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Influence of Integration Algorithms on Results of Simulation Studies of a Ring of Coupled Chaotic Circuits

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ABSTRACT

In this paper we study the influence of the time step of the integration method on the results of simulations of a one–dimensional lattice of coupled electronic oscillators. We also discuss the feasibility of using computer simulations for studies of this system.

1. INTRODUCTION

Recently there has been a growing interest in studies of systems composed of coupled nonlinear oscillators located at the nodes of a one or two–dimensional lattice. Such systems are appropriate models of a variety of phenomena including wave propagation, pattern formation, behavior of spatio–temporal chaotic processes [3, 4, 5, 6].

In our previous studies [1] we have investigated long-term (steady-state) behavior of a one-dimensional array of chaotic circuits for different connection strength. Using computer experiments we have confirmed the existence of a very large number of stable steady–states depending only on the initial conditions applied in the individual cells and on the values of coupling coefficients.

A natural question which arises is whether these results depend on the integration algorithm used in the simulations of the system.

In this paper we investigate how the integration method and the time step in particular influence the results obtained. There are two different problems which we shall consider. The first problem is how well we can predict the state of the system after a certain time. In this context we want to investigate how the results change depending on the integration algorithm and the time step chosen. The second problem is the dependence of the steady state obtained on the integration algorithm. This is especially important in searching for attractors and investigations of their basins of attraction.

2. EXPERIMENTAL SETUP

Let us consider a one–dimensional lattice of simple third–order electronic oscillators (Chua's circuits). The oscillators are coupled bi-directionally by means of two resistors cross-connected between the capacitors C_1 and C_2 of the neighboring cells (compare Fig. 1a). Every cell is connected with two nearest neighbors. The first and the last

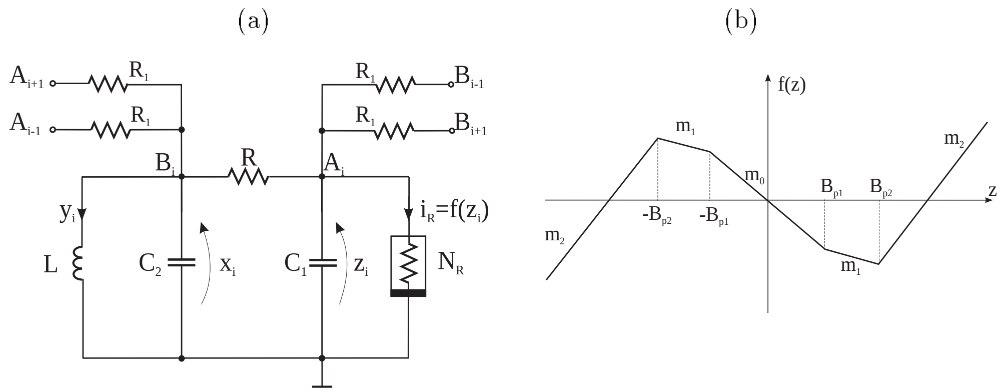

Fig. 1: (a) A one–dimensional array of simple third–order oscillators, (b) A five–segment piecewise linear function.

cells are also connected and the lattice forms a ring. The dynamics of the lattice composed of n cells can be described by the following set of ordinary differential equations [3], [4]:

$$\left\{\begin{array}{rcl} C_2\dot{x}_i &=& -Gx_i - y_i + Gz_i + G_1(z_{i-1} - x_i) + G_1(z_{i+1} - x_i) \\ L\dot{y}_i &=& x_i \\ C_1\dot{z}_i &=& Gx_i - Gz_i - f(z_i) + G_1(x_{i-1} - z_i) + G_1(x_{i+1} - z_i) \end{array}\right. \quad \text{for i=1,2,...,n} \tag{1}$$

where $x_0 := x_n$, $z_0 := z_n$, $x_{n+1} := x_1$ and $z_{n+1} := z_1$ and f is a five–segment piecewise linear function (compare Fig. 1b):

$$f(z) = m_2 z + \frac{1}{2}(m_1 - m_2)(|z + B_{p_2}| - |z - B_{p_2}|) + \frac{1}{2}(m_0 - m_1)(|z + B_{p_1}| - |z - B_{p_1}|). \tag{2}$$

We consider the array of size 31. As in our previous studies we use typical parameter values for which an isolated Chua's circuit generates chaotic oscillations — the "double scroll" attractor ($C_1 = 1/9F$, $C_2 = 1F$, $L = 1/7H$, $G = 0.7S$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$). For these parameter values together with a chaotic attractor there exist periodic orbit with a large amplitude. In the experiments we have considered the uniform coupling $G_1 \in [0.01, 0.1]$.

