INVESTIGATIONS OF PERIODIC ORBITS IN ELECTRONIC CIRCUITS WITH INTERVAL NEWTON METHOD

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ABSTRACT

We study the possibility of using interval Newton method for proving the existence of periodic orbits for models of electronic circuits. The advantages and limitations of the method are discussed. As an example a simple third order electronic circuit is considered, for which the existence of low–period unstable periodic orbits is proved.

1. INTRODUCTION

The problem of existence and stability of periodic orbits is very important in analysis of nonlinear systems and also in many applications. In chaotic systems it is even of higher importance. Chaotic systems are characterized by the existence of infinitely many unstable periodic orbits. One of the well-known methods for controlling such systems is to stabilize one of periodic orbits embedded in the chaotic attractor [6].

A method to find periodic solutions form a time series was developed in [5]. In this method one searches for parts of a trajectory which are almost periodic (the trajectory returns close to the initial point). We believe that in the neighborhood of such fragment there exist a real periodic orbit. But one never knows if a real trajectory actually exists. For example in a quasiperiodic motion defined on the two-dimensional torus the method of close returns would find many periodic orbits but we know that there exist no periodic orbit for this system.

There are several methods for proving the existence of periodic orbits. One of them is based on the Schauder's fixed point theorem, which states that if a convex compact set $X \subset \mathbb{R}^n$ is mapped by a continuous map $\mathbf{f}$ into itself then there exist a point $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$. With this theorem one can easily prove the existence of a periodic orbit of a continuous–time dynamical system when the periodic orbit is stable. In such case one can always choose a plane Σ transversal to the periodic orbit and find a neighborhood U of the point at which the periodic orbit intersects the plane such that $P(U) \subset U$, where P is the return map defined by Σ. Similarly one can prove the existence of a periodic orbit which is unstable in all directions (it becomes stable when the direction of time is changed). Unfortunately this method cannot be used directly for proving the existence of saddle type orbits.

Another class of methods is based on the fixed point index properties. In one of the methods one has to prove the topological conjugacy of the Poincaré map in the neighborhood of the fixed point with a linear map possessing a saddle–type fixed point [3]. The second method involves computation of an integral of a certain function over a circle surrounding a fixed point on the Poincaré map. If this integral is non-zero then the existence of a fixed point is ensured [4]. This last method can be used when the Poincaré map is two dimensional. Both methods allow to prove the existence of all types of periodic orbits (also of the saddle–type). Their main drawback is non-efficiency — one has to perform a lot of calculation in order to prove the assumptions of the existence theorem and control the computational error (in case of computer assisted proof).

In this paper we investigate another method for proving the existence of periodic orbits, namely the *interval Newton method* [1]. This method is based on interval arithmetic tools [2], where intervals are used instead of real numbers. When interval arithmetic is implemented on a computer the rounding of every operation is directed outwards. In this way we are sure that the result obtained encloses the true solution (together with the rounding error). Thus interval arithmetic overcome the usual problem of computer calculations — the existence of rounding errors makes it difficult or even impossible to find the relation between true solutions and approximations obtained using standard computational methods.

The interval Newton method uses fixed point theorem and belongs to the class of *self validating algorithms*. This method allows to find zeros of a function

$$\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n. \quad (1)$$

In order to investigate the existence of zeros of $\mathbf{f}$ in an n-dimensional interval $\mathbf{X}$ one has to evaluate the *interval New-*

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ton operator

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - (\mathbf{Df}(\mathbf{X}))^{-1}\mathbf{f}(\mathbf{x}_0), \qquad (2)$$

where $\mathbf{x}_0$ is an arbitrary point belonging to the interval $\mathbf{X}$. One usually chooses $\mathbf{x}_0$ to be the center of $\mathbf{X}$.

The following lemma [1] states the relation between the zeros of $\mathbf{f}$ in $\mathbf{X}$ and the position of $\mathbf{N}(\mathbf{X})$ with respect to $\mathbf{X}$.

Theorem 1.

1. *If* $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ *then there exist exactly one point* $\mathbf{x} \in \mathbf{X}$ *such that* $f(\mathbf{x}) = 0$.

2. *If* $\mathbf{N}(\mathbf{X}) \cap \mathbf{X} = \emptyset$ *then there are no zeros of* $\mathbf{f}$ *in* $\mathbf{X}$.

