INTERVAL NEWTON METHOD AND BACKWARD SHOOTING FOR FINDING LONG PERIODIC ORBITS IN 1D CHAOTIC MAPS

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Abstract— *In this paper we present a method to find very long periodic orbits for one–dimensional maps. This new approach is a combination of the interval Newton method and the shooting technique. We also describe how to use this approach to find better approximation of position of computer generated pseudoperiodic orbit. Using this method we find very long periodic orbits for the logistic map and we calculate Lyapunov exponents of these orbits.*

I. Introduction

Finding periodic orbits of nonlinear systems is an important problem which is frequently encountered in a variety of fields. Usually periodic orbits are found in numerical studies but there is no guarantee that there exists a true periodic trajectory that stays near a computer generated one. This problem is especially important for chaotic systems, as due to sensitive dependence on initial conditions usually after certain number of iterations (100 or so) the computer generated trajectory becomes uncorrelated with the true trajectory. A very important question is whether there really exists a true periodic trajectory in the neighborhood of a computer generated one.

In the present work we develop a technique for proving the existence of true periodic orbits near pseudo–periodic orbits obtained in computer simulations. This approach is based on the *interval Newton method* [1], [8] and the shooting technique. An introduction to the interval arithmetic is given in [2]. In interval analysis we are sure that the result obtained encloses the true solution (together with the rounding error). In this paper we use boldface letters to denote intervals and usual math italic letters to denote point quantities.

The interval Newton method allows to prove the existence of zeros of n–dimensional maps. In this method in order to investigate the existence of zeros of a function $\mathbb{R}^m \ni x \mapsto f(x) \in \mathbb{R}^m$ in an m-dimensional interval $\mathbf{x}$ one computes the interval Newton operator:

$$\mathbf{N}(\mathbf{x}) = x_0 - f'(\mathbf{x})^{-1} f(x_0), \qquad (1)$$

where $f'(\mathbf{x})^{-1}$ is the interval matrix containing all Jacobian matrices of f of the form $f'(x)^{-1}$ for $x \in \mathbf{x}$ and x_0 is an arbitrary point belonging to the interval vector $\mathbf{x}$. Usually one chooses x_0 as the center of $\mathbf{x}$. The key property of the interval Newton operator is following: if $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ then there exists exactly one point $x \in \mathbf{x}$ such that $f(x) = 0$.

The interval Newton method can be used to find all low period cycles for discrete time dynamical systems [4]. It can be also used for proving the existence of periodic orbits for continuous–time systems [5], [6]. When we try to use this method directly for very long periodic orbits we face the problem of efficient computation of the interval operator.

In this paper we use shooting technique for evaluation of the interval Newton operator, which makes the method applicable to long periodic orbits. As an example we consider the logistic map. For this map we find extremely long periodic orbits.

II. Interval Newton Method for 1D maps

In this section we present a combination of the global interval Newton method and backward shooting which may be useful for proving the existence of very long periodic orbits in one–dimensional maps. Backward shooting is a popular technique and was used in the context of finding true periodic trajectories shadowing computer generated ones [3]. Here we apply this technique for efficient computation of interval Newton operator.

Let us consider a function $\mathbb{R} \ni x \mapsto f(x) \in \mathbb{R}$. In order to prove the existence of a period-n cycle of f one may apply the interval Newton operator to the global map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by

$$\begin{aligned} [F(z)]_1 &= x_2 - f(x_1), \\ [F(z)]_2 &= x_3 - f(x_2), \\ &\dots \\ [F(z)]_n &= x_1 - f(x_n), \end{aligned} \tag{2}$$

where $z = (x_1, \dots, x_n)$. Observe that z is a zero of F if and only if x_1 is a fixed point of f^n. The Jacobian of F at $z = (x_1, \dots, x_n)$ can be computed as

$$F'(z) = \tag{3}$$
$$\begin{pmatrix} -f'(x_1) & 1 & 0 & \dots & 0 \\ 0 & -f'(x_2) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -f'(x_n) \end{pmatrix}.$$

In order to prove the existence of a periodic orbit in the interval vector $\mathbf{z}$ we choose $z_0 \in \mathbf{z}$ and show that the interval vector $\mathbf{N}(\mathbf{z}) = z_0 - F'(\mathbf{z})^{-1}F(z_0)$ is enclosed in $\mathbf{z}$. The main problem we encounter is the necessity of computation of $F'(\mathbf{z})^{-1}F(z_0)$. Computation of the interval matrix inverse of $F'(\mathbf{z})$ can be done only for small n. Here because the matrix $F'(\mathbf{z})$ has a special form we can use the technique described below to find $\mathbf{N}(\mathbf{z})$.

