CNN ARRAY WITH BALANCED CHAOTIC CELLS — STABILITY OF THE SYNCHRONOUS MOTION

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Abstract— We investigate the stability of synchronous motion in an array of bi–directionally coupled electronic circuits. We compute Lyapunov exponents of the generic variational equation associated with directions transversal to the synchronization subspace. Using this results we derive conditions for the coupling strength for which the stable synchronous solution exists. We also find the limit on the size of the network, which can sustain stable synchronous motion. Theoretical results are compared with the results of numerical experiments.

I. INTRODUCTION

Cellular networks of coupled oscillators provide a versatile model for a variety of phenomena observed in real systems in such areas as physics, biology and medicine. Dynamics of such systems is one of very lively studied topics [1, 2, 3, 4].

Depending on dynamics of individual cells in the network and the coupling between them a variety of interesting behaviors can be observed, including hyper–switching and clustering [1], attractor crowding and various kinds of spatial, temporal or spatio–temporal orderly structures referred to as self–organization [5].

One of particularly interesting type of dynamical behavior that occurs in networks of coupled systems is the synchronization behavior when all cells behave in the same manner. In this context stability of the synchronous motion is a very important problem. In this paper we study the stability of synchronous motion in a one–dimensional lattice of bi–directionally coupled chaotic circuits.

II. DYNAMICS OF THE NETWORK

Let us consider a one–dimensional cellular neural network composed of simple third–order electronic oscillators. The cells are coupled bi-directionally by means of two resistors cross-connected between the capacitors C_1 and C_2 of the neighboring cells. Every cell is connected with two nearest neighbors. The dynamics of the one–dimensional lattice composed of n cells can be described by the following set of ordinary differential equations:

$$\begin{aligned} C_2\dot{x}_i &= -y_i + (G-2G_1)(z_i - x_i) + \\ &\quad + G_1(z_{i-1} - x_i) + G_1(z_{i+1} - x_i), \\ L\dot{y}_i &= x_i, \\ C_1\dot{z}_i &= (G-2G_1)(x_i - z_i) - f(z_i) + \\ &\quad + G_1(x_{i-1} - z_i) + G_1(x_{i+1} - z_i), \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, n$ and the lattice forms a ring ($x_{n+1} = x_1$, $z_{n+1} = z_1$, $x_0 = x_n$, $z_0 = z_n$). f is a five–segment piecewise linear function

$$\begin{aligned} f(z) = m_2 z &+ \frac{1}{2}(m_1 - m_2)(|z + B_{p_2}| - |z - B_{p_2}|) \\ &+ \frac{1}{2}(m_0 - m_1)(|z + B_{p_1}| - |z - B_{p_1}|). \end{aligned} \tag{2}$$

In our study we use typical parameter values for which an isolated cell generates chaotic oscillations — the "double scroll" attractor ($C_1 = 1/9F$, $C_2 = 1F$, $L = 1/7H$, $G = 0.7S$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$). For the integration of the system the fourth-order Runge-Kutta method was used with the time step $\tau = 0.1$.

The setup described above is slightly different from the one used in our previous experiments [3, 4]. We use balanced chaotic cells, where the value of the resistor connecting capacitors C_1 and C_2 in a single cell is modified. This ensures the existence of a synchronized chaotic solution. If we apply identical initial conditions to every cell in the array ($x_i(0) = x(0)$, $y_i(0) = y(0)$, $z_i(0) = z(0)$ for $i = 1, \ldots, n$) then all the cells oscillate synchronously and the equations describing the array can be written as

$$\begin{aligned} C_2\dot{x} &= -y + G(z - x), \\ L\dot{y} &= x, \\ C_1\dot{z} &= G(x - z) - f(z), \end{aligned} \tag{3}$$

where $x_i = x$, $y_i = y$ and $z_i = z$ for $i = 1, \ldots, n$. Hence in the case of equal initial conditions the network as a whole behaves chaotically as a single uncoupled cell. Our aim in this paper is the investigation of stability of this synchronous solution.

III. STABILITY OF THE SYNCHRONOUS MOTION

In order to investigate stability of the synchronous motion we use Lyapunov exponents. We follow the method introduced in [6].