The above theorem can be used to prove both the existence and uniqueness of zeros. By iterating the method one can easily sharpen the bounds of solutions.

2. ELECTRONIC CIRCUIT AND BASIC NOTATIONS

As an example let us consider the Chua's circuit, a simple third–order system defined by:

$$\begin{aligned} C_1\dot{x} &= G(y-x) - g(x), \\ C_2\dot{y} &= G(x-y) + z, \\ L\dot{z} &= -y - R_0 z, \end{aligned} \qquad (3a)$$

where $g(\cdot)$ is a three-segment piecewise-linear function

$$g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|). \qquad (3b)$$

For parameters: $C_1 = 1$, $C_2 = 9.3515$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$ the system (3) has a well-known "double–scroll" chaotic attractor. The state space $\mathbb{R}^3$ can be divided into three open regions $U_\pm = \{\mathbf{x} = (x,y,z)^T \in \mathbb{R}^3 : \pm x > 1\}$, $U_0 = \{\mathbf{x} = (x,y,z)^T \in \mathbb{R}^3 : |x| < 1\}$ separated by planes $V_\pm = \{\mathbf{x} \in \mathbb{R}^3 : x = \pm 1\}$. In the regions U_0 and $U_\pm$ the system is linear. The state equation can be rewritten as: $\mathbf{x} = \mathbf{A}_0\mathbf{x}$ if $x \in U_0$ and $\dot{\mathbf{x}} = \mathbf{A}_\pm(\mathbf{x} \mp \mathbf{p})$ if $x \in U_\pm$, where $\mathbf{A}_0$, $\mathbf{A}_+ = \mathbf{A}_-$ are matrices with real coefficients. In the regions $U_0, U_\pm$ the solution has the form $\mathbf{x}(t) = e^{\mathbf{A}_0 t}\mathbf{x}$ and $\mathbf{x}(t) = e^{\mathbf{A}_\pm t}(\mathbf{x} \mp \mathbf{p}) \pm \mathbf{p}$ respectively.

Let us define a *Poincaré map* $\mathbf{H} : V_+ \mapsto V_+$:

$$\mathbf{P}(\mathbf{x}) = \phi_{\tau(\mathbf{x})}(\mathbf{x}), \qquad (4)$$

where $\phi_t(\mathbf{x})$ is a trajectory of the system (3) based at $\mathbf{x}$ and $\tau(\mathbf{x})$ is the time needed for the trajectory $\phi_t(\mathbf{x})$ to return to V_+. Similarly we define a *halfmap* $\mathbf{H} : V_- \cup V_+ \mapsto V_- \cup V_+$:

$$\mathbf{H}(\mathbf{x}) = \phi_{\tau(\mathbf{x})}(\mathbf{x}), \qquad (5)$$

n	$n_\mathbf{H}$	Length	λ_1	λ_2
1	2	7.38	-3.1798	-0.00412
2	4	14.38	-9.09	$-5.39 \cdot 10^{-6}$
2	4	24.70	63.64	$3.93 \cdot 10^{-10}$
2	8	32.99	-4.521	$-6.01 \cdot 10^{-10}$
3	6	28.62	45.57	$3.69 \cdot 10^{-12}$

Table 1: Periodic orbits of $\mathbf{P}$. n is the period of the orbit. $n_\mathbf{H}$ is the number of the regions U_0 and $U_\pm$ visited by the orbit (with multiplicities — every region may be visited several times), $\lambda_{1,2}$ are the eigenvalues of $\mathbf{P}^n$

where $\tau(\mathbf{x})$ is the time needed for the trajectory $\phi_t(\mathbf{x})$ to reach one of the planes V_- or V_+.

