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OPF Control of Chaotic Systems — Analytical and Simulation Results

ABSTRACT

In this paper we present some theoretical results of the possibilities for controlling a given chaotic system using the (occasional proportional feedback) OPF method. It is usually claimed that the OPF method can be used only if the return map is almost one–dimensional and that the OPF control lacks a theoretical background for the choice of the feedback gain. We show how to find the gain value for which the stabilization can be obtained. We prove that the possibility of stabilization depends on the behavior of the system in the neighborhood of the periodic orbit and not on the dimensionality of the attractor on the Poincaré surface.

1. INTRODUCTION

It is well known that by perturbing a chaotic system in the right way one can force the system to behave more regularly. Various methods of controlling chaotic systems have been developed. They can be divided into two main categories. The first class consists of nonfeedback methods, where chaos is supressed by applying some external forcing: constant, periodic or even random noise. Feedback methods fall in the second class. In one of these methods the control force is proportional to the difference between a desired oscillation (generated by a special oscillator) and the scalar output of the chaotic system. In another one stabilization is achieved by applying a control signal proportional to the difference between scalar variable and its delayed version. An important category of feedback methods uses the existence of infinitely many unstable periodic orbits embedded within a chaotic attractor. These methods are of special interest because they use specific properties of chaotic systems and only very small control signals are required. A comprehensive review of the methodologies for controlling chaos is presented in [3].

Let us start with the description of two feedback methods, namely OGY [1] and OPF [2] control. One of the special properties of chaotic attractors is that they contain an infinite number of unstable periodic orbits. As it was shown in [1], any of these periodic orbits can be stabilized by applying small perturbations to one of the system parameters.

Let us assume that we have a three-dimensional autonomous continuous time system of first-order ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, p), \tag{1}$$

where $\mathbf{x} \in I\!R^3$ and $p \in I\!R$ is a system parameter which we can change. We also assume that parameter p can be modified within a small interval around its nominal value p_0. We choose a two-dimensional transversal section Σ which defines a Poincaré map $\mathbf{P}$. Since the vector field $\mathbf{F}$ depends on p, the Poincaré map $\mathbf{P}$ also depends on this parameter p. Thus, we have

$$\mathbf{P}\colon I\!R^2 \times I\!R \ni (\xi, p) \longrightarrow \mathbf{P}(\xi, p) \in I\!R^2, \tag{2}$$

where $\xi = (\xi_1, \xi_2)^T$. $\mathbf{P}(\xi, p)$ is the point at which the trajectory starting from $\xi \in \Sigma$ intersects Σ for the first time. Let us assume that $\mathbf{P}$ is differentiable. Say we have selected one of the unstable periodic orbits embedded in the system's attractor as the goal of our control because, for example, it offers an improvement in system performance over the original chaotic behavior. For simplicity, we assume that this is a period-1 orbit (a fixed point of $\mathbf{P}$). Let us denote by $\xi_F = (\xi_{F1}, \xi_{F2})^T$ an unstable fixed point of $\mathbf{P}$ for $p = p_0$ (i.e., $\mathbf{P}(\xi_F, p_0) = \xi_F$). Let the first-order approximation of $\mathbf{P}$ in the neighborhood of (ξ_F, p_0) be of the form

$$\mathbf{P}(\xi, p) \approx \mathbf{P}(\xi_F, p_0) + \mathbf{A} \cdot (\xi - \xi_F) + \mathbf{w} \cdot (p - p_0), \tag{3}$$

where $\mathbf{A}$ is a Jacobian matrix of $\mathbf{P}(\cdot, p_0)$ at ξ_F, and $\mathbf{w} = \frac{\partial \mathbf{P}}{\partial p}(\xi_F, p_0)$ is the derivative of $\mathbf{P}$ with respect to the parameter p.