Let us denote the variables of the ith cell by $\mathbf{x}_i = (x_i, y_i, z_i)^T$. Let $\mathbf{F}$ be the dynamics of the uncoupled cell, $\mathbf{x} = \mathbf{F}(\mathbf{x})$ as defined by Equation (3). Let us write the dynamics of the ith cell in the following form:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + G_1 \sum_j G_{ij} \mathbf{E} \mathbf{x}_j \,, \tag{4}$$

where G_1 is the coupling strength and $\mathbf{E}$ is the matrix that is used for the coupling. For the bidirectional coupling of our system the coupling matrix is

$$\mathbf{E} = \begin{pmatrix} 0 & 0 & 1/C_2 \\ 0 & 0 & 0 \\ 1/C_1 & 0 & 0 \end{pmatrix} \tag{5}$$

$\mathbf{G}$ is the matrix of couplings between cells. In our setup we use $G_{ij} = 1$ for adjacent cells, $G_{ii} = -2$ (this corresponds to the modification of G connecting capacitors C_1 and C_2 by $-2G_1$) and $G_{ij} = 0$ for other cells (no connection for distant cells).

$$\mathbf{G} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix} \tag{6}$$

The sum $\sum_j G_{ij} = 0$ and hence the synchronization manifold in invariant. For $\mathbf{x}_i = \mathbf{x}_j$ for all i, j the last term in Equation (4) disappears.

Then we create the variational equation of Equation (4) and diagonalize $\mathbf{G}$ obtaining a block diagonalized variational equation with each block having the form (for the details see [6]):

$$\dot{\xi}_k = (D\mathbf{F} + G_1 \gamma_k \mathbf{E}) \xi_k \,, \tag{7}$$

where γ_k are eigenvalues of $\mathbf{G}$ ($k = 0, \ldots, N-1$). The above equation for $k = 0$ is the variational equation for the synchronization manifold. Other eigenvalues correspond to transverse eigenvectors.

The idea, introduced in [6], is to compute the maximum Lyapunov exponent for the generic variational equation

$$\dot{\xi} = (D\mathbf{F} + \gamma \mathbf{E}) \xi, \tag{8}$$

as a function of γ. In general case Equation (8) should be solved for γ from complex plane. In our case however as the matrix $\mathbf{G}$ has only real eigenvalues it suffices to solve it for real line only.

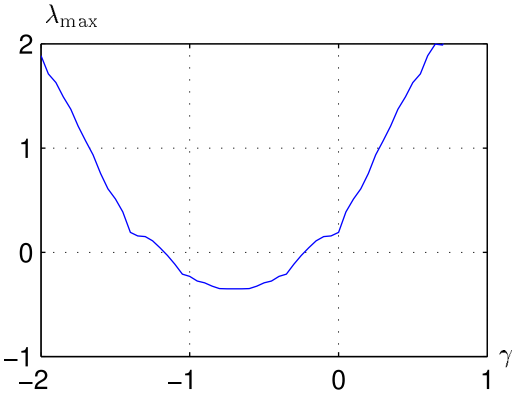

Figure 1: Maximum Lyapunov exponent $\lambda_{\max}$ of the generic variational equation for real γ.

We have solved the generic variational equation for 60 equidistant points from the interval $(-2, 1)$. The results are shown in Fig. 1.

Now we describe how to use the solution of the variational equation plotted in Fig. 1 for investigation of stability of synchronous motion. Let us assume that we have n cells in the array and the coupling strength is G_1. First we compute the eigenvalues $\gamma_k = -4\sin^2 \pi k/n$ of $\mathbf{G}$. We pick up the eigenvalues associated with transverse eigenvectors $(\gamma_1, \ldots, \gamma_{n-1})$. The synchronous motion is stable if $\lambda_{\max}$ is negative for $\gamma = G_1\gamma_k$, where $k = 1, \ldots, n-1$. We can read this information from Fig. 1.

Now we derive the conditions for G_1 ensuring stable synchronous state and find the array sizes for which it is possible to obtain stable synchronous solution. Let us assume that there exist an interval of γ for which the maximum Lyapunov exponent of Equation (8) is negative. From Fig. 1 one can see that this assumption is fulfilled. Let us denote the ends of this interval by α and β (with $\alpha < \beta < 0$). We have estimated that in our case $\alpha \approx -1.171$ and $\beta \approx -0.229$.