In our computation we will need to evaluate the Jacobian matrix of the Poincaré map. Let us assume that the trajectory based at $\mathbf{x} \in V_+$ visits k linear regions before returning to V_+. Hence $\mathbf{P}(\mathbf{x}) = \mathbf{H}^k(\mathbf{x})$ and the Jacobian of the full Poincaré map is

$$\mathbf{DP}(\mathbf{x}) = \mathbf{DH}(\mathbf{H}^{k-1}(\mathbf{x})) \cdots \mathbf{DH}(\mathbf{H}(\mathbf{x})) \cdot \mathbf{DH}(\mathbf{x}). \quad (6)$$

For the evaluation of the Jacobian of $\mathbf{H}$ we will use the following lemma [3]:

Lemma 1. *Let* $\mathbf{x}_0 \in V_- \cup V_+$. *Let us assume that the solution of the system for* $t \in [0, t_0]$ *is given by* $\mathbf{x}(t) = e^{\mathbf{A}t}(\mathbf{x}_0 - \mathbf{p}) + \mathbf{p}$. *Let* $\mathbf{y}_0 = \mathbf{H}(\mathbf{x}_0) = \mathbf{x}(t_0)$. *Let us also assume that the intersections of* V_- *and* V_+ *with the trajectory* $\mathbf{x}(t)$ *at* $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *are transversal. Then the Jacobian of the halfmap* $\mathbf{H}$ *at* $\mathbf{x}_0$ *is the principal minor (created by removing the first row and the first column) of the matrix*

$$\left[\mathbf{I} - \frac{\mathbf{A}(\mathbf{y}_0 - \mathbf{p})\mathbf{e}_1^T}{\mathbf{e}_1^T\mathbf{A}(\mathbf{y}_0 - \mathbf{p})}\right] e^{\mathbf{A}t_0}, \qquad (7)$$

where $\mathbf{e}_1 = (1,0,0)^T$ *and* $\mathbf{I}$ *is the* 3×3 *identity matrix.*

3. PERIODIC ORBITS

In this section we show how interval Newton method can be used to prove the existence and uniqueness of periodic orbits for continuous–time systems. For the extraction of periodic orbits we have used the combination of the method of close returns [5] and standard Newton method. We have found several periodic points of the Poincaré map $\mathbf{P}$ associated with the continuous flow. Some of them are shown in Fig. 1. Their approximate position and other parameters are collected in Table 1.

Now we describe how to prove the existence of a periodic orbit by means of the interval Newton method. Let n be the period of the orbit. First one chooses a rectangle $\mathbf{X}$ on the Poincaré map which encloses the periodic point found numerically. Then one evaluates the image of the center $\mathbf{x}_0$