Let

$$\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n), \tag{4}$$

and $z_0 = (x_1, x_2, \dots, x_n)$ be an arbitrary point in $\mathbf{z}$. In order to evaluate the interval Newton operator we first find the interval vector $\mathbf{h} = F'(\mathbf{z})^{-1}F(z_0)$. This is equivalent to solving the following equation:

$$F'(\mathbf{z})\mathbf{h} = F(z_0). \tag{5}$$

We can rewrite this equation in the following form:

$$\begin{aligned} -f'(\mathbf{x}_1)\mathbf{h}_1 + \mathbf{h}_2 &= x_2 - f(x_1), \\ -f'(\mathbf{x}_2)\mathbf{h}_2 + \mathbf{h}_3 &= x_3 - f(x_2), \\ &\dots \\ -f'(\mathbf{x}_n)\mathbf{h}_n + \mathbf{h}_1 &= x_1 - f(x_n), \end{aligned} \tag{6}$$

Let $\mathbf{a}_k = f'(\mathbf{x}_k)$ and $\mathbf{g}_k = f(x_k) - x_{k \bmod n+1}$. We assume that the intervals $\mathbf{a}_k$ do not contain zero so one can compute $\mathbf{a}_k^{-1}$. With this assumption equation (6) can be written as

$$\begin{aligned} \mathbf{h}_1 &= \mathbf{a}_1^{-1}(\mathbf{h}_2 + \mathbf{g}_1), \\ \mathbf{h}_2 &= \mathbf{a}_2^{-1}(\mathbf{h}_3 + \mathbf{g}_2), \\ &\dots \\ \mathbf{h}_n &= \mathbf{a}_n^{-1}(\mathbf{h}_1 + \mathbf{g}_n). \end{aligned} \tag{7}$$

In order to find $\mathbf{h}_1$ we substitute $\mathbf{h}_i$ from the ith equation into the $(i-1)$th equation, starting from the last equation. After $n-1$ substitutions we obtain

$$\begin{aligned} \mathbf{h}_1 =& \mathbf{a}_1^{-1}(\mathbf{a}_2^{-1}(\mathbf{a}_3^{-1}(\dots \mathbf{a}_{n-1}^{-1}(\mathbf{a}_n^{-1}(\mathbf{h}_1 + \mathbf{g}_n) + \\ &+ \mathbf{g}_{n-1}) \dots + \mathbf{g}_3)) + \mathbf{g}_2) + \mathbf{g}_1). \end{aligned} \tag{8}$$

Hence $\mathbf{h}_1$ can be computed as

$$\mathbf{h}_1 = \left(1 - \mathbf{a}_1^{-1}\mathbf{a}_2^{-1} \dots \mathbf{a}_n^{-1}\right)^{-1} \sum_{i=1}^{n} \mathbf{a}_1^{-1} \dots \mathbf{a}_i^{-1}\mathbf{g}_i. \tag{9}$$

Now we can find $\mathbf{h}_n, \mathbf{h}_{n-1}, \dots, \mathbf{h}_2$ by backward recursion using formula (7). Finally we evaluate $\mathbf{N}(\mathbf{z}) = z_0 - \mathbf{h}$.

There are several methods which can be used to solve equation (6). For a chaotic map we expect that $\mathbf{a}_k = f'(\mathbf{x}_k)$ has on average the absolute value larger than 1. In this case multiplication by $\mathbf{a}_k^{-1}$ reduces the diameter of the product and we avoid the "wrapping effect". Hence by using formula (9) we obtain very narrow enclosure of $\mathbf{h}_1$. Also the backward recursion does not produce wide intervals due to the existence of the factor $\mathbf{a}_k^{-1}$ in (7).

The method requires relatively small amount of memory (only few doubles per iteration need to be stored). The computation time depends linearly on the length of the orbit. Hence periodic orbits of length of order 1000000 are easily handled.