The conditions for stability of synchronous motion are $\alpha < G_1\gamma_k < \beta$ for $k = 1, \ldots, n-1$. Using the formulas for γ_k one can easily obtain the range of G_1, for which synchronization is ensured:

$$G_1 \in \left(\frac{-\beta}{4\sin^2 \frac{\pi}{n}}, \frac{-\alpha}{4} \right) \qquad \text{for even } n, \tag{9}$$

$$G_1 \in \left(\frac{-\beta}{4\sin^2 \frac{\pi}{n}}, \frac{-\alpha}{4\sin^2 \frac{(n-1)\pi}{2n}} \right) \qquad \text{for odd } n. \tag{10}$$

For a given network size n synchronization is possible if the above intervals are not empty. One can easily obtain the following conditions:

$$n < \frac{\pi}{\arcsin\sqrt{\beta/\alpha}} \approx 6.86 \qquad \text{for } n \text{ even}, \quad (11)$$

$$n < \frac{\pi}{2\arcsin 0.5\sqrt{\beta/\alpha}} \approx 7.05 \qquad \text{for } n \text{ odd}. \quad (12)$$

n	G_1
3	$(0.0763, 0.3903)$
4	$(0.1145, 0.2928)$
5	$(0.1657, 0.3237)$
6	$(0.2290, 0.2928)$
7	$(0.3041, 0.3080)$
8	$(0.3909, 0.2928) = \emptyset$
9	$(0.4894, 0.3019) = \emptyset$

Table 1: Coupling strength G_1 for which the synchronous state is stable.

Hence, synchronization is possible for $n = 3, \ldots, 7$. In Table 1 we collect the values of the coupling strength G_1 for which the synchronous state is stable. They were obtained using formulas (9) and (10).

(a) (b) (c) (d)

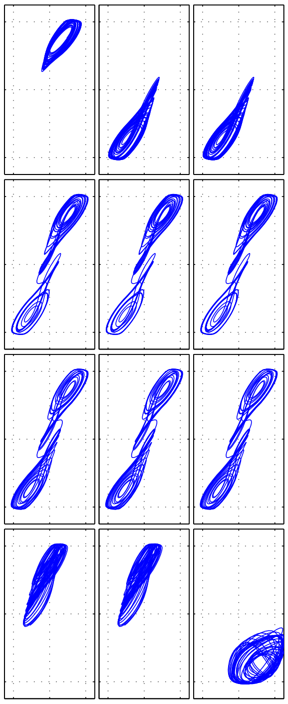

Figure 2: Steady–state of disturbed synchronous motion in an array composed of $n = 3$ circuits for different coupling strength: (a) $G_1 = 0.05$, (b) $G_1 = 0.1$, (c) $G_1 = 0.35$, (d) $G_1 = 0.4$.

IV. COMPUTER SIMULATIONS

In this section we outline the results of computer experiments on stability of synchronous solution.

(a) (b) (c) (d)

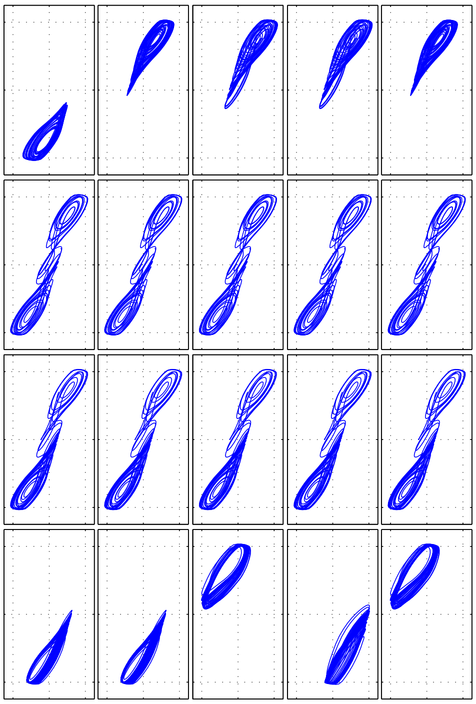

Figure 3: Steady–state of disturbed synchronous motion in an array composed of $n = 5$ circuits for different coupling strength: (a) $G_1 = 0.15$, (b) $G_1 = 0.17$, (c) $G_1 = 0.32$, (d) $G_1 = 0.325$.