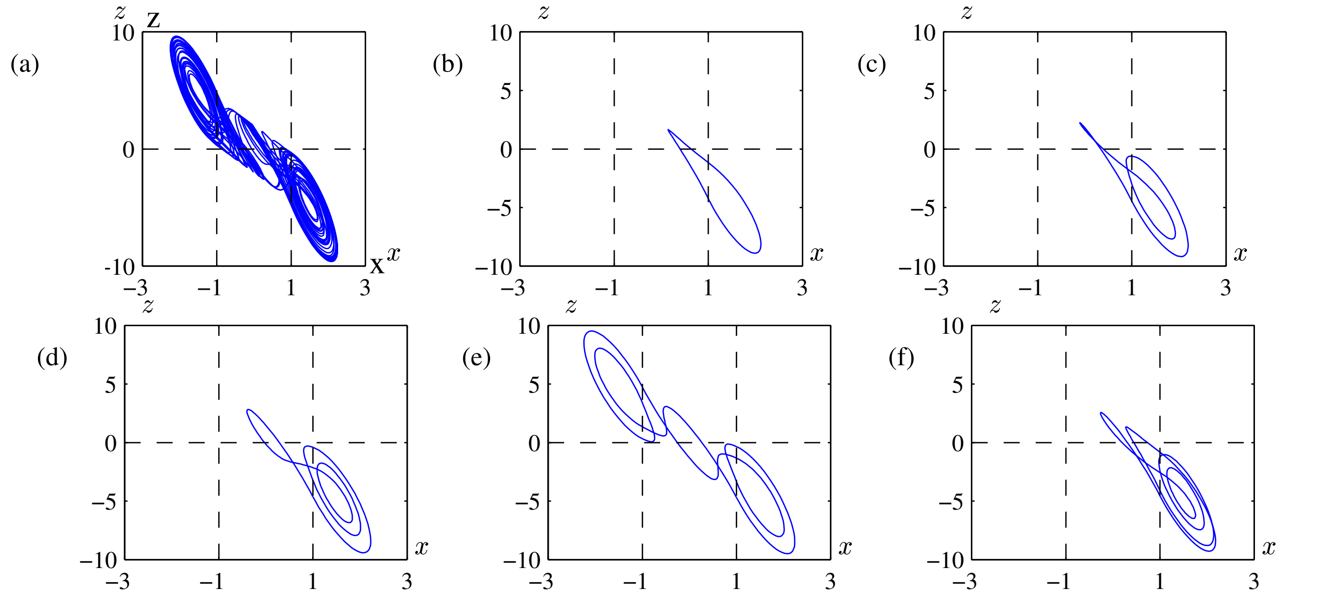

Figure 1: Chaotic trajectory (a) and periodic orbits of the Chua's circuit, period-1 orbit (b), period-2 orbits (c)–(e), period-3 orbit (f)

of $\mathbf{X}$ under the n^{th} iteration of the Poincaré map. We also need to compute the Jacobian matrix of $\mathbf{P}^n$ at the interval $\mathbf{X}$. Finally the interval Newton operator for the map $\mathrm{id}-\mathbf{P}^n$ is computed:

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - \left(\mathbf{I} - \mathbf{DP}^n(\mathbf{X})\right)^{-1}\left(\mathbf{x}_0 - \mathbf{P}^n(\mathbf{x}_0)\right). \quad (8)$$

If $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ then there exists exactly one periodic point of $\mathbf{P}$ with period n belonging to $\mathbf{X}$. In the opposite case one has to modify the initial rectangle $\mathbf{X}$ and repeat the computations.

3.1. Fixed point

First let us consider a fixed point of $\mathbf{P}$ (compare Fig. 1(b)). Using the Newton method we were able to prove the existence and uniqueness of the fixed point of $\mathbf{P}$ within the rectangle $(y,z) = (-0.333^{2181}_{0109}, -4.239^{9987}_{7915})$. Its diameter is greater than 0.0002 in both y and z. By using the techniques described later it is possible to improve this result and obtain even greater rectangle (this is important from the point of view of uniqueness).

By applying the interval Newton operator iteratively (3 iterations) we were able to prove the existence of the fixed point within the interval

$$(y,z) = (-0.333114482^{207}_{009}, -4.23989511^{63}_{47}). \quad (9)$$

In this way we have sharpened the bounds of the result to an uncertainty under $1.98 \cdot 10^{-10}$ in y and $1.6 \cdot 10^{-9}$ in z. Using this result we have computed the Jacobian matrix of the Poincaré map at the fixed point and the eigenvalues of this Jacobian. The eigenvalues belong to the intervals: -3.1798^{239}_{308} and -0.0041^{23}_{30}. The uncertainty is below $7 \cdot 10^{-6}$. Hence the fixed point is of a saddle type.

3.2. Period-2 periodic orbit

Let us now consider the period–2 orbit shown in Fig. 1(c). This orbit has four intersections with the planes $V_{\pm}$.

First we have tried to use directly theorem 1 to prove the existence of this periodic orbit. Starting with intervals $\mathbf{X}$ of different size we have computed $\mathbf{N}(\mathbf{X})$ according to equation (8). When the diameter of $\mathbf{X}$ is smaller than 10^{-8} the diameter of $\mathbf{N}(\mathbf{X})$ is almost constant and greater than $2 \cdot 10^{-7}$. For greater $\mathbf{X}$ ($\mathrm{diam}(\mathbf{X}) > 10^{-7}$) it is not possible to compute $\mathbf{N}(\mathbf{X})$ (the Jacobian matrix is not invertible). Hence it is not possible to prove the existence of periodic point using the interval Newton method directly. The reason is the "wrapping effect" that causes quick growth of initial rectangle when we compute its trajectory using interval arithmetic.

One method to overcome this problem is to divide the initial rectangle into several subsets $\mathbf{X}_i$ and compute $\mathbf{N}(\mathbf{X}_i)$ for each of them. In order the prove the existence of the orbit is suffices to show that $\mathbf{N}(\mathbf{X}_i) \subset \mathbf{X}$ for each i. We have estimated that in order to fulfill this condition we should divide $\mathbf{X}$ into more than $2 \cdot 10^5$ rectangles.