The above method is very efficient for evaluation of interval Newton operator. Another problem is how to obtain a good candidate for $\mathbf{z}$. It appears that we can use the same technique for this task. We propose to use the following procedure. First by iterating the map we obtain a quasi-periodic orbit $z = (x_1, x_2, \dots, x_n)$. This orbit usually cannot serve as a center of a candidate because the distance between $f(x_n)$ and x_1

is usually large. In order to obtain better approximation of position of periodic orbit with a smaller global error

$$e_{\mathrm{glob}}(z) = \max_{i=1,\dots,n} |f(x_{i \bmod n+1}) - x_i| \qquad (10)$$

we evaluate the interval Newton operator over the point interval vector $z = ([x_1, x_1], \dots, [x_n, x_n])$. If the initial approximation z is good enough then the center of $\mathbf{N}(z)$ is better approximation of the position of a true periodic orbit. We may repeat this procedure several times in order to further improve the approximation $(x_1, x_2, \dots, x_n)$. Usually after 3 to 5 iterations we do not observe further improvements. Finally as a candidate we choose the interval vector $\mathbf{z} = (\mathbf{x}_1, \dots, \mathbf{x}_n) = ([x_1 - \varepsilon, x_1 + \varepsilon], \dots, [x_n - \varepsilon, x_n + \varepsilon])$ centered around the position of the pseudo–periodic orbit we obtained form the interval Newton operator. We use the uniform radius ε of the interval $\mathbf{x}_i$ along the orbit. The radius ε must be chosen in such a way that the condition $0 \notin f'(\mathbf{x}_i)$ holds for all i. In case of the logistic map with one maximum at $c = 0.5$ we have to choose

$$\varepsilon < \varepsilon_{\max} = \min_{i=1,\dots,n} |x_i - c|. \qquad (11)$$

If we do not have any clues for the initial value of ε we may start with arbitrary ε. If we do not succeed in proving the existence condition we may modify ε and repeat the computations.

III. Computation of the Lyapunov exponent of periodic orbit

Lyapunov exponents of periodic orbits describe the behavior of dynamical system in the neighborhood of the orbit. If one of the Lyapunov exponents is positive then the periodic orbit is unstable and typical trajectory is repelled from the orbit. The Lyapunov exponent of the periodic orbit $(x_k)_1^n$ for the one–dimensional map f is defined by

$$\lambda = \frac{1}{n} \log |\mathrm{D}f^n(x_1)| \qquad (12)$$

$$= \frac{1}{n} \log |f'(x_n) \dots f'(x_2) f'(x_1)| \qquad (13)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log |f'(x_i)|. \qquad (14)$$

If we know that the point x_i belongs to the interval $\mathbf{x}_i$ we may easily find the interval enclosing the Lyapunov exponent of the orbit using the last equation. Observe that although formulas (13) and (14) are mathematically equivalent the second one is much more useful for computation of the accurate bound for the Lyapunov exponent of the orbit. The reason is that the first formula contains the product of n intervals, which when evaluated in interval arithmetic may produce very large errors.

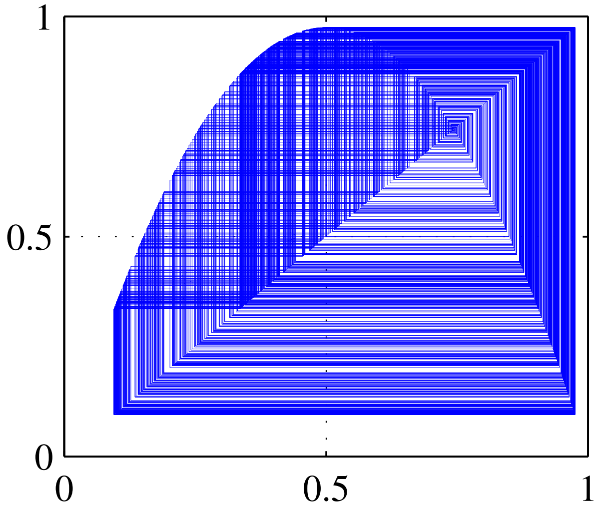

Fig. 1. Trajectory of the logistic map composed of 1000 points.

IV. Long periodic orbits for the logistic map

We apply the method described in the previous section to the logistic map

$$f(x) = ax(1-x), \qquad (15)$$

with the parameter value $a = 3.9$. For this value of the parameter a chaotic behavior is observed. A trajectory of the logistic map consisting of 1000 points is plotted in Fig. 1.