$G_1\backslash\tau$	0.01	0.02	0.05	0.1	0.15	0.2	0.25	τ_∞
0.01	A1	A2	A3	A4	A5	A6	A7	0.95
0.02	B1	B2	B3	B4	B5	B6	B7	0.96
0.03	C1	C2	C3	C4	C5	C6	C7	0.98
0.04	D	D	D	D	D	D	D	0.86
0.05	E	E	E	E	E	E	F	0.34
0.06	G	G	G	G	G	G	H	0.33
0.07	=	=	=	=	=	=	=	0.29
0.08	←	=	=	←	=	←	←	0.29
0.09	←	←	←	←	=	←	←	0.29
0.10	=	=	=	=	=	=	←	0.30

Table 1: Small initial conditions. Steady states: A1–A7 – small amplitude chaotic "double–scroll" attractors, B1–B7 – small amplitude periodic attractors, C1–C7 – small amplitude chaotic "double–scroll" attractors with very infrequent switchings between the scrolls, D1–D7 – small amplitude chaotic Rössler type attractors, E – the steady state with two clusters of cells with large oscillations (8–11 and 29–31), F – the steady state with two clusters of cells with large oscillations (8–10 and 29–31), G – 4 cells with very small oscillations, H – five cells with very small oscillations, "=" denotes synchronized behavior, "←" denotes a single hump wave traveling in the left direction, τ_∞ – the time step, for which the trajectory escapes to infinity

3. SIMULATION RESULTS

In our simulations we have used standard integration algorithms: Euler, Heun, modified Euler, fourth–order Runge–Kutta method and second–order predictor–corrector method [7]. In all cases fixed integration step has been used. Below we outline the results for the simulations using Runge–Kutta method only, as this was the one used in our previous studies. For other methods the dependence on the time step was similar.

We have performed several experiments starting the network from different initial conditions and using several values of the integration step: $\tau \in \{0.01, 0.02, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3\}$. The time step $\tau = 0.1$ was used in our previous studies.

In the first experiment we applied random initial conditions of small amplitude (0.01) in all cells. The system was integrated for the time $T = 500$, which requires 50000 iterations for $\tau = 0.01$ and 2000 iterations for $\tau = 0.25$. We have repeated the simulations for different values of the coupling coefficient G_1. The results are summarized in Table 1.

For $G_1 = 0.01$ and $G_1 = 0.03$ in the steady state all cells display the "double-scroll" attractors. The integration results (after $T = 500$) for different timesteps are uncorrelated. This kind of behavior is quite natural for chaotic systems. Due to sensitive dependence on initial conditions trajectories after sufficiently long time are uncorrelated. Change of the time step introduces small deviation, which is amplified by the "butterfly" effect.

$G_1 \backslash \tau$	0.01	0.02	0.05	0.1	0.15	0.2	0.25
0.01	A	A	A	A	A	A	A'
0.02	B	B	B	B	B	B	B'
0.03	C	C	C	C	C	C	C'
0.04	D	D	D	D	D	D	E
0.05	F	F	F	F	F	F	F'
0.06	=	=	=	I	I	H	G
0.07	=	=	=	=	=	=	I
0.08	←	←	←	←	←	←	←
0.09	=	=	=	=	=	←	←
0.10	=	=	=	=	=	=	=

Table 2: Medium initial conditions. Steady states: "=" denotes synchronized behavior, "←" denotes a single hump wave traveling in the left direction, A—I denotes a steady state with different numbers of cells with small amplitude oscillations (A – 14, B – 11, C – 8, D – 6, E – 5, F – 4, G – 3, H – 2, I – 1), X' — a steady state similar to X (where X is one of A, B, C, F) but oscillations are of smaller amplitude.

For $G_1 = 0.02$ in all experiments the steady–state is periodic of small amplitude. Each cell oscillates in upper or lower halfspace. For different time steps the halfspace patterns are different which indicates that the system converges to different attractors. The situation here is different from the case $G_1 = 0.01$. There is no sensitive dependence on initial conditions as the attractor is periodic. There coexist however many attractors and their basins of attraction are probably riddled. Small change of initial conditions or small change of dynamical system may force the trajectory to converge to a different attractor.

