Fig 2(e)).

From the experiments described above it follows that the synchronized chaotic motion is not stable for any of the values of $G_1$.

### Stability of periodic synchronous solution

We have performed similar experiments for periodic synchronous solution. In all the experiments the synchronous solution has been perturbed by additive noise with amplitude 0.1. We have observed that for coupling $G_1 < 0.35$ this solution is stable. In all the experiments we have observed perfect synchronization. For large coupling ($G_1 > 0.35$) the periodic synchronous solution becomes unstable. The steady–state solutions are the same as for the case of perturbed chaotic synchronous solution.

## 4. SYNCHRONIZATION OF CELLS

In this section we investigate the synchronization of neighboring circuits in the steady state for different coupling values. In Fig. 3 we plot three examples. For every cell we plot $z_i$ versus $z_{i-1}$.

For $G_1 = 0.02$ the neighboring circuits do not synchronize. One can determine whether the neighboring cells are operating in the same half-plane by checking if the plot $z_i, z_{i-1}$ belongs to the first or the third quadrant of the plane.

For $G_1 = 0.2$ and $G_1 = 0.3$ one can see weak synchronization. If the circuits are operation in the same half–space then the plot $z_i, z_{i-1}$ is close to the diagonal line $z_i = z_{i-1}$. If the neighboring cells are operating in different half-planes than the plot is contained in the second or the fourth quadrant and its deviation from a line is much stronger. In this case neighboring cells operation is the same half–space form clusters of nearly coherent behavior.

For very large coupling ($G_1 = 0.4$) and even number $n$ neighboring cells are operating in an opposite phase ($x_i = -x_{i+1}$, $y_i = -y_{i+1}$, $z_i = -z_{i+1}$). In Fig. 2(b) one can see that cells are perfectly synchronized with opposite phase (after a short transient $x_i = -x_{i+1}$, $y_i = -y_{i+1}$ and $z_i = -z_{i+1}$ for all $i = 1, \ldots, n$). A different behavior is observed when the number of cells in the network is odd. In this case the network cannot sustain opposite-phase synchronization. One can observe a disturbance traveling in the right or left direction. As an example we have considered the lattice composed of 31 cells. Projections of the trajectory on

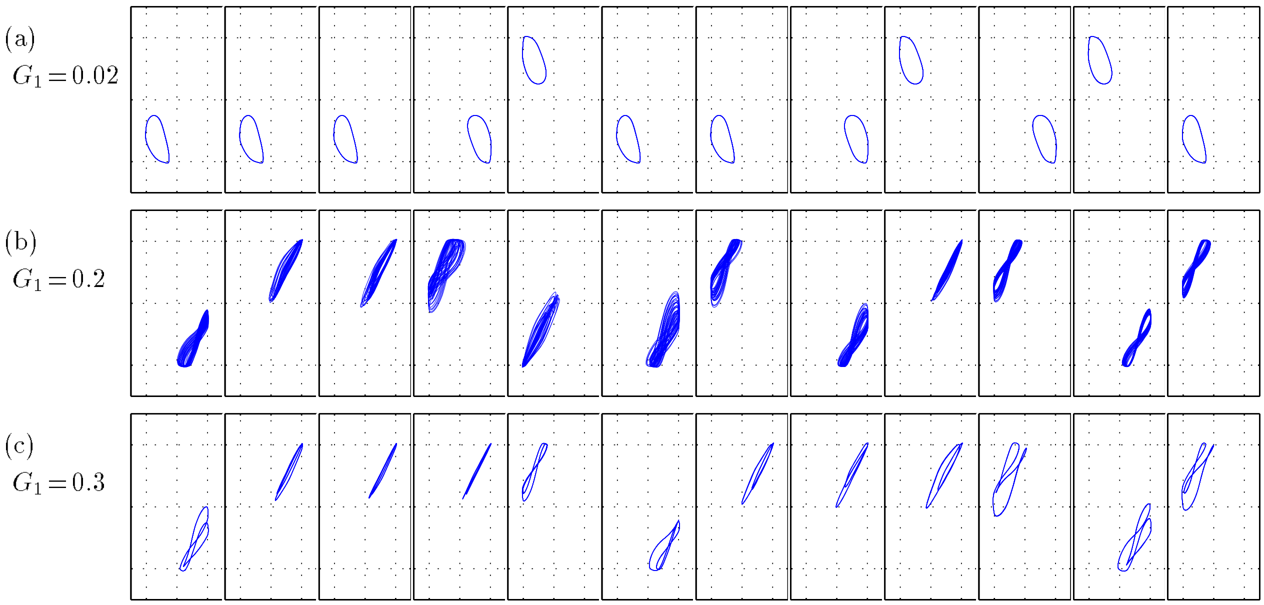


**Figure 3:** Synchronization of the neighboring cells for the steady–state behavior of the lattice composed of $n = 12$ chaotic cells for different coupling conductance $G_1$. In the $i$th column the projection of the steady–state onto the $z_i, z_{i-1}$ plane is shown ($z_0 = z_{12}$), $z_i, z_{i-1} \in [-3, 3]$

the different planes are shown in Fig 2(c,d). A single cell oscillates periodically, with large amplitude of oscillations (similarly to the previous case (compare Fig. 2(a)). Neighboring cells are not perfectly synchronized (see Fig. 2(d)).

## 5. CONCLUSIONS

In this paper we have studied the influence of initial conditions and coupling strength on long-term behavior of the array of locally coupled chaotic circuits. We have observed that the chaotic synchronous behavior is not stable for any value of the coupling coefficient. On the other hand the periodic synchronous solution is stable in a wide range of $G_1$. We have also studied the formation of clusters of cells oscillating almost synchronously. Another interesting phenomena is the existence of opposite–phase synchronization for very large value of coupling coefficient.

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