In order to find a quasi-periodic orbit we have chosen $x_1 = 0.66$ as an initial point. After $n = 5480633$ iterations the trajectory returns to the small neighborhood of the initial point

$$\delta_1 = |x_{5480634} - x_1| < 2.21 \cdot 10^{-7}.$$

After two iterations of the interval Newton operator with point interval as an input we have obtained a better approximation $(x_1, x_2, \dots, x_n)$ of the periodic orbit with the global error near the

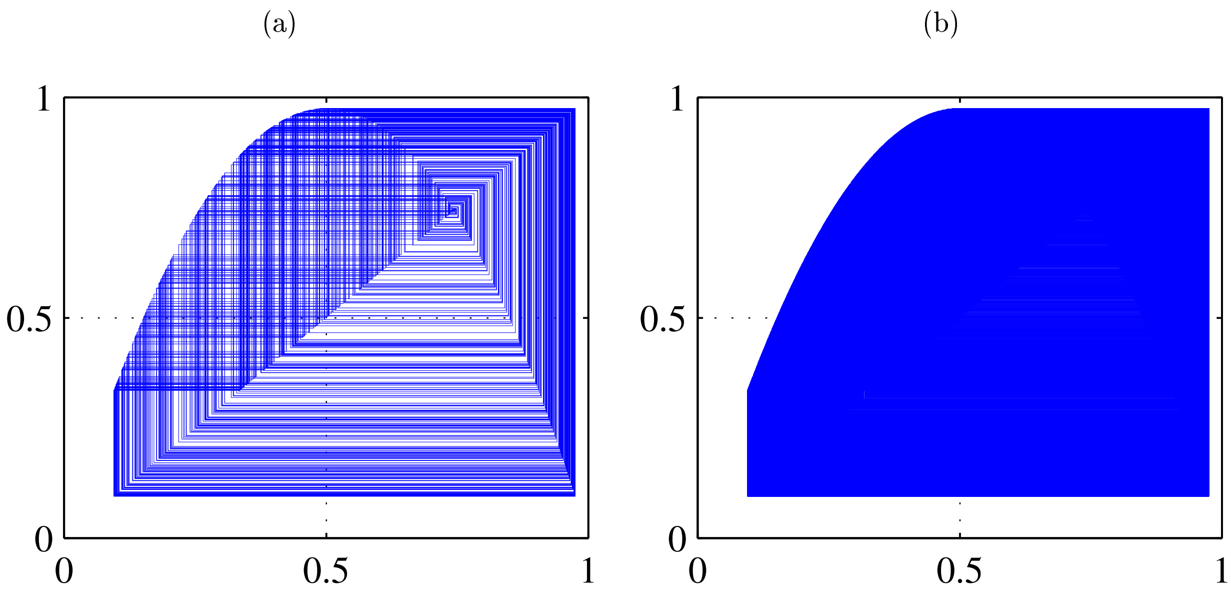

Fig. 2. Periodic orbits for the logistic map, (a) period 701 orbit, (b) period 11348 orbit.

machine precision

$$\delta_2 = \max_{i=1,\dots,n} |f(x_i) - x_{i \bmod n+1}| < 4.16 \cdot 10^{-16}.$$

The distance between the point $c = 0.5$ and the pseudo–periodic orbit is

$$\varepsilon_{\max} = \min_{i=1,\dots,n} |x_i - c| = 4.9 \cdot 10^{-8}.$$

Hence we know that as the radius of the interval vector have to be smaller than $4.9 \cdot 10^{-8}$. Next we have created the interval vector $\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ where $\mathbf{x}_i = [x_i - \varepsilon, x_i + \varepsilon]$ with $\varepsilon = 10^{-8}$. Finally we have checked that $\mathbf{N}(\mathbf{z}) \subset \mathbf{z}$ and proved that there exist a unique periodic orbit within $\mathbf{z}$. By iterating the interval Newton operator twice we have obtained the interval vector containing the orbit with uncertainty smaller than $\delta_3 = 2.08 \cdot 10^{-10}$ at every point along the orbit. As 5480633 is a prime number we are sure that this is the principal period of the orbit. In this way we have shown that there exists a true periodic trajectory near the computer generated one. In particular we know that the interval

$$\mathbf{x}_1 = [0.6599999999999998, 0.66000000000000026]$$

contains a periodic point with period 5480633. The time necessary to complete the existence proof is 87.14 seconds. Its main part is the time needed to evaluate the interval Newton operator. Computations were performed on a Pentium III computer with 500MHz processor. Program was written in C++ (gcc version egcs-2.91.66). In the program we used the Bias/Profil packages for interval computations [7].

In a similar way we have found several other long periodic orbits for the logistic map. Two orbits with periods 701 and 11348 respectively are plotted in Fig 2. Computation details are collected in Table I. n is the period of the orbit, x_1 is a start point of the computer generated δ_1–pseudoperiodic orbit $(x_k)_{k=1}^n$ ($|f(x_n) - x_1| < \delta_1$). After p iterations of Newton operator we obtain a very good approximation of periodic orbit with errors $|f(x_{k \bmod n+1}) - x_k| < \delta_2$ for $k = 1, \dots, n$. ε is the uniform radius of the interval vector $\mathbf{z}$ ($\mathbf{x}_i = [x_i - \varepsilon, x_i + \varepsilon]$) used as an input to the interval Newton operator and δ_3 is the maximum radius of the intervals for which the existence of the periodic orbit was proved — it is the accuracy of the position of the periodic orbit found. In the last column we give the time necessary to complete the existence proof.