Stabilization of the fixed point is achieved by realizing feedback of the form

$$p(\xi) = p_0 + \mathbf{c}^T(\xi - \xi_F). \tag{4}$$

In the original description of the OGY method [1], the vector $\mathbf{c}$ is computed using the expression

$$\mathbf{c} = -\frac{\lambda_u}{\mathbf{f}_u^T \mathbf{w}} \mathbf{f}_u^T, \tag{5}$$

where λ_u is the unstable eigenvalue and $\mathbf{f}_u$ is the corresponding contravariant eigenvector of $\mathbf{A}$.

The second method, we will describe is the one–dimensional version of the OGY method, called usually the occasional proportional feedback (OPF) control. In this method the parameter p is computed using only one variable, for example ξ_1:

$$p(\xi) = p_0 + c(\xi_1 - \xi_{F1}). \tag{6}$$

There are several possibilities of implementation of the OPF control. One of them is the Hunt's implementation [2]. Hunt uses the peaks of one of the system variables to generate the one–dimensional map. He uses a window around a fixed level to set the region where control is applied. This approach means that his controller needs just one of the system variables as input. In order to find the peaks, Hunt's scheme uses a synchronizing generator. The frequency, delay, control pulse width, window position, width and gain are all adjustable. All these parameters are found by trial and error. One of the major advantages of Hunt's controller over OGY is that the control law depends on only one variable and does not require any complicated calculations in order to generate the required control signal.

In another implementation of the OPF method (compare [6]) one takes the derivative of the input signal and generates a pulse when it passes through zero. This pulse is used instead of Hunt's external driving oscillator as the "synch" pulse for the Poincaré map. This obviates the need for the external generator and makes the controller simpler.

2. THEORETICAL RESULTS

In this section, we address the problem of whether or not it is possible to stabilize a given periodic orbit using the one–dimensional control method. We describe our approach for the case of stabilizing a fixed point of a Poincaré map.

The control signal is computed using equation (6). We want to find values of c for which ξ_F is an asymptotically stable fixed point of the map $\xi \mapsto \mathbf{P}(\xi, p(\xi))$.

Theorem 1 *Let*

$$\mathbf{f}(\xi, p) = \mathbf{A}\xi + \mathbf{w}p, \tag{7}$$

where $\mathbf{A}$ *is a two-dimensional square matrix,* $\xi = (\xi_1, \xi_2)^T$*,* $\mathbf{w} = (w_1, w_2)^T$ *and* $p \in I\!R$*. Let us denote* $\mathbf{A} = (a_{ij})_{i,j=1,2}$*. Let* $\mathrm{tr}\mathbf{A} = a_{11} + a_{12}$ *and* $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ *denote the trace and determinant of matrix* $\mathbf{A}$ *respectively. If*

$$\begin{aligned} 1 + \mathrm{tr}\mathbf{A} + \det \mathbf{A} + c(w_1 + w_1 a_{22} - w_2 a_{12}) &> 0 \\ 1 - \det \mathbf{A} + c(-w_1 a_{22} + w_2 a_{12}) &> 0 \\ 1 - \mathrm{tr}\mathbf{A} + \det \mathbf{A} + c(-w_1 + w_1 a_{22} - w_2 a_{12}) &> 0 \end{aligned} \tag{8}$$

then $(0,0)^T$ *is a stable fixed point of*

$$\mathbf{f}(\xi) \stackrel{\mathrm{df}}{=} \mathbf{f}(\xi, p(\xi)) = \mathbf{f}(\xi, c\xi_1) = \mathbf{A}\xi + \mathbf{w}c\xi_1. \tag{9}$$

Proof:

$$\begin{aligned} \mathbf{f}(\xi) &= \mathbf{A}\xi + \mathbf{w}c\xi_1 = \mathbf{A}\xi + \mathbf{w}\,(c \quad 0)\,\xi \\ &= \begin{pmatrix} a_{11} + w_1 c & a_{12} \\ a_{21} + w_2 c & a_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \stackrel{\mathrm{df}}{=} \mathbf{W}_1 \xi. \end{aligned}$$

Now $(0,0)^T$ is an asymptotically stable fixed point of $\mathbf{f}$ if both eigenvalues of $\mathbf{W}_1 = \mathbf{A} + \mathbf{w}\,(c \quad 0)$ lie within the unit circle. One can easily see using for example the Hurwitz criterion that this is equivalent to (8). □

The above theorem is formulated for the case of a linear map $\mathbf{f}$ with the fixed point $\xi_F = (0,0)^T$. The next theorem extends this result to a nonlinear map $\mathbf{P}$ with an arbitrary fixed point ξ_F.