In order to test the stability of a particular solution one can perturb this solution by a random additive signal with a small amplitude and observe the steady–state behavior of the system. If the system converges to the solution under consideration one claims that the solution is stable. In all experiments the chaotic synchronous solution is perturbed by a random additive signal of amplitude 0.01.

First let us consider the case of network with three cells. We consider four different coupling values $G_1 = 0.05, 0.1, 0.350.4$. In each case the system is integrated for $T = 500$ after the disturbance. After the transient $T > 500$ we observe the steady–state behavior, which is plotted in Fig. 2.

One can clearly see that for $G_1 = 0.1$ and $G_1 = 0.35$ the steady–state is the synchronous motion, while for other two cases the steady–state is different. Thus, the results observed numerically compare very well with theoretical prediction, from which it follows that synchronous state is stable for $G_1 \in (0.0763, 0.3903)$ (compare Table 1).

We obtain similar results for $n = 5$. For $G_1 = 0.17$ and $G_1 = 0.32$ we observe stability of synchronous behavior, while for $G_1 = 0.15$ and $G_1 = 0.325$ the synchronous mode is not stable (see Fig. 3). The experimental results agree very well with theoretical predictions (compare Table 1).

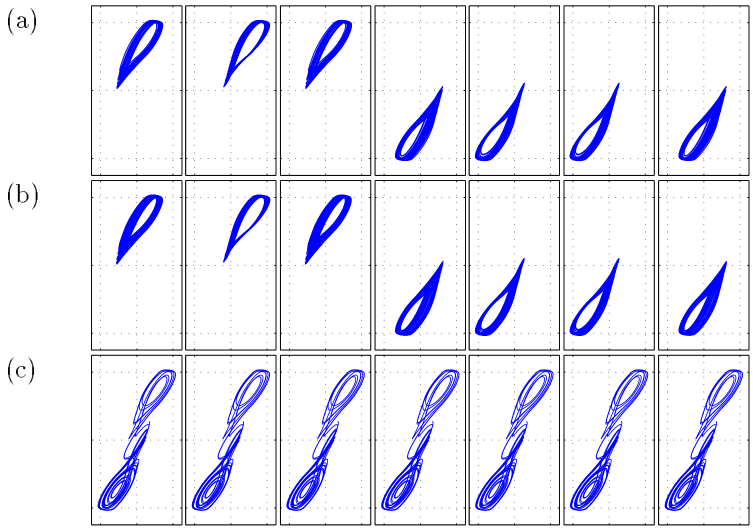

Figure 4: Steady–state of disturbed synchronous motion in an array composed of $n = 7$ circuits for different coupling strength: (a) $G_1 = 0.30$, (b) $G_1 = 0.306$, time step $\tau = 0.1$ (c) $G_1 = 0.306$, time step $\tau = 0.02$.

Finally let us consider the network composed of 7 cells. In this case the interval of coupling strength with stable synchronous motion is very narrow $G_1 \in (0.304, 0.308)$. We choose two values of coupling coefficients: $G_1 = 0.30$ and $G_1 = 0.306$. The trajectory of the system after time $T = 1000$ is shown in Fig. 4(a),(b). In the steady–state the system is not in the synchronous mode. This is in contrast to theoretical predictions, as for the second case we expect the synchronization behavior. The escape from synchronization manifold is very slow. For $G_1 = 0.30$ we observe loss of synchronous behavior after $T = 400$. For $G_1 = 0.306$ the escape time is even longer: $T = 800$. We believe that the reason for this disagreement is small noise, coming from the integration procedure, that causes desynchronization bursts. The stability of synchronous behavior is not robust (the maximum Lyapunov exponents corresponding to transversal directions is negative but very close to zero). We have repeated the experiment using smaller integration step ($\tau = 0.02$ instead of $\tau = 0.1$). The results are shown in Fig. 4(c). After a very long time $T = 10000$ one still observes synchronization behavior.

V. CONCLUSIONS

In this paper we have investigated stability of synchronous solution of a one–dimensional array of bi–directionally coupled chaotic circuits. We have found the upper limit on the size of the network, that can sustain stable synchronous motions. For different array sizes we have found the ranges of the coupling strength, for which the synchronous motion is stable. We have confirmed that the theoretical predictions of the existence of the stable synchronous solution compare very well with the results of computer simulations.

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