Much better results can be obtained by using the method of intermediate sections. In this method one chooses several sections along the trajectory (in our case it is natural to choose sections $x = \pm 1$) and makes a division into a certain number of rectangles at each section. It corresponds to use the previous method for each halfmap independently. This may reduce significantly the diameter of the computed Jacobian matrix. Similarly we can use intermediate sections for the computation of $\mathbf{P}^n(\mathbf{x}_0)$. It makes no sense to divide the initial interval $\mathbf{x}_0$ as it is already a point interval. But its images under $\mathbf{H}^i$ have nonzero diameter and the method of intermediate sections may reduce the diameter of $\mathbf{P}^n(\mathbf{x}_0)$.

In the table below we present the influence of using different number of covering rectangles at each intermediate section on the diameter of the Jacobian matrix and the diameter of the image of the center. In all cases we started from the rectangle: $\mathbf{X} = (-0.351500^8_9, -4.43901^{119}_{245})$.

r	$\mathrm{diam}(\mathbf{DP}^n(\mathbf{X}))$	$\mathrm{diam}(\mathbf{P}^n(\mathbf{x}_0))$
1	no inverse	$1.34 \cdot 10^{-6}$
4	7.38	$2.93 \cdot 10^{-7}$
4^2	0.761	$8.41 \cdot 10^{-8}$
4^3	0.156	$1.64 \cdot 10^{-8}$
4^4	0.045	$1.64 \cdot 10^{-8}$

For $r = 1$ (no divisions) the inverse of the Jacobian does not exists. For $r \geq 4$ the condition $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ holds and hence the existence and uniqueness of period two orbit follow from Theorem 1. For example when we use division into 4 rectangles at each intermediate section for the evaluation of both $\mathbf{DP}^n(\mathbf{X})$ and $\mathbf{P}^n(\mathbf{x}_0)$ we obtain

$$\mathbf{N}(\mathbf{X}) = (-0.3515008^{12}_{79}, -4.43901^{19}_{25}) \qquad (10)$$

with diameter $6.7 \cdot 10^{-8}$ in y and $6 \cdot 10^{-7}$ in z. If we divide into 4^4 rectangles we obtain $\mathbf{N}(\mathbf{X})$ with diameter $1.9 \cdot 10^{-9}$ in y and $1.27 \cdot 10^{-8}$ in z. Hence the method of intermediate sections can also be used for sharpening the bounds of the position of the periodic orbit.

3.3. Other periodic orbits

We have also tried to use the interval Newton method for proving the existence of other periodic orbits from Table 1. We were not able to do this in a reasonable time. We have estimated that in order to prove the existence of the period–3 orbit (compare Fig. 1(f)) it would be necessary to use 4^7 rectangles at each intermediate section. For the proof of the existence of period–2 "symmetric" orbit (Fig. 1(e)) we need 4^9 rectangles at each section. For the proof of the existence of period–2 orbit with three scrolls (see Fig. 1(d)) we need 4^{10} covering rectangles. This orbit is the most difficult probably due to large eigenvalues (compare Table 1). The time necessary to complete the proof for the first of these orbits is more than 10 hours, using Sun-ULTRA 1 computer. This is the cost we pay for the necessity of computing exact Jacobian matrix. However it may by worth doing as we gain the uniqueness of the orbit.

It is possible that the assumptions of the method could be checked in a reasonable time if we use interval arithmetic based on higher precision (we have used double precision).

4. CONCLUSIONS

We have studied the possibility of using interval Newton method for proving the existence and uniqueness of periodic orbits continuous–time systems. This method is very powerful in a sense that it allows to prove also the uniqueness of the obtained periodic solution. It is also very efficient as in order to prove the existence and uniqueness we need to evaluate the map under investigation only at one interval.

As an example we have considered a simple third-order nonlinear circuit. We were able to prove the existence and uniqueness (in a certain region) of two unstable periodic orbits. By iterating the interval Newton method we sharpened the bounds of the fixed point on the Poincaré map to an uncertainty less than $1.6 \cdot 10^{-9}$. For longer periodic orbits we have estimated the time necessary to perform the proof.

It seems that the interval Newton method can be successfully used for proving the existence and uniqueness of periodic orbits with low periods. In case of longer periodic orbits when it is not possible to prove the assumptions of the method in one step one can divide the initial interval into several subsets obtaining smaller bounds for the result of the Newton operator. Even better one can use the method of intermediate sections. For very long periodic orbits the time necessary to prove the assumptions of the method is usually very long which makes the method unusable.

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