Theorem 2 *Let* $\mathbf{P}$ *be the map defined in* (2) *and let* ξ_F *be a fixed point of* $\mathbf{P}$*. Let the linear approximation of* $\mathbf{P}$ *be of the form* (3)*. Define*

$$\mathbf{P}(\xi) \stackrel{\mathrm{df}}{=} \mathbf{P}(\xi, c\xi_1) \tag{10}$$

If conditions (8) *are satisfied then there exists a neighborhood* U *of* ξ_F *such that* $\mathbf{P}^n(\xi) \stackrel{n\to\infty}{\longrightarrow} \xi_F$ *for all* $\xi \in U$ *(i.e.* ξ_F *is asymptotically stable for the map* (10)*).*

Proof: Apply Theorem 1 to the linear approximation of the map $\mathbf{P}$ at ξ_F. □

From the above theorem one can find values of c for which successful control is possible. Similar results can be obtained for the case when the second variable is used for the computation of the control signal (parameter is modified according to $p(\xi) = p_0 + c(\xi_2 - \xi_{F2}$, this time one has to check if eigenvalues of $\mathbf{W}_2 = \mathbf{A} + \mathbf{w}\,(0 \quad c)$ lie within the unit circle).

The most important conclusion which can be drawn from the results presented in this section is that the possibility of succesfull control using the OPF technique depends on the form of the linear approximation of the system's behavior in the neighborhood of the periodic orbit.

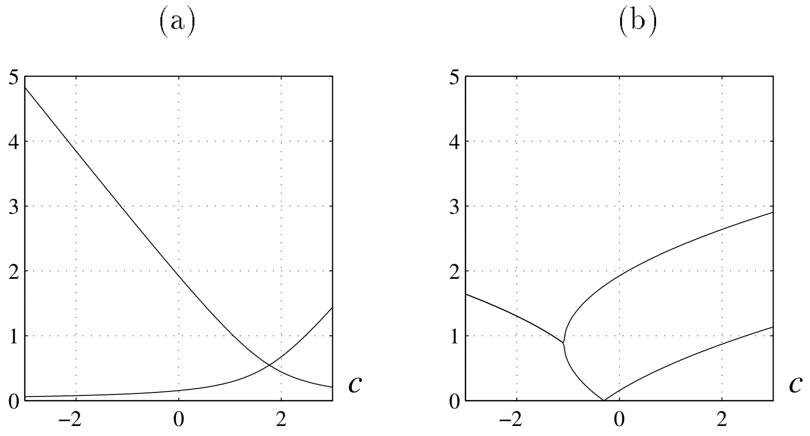

Figure 1. Eiqenvalues of the controlled Hénon system at the fixed point for different values of c, (a) control signal computed using x state, eigenvalues of $\mathbf{W}_1 = \mathbf{A} + \mathbf{w}\,(c \quad 0)$ (b) control signal computed using y state, eigenvalues of $\mathbf{W}_2 = \mathbf{A} + \mathbf{w}\,(0 \quad c)$