For $G_1 = 0.04$ trajectories of individual cells form the Rössler type attractor enclosed in one of the halfspaces. For all time steps the steady state is the same — pattern of halfspaces agree perfectly, but the state of the system after the time $T = 500$ is different for all cases. This can be again explained by "sensitive dependence on initial conditions".

For $G_1 = 0.05, 0.06, 0.07, 0.10$ and for small time step $\tau \leq 0.15$ the steady state observed in simulations does not depend on the integration step.

Very integesting phenomena are found for $G_1 = 0.08$. In this case even for small time steps we observe different steady states (synchronization $\tau = 0.01, 0.1$ and traveling wave $\tau = 0.02, 0.05, 0.15$). Probably for this coupling the initial state lies close to the boundary of the basins of attraction of the two attractors.

We have also searched for the value of time step τ_∞ for which the trajectory of the system escapes to infinity. The results are collected in the last column of Table 1. When all cells display oscillations of small amplitude τ_∞ is close to one. If some cells are operating in a large amplitude regime τ_∞ is close to 0.3.

In the second series of simulations we have applied random initial conditions of amplitude 1 in every cell. As in the previous experiment the system was integrated until $T = 500$ using different time steps and for different choices of coupling coeffi-

$G_1\backslash\tau$	0.01	0.02	0.05	0.1	0.15	0.2	0.25
0.01	C	C	C	C	C	A	E
0.02	C	C	C	C	C	C	E
0.03	C	C	C	C	C	C	E
0.04	B	B	B	B	B	C	D
0.05	B	B	B	B	B	C	D
0.06	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$
0.07	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	∞
0.08	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	∞
0.09	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	∞
0.10	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	∞

Table 3: Large initial conditions The integration result after time $T = 500$ for $\tau \leq 0.15$ is the same or very close to the "ideal" result obtained for $\tau = 0.01$, $\tau = 0.2$ — larger deviation from the "ideal" case (possible different steady–state), $\tau = 0.25$ — different steady–state or trajectory escapes to infinity. Steady states: "$\leftarrow$" denotes a single hump wave traveling in the left direction, "∞" denotes the trajectory escaping to infinity, A–E coexistence of small amplitude oscillations (31st cell), synchronous behavior of some cells and traveling waves with different number of wave lengths along the ring: A – 1.5, B – 2, C – 2.5, D – 3, E – 3.5.

cients. The results are summarized in Table. 2. We have observed that usually for $\tau \leq 0.15$ the integration result and also the steady state are the same as for the "ideal" case $\tau = 0.01$. The only exception is $G_1 = 0.06$, where for $\tau = 0.1, 0.15$ the steady state is different from the one obtained in the "ideal" case. For $\tau = 0.2$ we have obtained the integration result with larger deviation but with the same type of steady state behavior (with the exception of $G_1 = 0.06, 0.09$.

Finally let us consider the case of large initial conditions (of amplitude 5). From the simulations it follows that the result of integration and the steady state does not depend on the time step if the time step is small ($\tau \leq 0.15$). For larger time steps the deviation of the integration result after time $T = 500$ with comparison to the case $\tau = 0.01$ is larger. Also the steady state may be different or the trajectory may escape to infinity (compare Table 3).

4. CONCLUSIONS

In this paper we investigated the influence of the integration step on the results of simulations of a network composed of coupled chaotic systems.

In the case of weak coupling and small initial conditions one cannot predict the behavior of the ring even for small integration time. This corresponds to sensitive dependence of chaotic systems on initial conditions. In this case using simulations we can investigate the whole attractor and not individual trajectories. If there exist many attractors one should be very careful when using simulations for the investigation of basins of attraction. However the search for attractors using different random initial conditions should work fine.

For larger coupling, when the network eventually enters the large amplitude oscillations the results of integration are more predictable and do not depend on the time step if it is chosen below some threshold value (in our experiments reasonable choice is $\tau \in [0.1, 0.15]$). One should also be aware of the problem of riddled basins of attraction.

In order to overcome these problems one may use an integration method with constant monitoring of integration error and modify the time step automatically. This complicates the integration algorithm but the results are more reliable [7]. In the case of sensitive dependence on initial conditions or when there exist riddled basins of attraction one has to be vary careful with interpretation of simulation results.

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