In Table II several parameters of the orbits are given. We have computed the interval $\boldsymbol{\lambda}$ containing the Lyapunov exponent of the orbit (for example 0.496142_{49}^{51} denotes the interval $[0.49614249, 0.49614251]$). In order to see how densely the orbit fills the attractor we compute two numbers. The value of $d_{\max}$ which is the maximum gap between the points belonging to

	n	x_1	δ_1	p	δ_2	ε	δ_3	t [s]
1	701	0.70	$4.20 \cdot 10^{-5}$	3	$3.05 \cdot 10^{-16}$	10^{-3}	$9.50 \cdot 10^{-15}$	0.01
2	11348	0.70	$3.45 \cdot 10^{-6}$	2	$3.89 \cdot 10^{-16}$	10^{-6}	$3.38 \cdot 10^{-12}$	0.12
3	723543	0.70	$2.23 \cdot 10^{-7}$	2	$4.02 \cdot 10^{-16}$	10^{-6}	$1.15 \cdot 10^{-11}$	7.96
4	1921687	0.60	$1.56 \cdot 10^{-7}$	2	$4.16 \cdot 10^{-16}$	10^{-7}	$1.61 \cdot 10^{-11}$	20.95
5	2444017	0.64	$2.56 \cdot 10^{-7}$	3	$4.16 \cdot 10^{-16}$	10^{-8}	$2.08 \cdot 10^{-10}$	27.29
6	5480633	0.66	$2.21 \cdot 10^{-7}$	2	$4.16 \cdot 10^{-16}$	10^{-8}	$2.08 \cdot 10^{-10}$	87.14
7	8076157	0.62	$8.18 \cdot 10^{-8}$	2	$4.16 \cdot 10^{-16}$	10^{-8}	$7.03 \cdot 10^{-10}$	166.00

TABLE I

Examples of long periodic orbits for the logistic map, n is the period of the orbit, x_1 is a start point of the computer generated δ_1–pseudoperiodic orbit, p is the number of iterations of Newton operator after which we obtain a very good approximation of periodic orbit with uniform error δ_2, ε is the uniform radius of the interval vector $\mathbf{z}$, δ_3 is the maximum radius of the intervals $\mathbf{x}_i$, t is the computation time necessary to prove the existence.

the orbit gives us the global information about the system. If there exists an attracting periodic orbit for the map then basin of its attraction cannot contain an interval with diameter larger than $d_{\max}$. $d_{\min}$ is the minimum distance between the centers of intervals $\mathbf{x}_i$. One can clearly see that the distance between the centers is much less than the maximum radius of these intervals (δ_3 in the Table I. This means that the intervals $\mathbf{x}_i$ overlap.

	$\boldsymbol{\lambda}$	$d_{\max}$	$d_{\min}$
1	0.503042830386^{9}_{7}	0.014	$2.83 \cdot 10^{-7}$
2	0.49837574464^{7}_{4}	0.0012	$2.08 \cdot 10^{-9}$
3	0.496148838^{6}_{4}	$2.75 \cdot 10^{-5}$	$2.02 \cdot 10^{-13}$
4	0.495602678^{6}_{4}	$1.11 \cdot 10^{-5}$	$7.92 \cdot 10^{-14}$
5	0.49553530^{9}_{5}	$9.11 \cdot 10^{-6}$	$1.40 \cdot 10^{-14}$
6	0.49589456^{9}_{5}	$4.70 \cdot 10^{-6}$	$1.75 \cdot 10^{-14}$
7	0.496142^{51}_{49}	$3.15 \cdot 10^{-6}$	$2.78 \cdot 10^{-15}$

TABLE II

Properties of long periodic orbits for the logistic map, $\boldsymbol{\lambda}$ is the enclosure of the Lyapunov exponent of the orbit, $d_{\max}$ is the maximum gap between the points belonging to the orbit, $d_{\min}$ is the minimum distance between the points in the orbit.

V. Conclusions

In this paper we have described how to prove the existence of a true periodic orbit in the neighborhood of a computer generated one using the interval Newton method and the backward shooting technique. We have applied this method to show the existence of very long periodic orbit for the logistic map.

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