3. SIMULATION RESULTS

First let us consider the Hénon system: $\mathbf{h}((x,y)^T) = (a - x^2 + by, x)^T$. This is a two-dimensional nonlinear map. For $a = 1.4$, $b = 0.3$ chaotic trajectories can be observed. The coordinates of the fixed point (x_F, y_F) and the Jacobian matrix $\mathbf{A}$ at the point $(x_F, y_F)^T$ can be computed analytically

$$x_F = y_F = \left((b-1) + \sqrt{(b-1)^2 + 4a}\right)/2$$

$$\mathbf{A} = \begin{pmatrix} -2x_F & b \\ 1 & 0 \end{pmatrix}.$$

Parameter a with nominal value $a_0 = 1.4$ was chosen as a control parameter. The derivative of $\mathbf{h}$ with respect to a is $\mathbf{w} = (1,0)^T$. Possibility of stabilization depends on the eigenvalues of the Jacobian matrix of the controlled system at the fixed point. Magnitudes of eigenvalues of $\mathbf{W}_1$ and $\mathbf{W}_2$ for different values of the gain c are shown in Fig. 1. The sufficient condition for the one–dimensional control method to work is that both eigenvalues are smaller than 1 in absolute value.

Let us first consider the case when the x state is used for the computation of the control signal. Using Theorem 1 one can easily prove that if $c \in (2x_F + b - 1, 2x_F - b + 1) \approx (1.06, 2.46)$ then both eigenvalues of the controlled system lie within the unit circle (compare also Fig. 1a). The eigenvalue, greater in magnitude decides about the quality of control. The smaller this value is the more robust control can be obtained. In our case the best results can be achieved for $c = 2x_F \approx 1.76$, where the curves representing two eigenvalues intersect (compare Fig. 1). For this c eigenvalues are $\lambda_{1,2} = \pm\sqrt{b} \approx \pm 0.547$. This control procedure was tested in computer simulations. We observed very quick convergence of the trajectory to the fixed point.

In the second case when we use the y variable it is much more difficult to stabilize the fixed point. According to Theorem 1 proper behavior of the method is guaranteed for $c \in [-1 - b, 1 - b - 2x_F] \approx [-1.3, -1.06]$. This interval is much smaller than in the previous case. Moreover the eigenvalues for the best choice of c are now greater, namely for $c = -b - x_F^2 \approx 1.07$ both eigenvalues are equal: $\lambda_{1,2} = -x_F \approx -0.88$. In computer simluations we observed that even small deviation from this value of c causes that the control method fails.

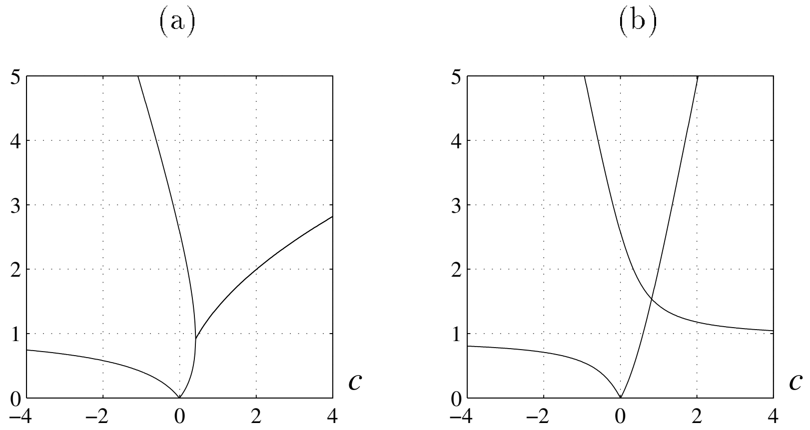

Figure 2. Eiqenvalues of controlled Chua's system at the fixed point for different values of c, control signal computed with (a) state y used for the computation of the control signal, eigenvalues of $\mathbf{W}_1 = \mathbf{A} + \mathbf{w}\,(c \quad 0)$ (b) state y used for the computation of the control signal, eigenvalues of $\mathbf{W}_2 = \mathbf{A} + \mathbf{w}\,(0 \quad c)$

The above example shows clearly that we can apriori predict which variable should be used for the computation of the control signal (if we have a choice) and what gain should be chosen in order to obtain robust stabilization of the periodic orbit.

As another example let us consider the Chua's circuit, which is a three-dimensional system described by the following state eqution:

$$\begin{aligned} C_1\dot{x} &= -g(x) + z, \\ C_2\dot{y} &= -Gy + z, \\ L\dot{z} &= -x - y - Rz, \end{aligned} \tag{11}$$

where $g(\cdot)$ is a piece-wise linear characteristic $g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|)$. We have considered the Chua's circuit with the following set of parameters: $C_1 = 1.02$, $C_2 = -0.632$, $G = -0.0033$, $L = -1.02$, $R = -0.33$, $G_a = -0.419$, $G_b = 0.839$. We have chosen the transversal plane $x = 1$ and the slope G_a of the nonlinear function g as the control parameter. As the aim of the stabilization procedure we chave chosen the period–1 orbit, corresponding to the fixed point on the transversal plane. From the three dimensional time series we have found the approximate position of the fixed point $\xi_F \approx (-1.39, -1.02)$ and the Jacobian matrix of the Poincaré map at the fixed point

$$\mathbf{A} \approx \begin{pmatrix} -1.67 & -1.15 \\ -1.31 & -0.91 \end{pmatrix}.$$

The derivative of the Poincaré map with respect to G_a were found to be $\mathbf{w} \approx (1.82, 3.15)^T$. In Fig. 2 the eigenvalues of matrices $\mathbf{W}_1 = \mathbf{A} + \mathbf{w}\,(c \quad 0)$ and $\mathbf{W}_2 = \mathbf{A} + \mathbf{w}\,(0 \quad c)$ are shown. For $c = 0.42$ both eigenvalues of the first matrix are aproximately 0.92. For the second matrix always at least one eigenvalue is greater than 1 in magnitude. It follows from Theorem 1 that if y is used for the computation of the control signal then we should succeed in stabilization of the chosen periodic orbit if we use c close to 0.42.

The one–dimensional control method was tested in simulations. The results for the case of y and $c = 0.42$ are shown in Fig. 3. We managed to stabilize the chosen orbit, but the stabilization was not robust against small change of c. For the second case (variable z) we were not able to stabilize the periodic orbit for any value of c.

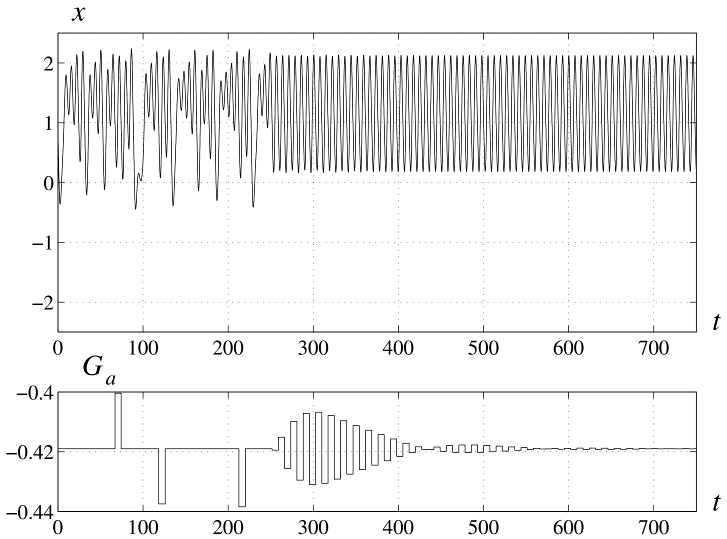

Figure 3. Control of Chua's circuit with the one–dimensional version of the OGY method, (a) state variable, (b) control parameter G_a

4. CONCLUSIONS

In this paper some analytical results concerning the possibilities for stabilizing periodic orbits in chaotic systems using one–dimensional control methods have been presented. We stress that the only essential difference between the OGY and OPF methods is the number of variables used for the computation of the control signal. Furthermore, we have shown that the usual assumption, when using OPF, that the Poincaré map is almost one–dimensional is *unnecessary*. The only assumptions that we need in order to determine if a given periodic orbit can be stabilized are concerned with the dynamics of the system in the neighborhood of the chosen orbit. We have shown that the OPF method does not lack the theoretical basis for the choice of the feedback gain. In order to compute it we can use similar methods as for the OGY control.

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