Proving the existence of long periodic orbits in 1D maps using interval Newton method and backward shooting

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Abstract

In this paper, we present a method that can be used to find very long periodic orbits for one–dimensional maps. This new approach is a combination of the interval Newton method and the shooting technique. We also describe how to use this approach to find better approximation of position of computer generated pseudo–periodic orbit. As an example, we find very long periodic orbits for the logistic map and we calculate the Lyapunov exponents of these orbits. Finally, we investigate the performance of this method used for finding all short period cycles.

Key words: periodic orbits, interval methods, shooting technique
MSC: 37E05, 65G20

1 Introduction

Periodic orbits carry a lot of vital information about the underlying dynamical system (Auerbach et al., 1987). A great deal of research on periodic orbits relies on computer simulations. It is usually an easy task to find numerically an approximate position of a periodic orbit using for example the standard Newton method. It is however much more difficult to prove the existence of a true periodic orbit in a neighborhood of the one generated by a computer. Since the round-off errors may introduce new kind of behavior, in general one cannot be certain that the true periodic orbit exists. This problem is especially pivotal for chaotic systems, as due to sensitive dependence on initial conditions

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chaotic trajectories starting from arbitrarily close initial conditions usually diverge exponentially from one another.

In the present work, we develop a technique for proving the existence of a true periodic orbits near a pseudo–periodic orbit found in computer simulations. This approach is based on the interval Newton method (Alefeld, 1994; Neumaier, 1990) and the shooting technique. A complete introduction to the interval arithmetic underlying the interval Newton method is given in Moore (1979) or Alefeld and Herzberger (1983). Interval computations allow to use a computer to obtain rigorous results, by ensuring that the results obtained enclose the true solution (together with the rounding errors). In this paper, we use boldface to denote intervals, interval vectors and matrices, and the usual math italics to denote point quantities.

The interval Newton method can be applied to prove the existence of zeros of m–dimensional maps. In this method in order to investigate the existence of zeros of a continuously differentiable function $f\colon \mathbb{R}^m \mapsto \mathbb{R}^m$ in an m-dimensional interval $\mathbf{x} \subset \mathbb{R}^m$ one evaluates the interval Newton operator:

$$\mathbf{N}(\mathbf{x}) = \hat{x} - f'(\mathbf{x})^{-1} f(\hat{x}), \tag{1}$$

where $f'(\mathbf{x})^{-1}$ is the interval matrix containing all matrices of the form $f'(x)^{-1}$ for $x \in \mathbf{x}$ and $\hat{x}$ is an arbitrary point belonging to the interval vector $\mathbf{x}$. Usually one chooses $\hat{x}$ to be the center of $\mathbf{x}$. The interval Newton method is based on the two following properties of the operator $\mathbf{N}$:

Theorem 1.
1. If $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ *then there exists exactly one point* $x \in \mathbf{N}(\mathbf{x})$ *such that* $f(x) = 0$.
2. If $\mathbf{N}(\mathbf{x}) \cap \mathbf{x} = \emptyset$ *then there is no zeros of* f *in* $\mathbf{x}$.

A simple proof of the above facts can be found in (Alefeld, 1994) or (Galias, 2001). The first property allows to prove the existence of a unique zero in a given m–dimensional interval vector $\mathbf{x}$, while the second one can be used as a simple non–existence criterion.

The interval Newton method combined with the generalized bisection can be used to find all low period cycles for discrete–time dynamical systems (Galias, 1999a, 2001), or to prove the existence of periodic orbits for continuous–time systems (Galias, 1999b,c). When we try to use this method directly for very long periodic orbits we face the problem of efficient computation of the interval operator.

In this paper, we use shooting technique for evaluation of the interval Newton operator, which makes the method applicable to long periodic orbits. As an example, we consider the logistic map, for which we find extremely long periodic orbits.

2 Interval Newton method for finding long periodic orbits of 1D maps

In this section, we present a combination of the global interval Newton method and backward shooting which may be useful for proving the existence of very long periodic orbits for one–dimensional maps $f\colon \mathbb{R} \mapsto \mathbb{R}$. Backward shooting is a popular technique and was used in the context of finding true periodic trajectories shadowing computer generated ones (Coomes et al., 1996). Here, we apply this technique for efficient evaluation of the interval Newton operator.

2.1 Interval Newton method

Let us consider a one–dimensional continuously differentiable map $f\colon \mathbb{R} \mapsto \mathbb{R}$. In order to prove the existence of a period–n cycle of f one may apply the interval Newton operator to the map $g(x) = x - f^n(x)$. This technique is useful for small n only. For larger n the intervals $g(\hat{x})$ and $g'(\mathbf{x})$ have large diameter, due to the "wrapping effect" and for unstable orbits additionally due to a positive Lyapunov exponent. In consequence the condition $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ is not satisfied, no matter how $\mathbf{x}$ is chosen. To handle long periodic orbits one can use the global map $F\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by

$$\begin{aligned} [F(z)]_1 &= x_2 - f(x_1), \\ [F(z)]_2 &= x_3 - f(x_2), \\ &\dots \\ [F(z)]_n &= x_1 - f(x_n), \end{aligned} \tag{2}$$

where $z = (x_1, \dots, x_n)$. Observe that z is a zero of F if and only if x_1 is a fixed point of f^n. The Jacobian of F at $z = (x_1, \dots, x_n)$ can be computed as

$$F'(z) = \begin{pmatrix} -f'(x_1) & 1 & 0 \dots & 0 \\ 0 & -f'(x_2) & 1 \dots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 1 & 0 & 0 \dots & -f'(x_n) \end{pmatrix}. \tag{3}$$

In order to prove the existence of a unique periodic orbit in the interval vector $\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$, we choose $\hat{z} \in \mathbf{z}$ and show that the interval vector

$$\mathbf{N}(\mathbf{z}) = \hat{z} - F'(\mathbf{z})^{-1} F(\hat{z})$$

is enclosed in $\mathbf{z}$. From the Theorem 1 it follows that there exists exactly one periodic orbit in $\mathbf{z}$. Once the existence is proved, it is possible to iterate the interval Newton operator to narrow down the set in which the periodic orbit exists.

The main problem we encounter is the necessity of computation of the term $F'(\mathbf{z})^{-1}F(\hat{z})$. Computation of the interval matrix inverse of $F'(\mathbf{z})$ can be done only for small n. Here, because the matrix $F'(\mathbf{z})$ has a special form, we can use the technique described below to find $\mathbf{N}(\mathbf{z})$.

2.2 Backward shooting

First, let us describe briefly the backward shooting technique for the computation of $h = F'(z)^{-1}F(\hat{z})$, where z and $\hat{z}$ are n–dimensional (real) vectors. This problem is equivalent to solving the equation

$$F'(z)h = F(\hat{z}). \tag{4}$$

We can rewrite the above equation in the following form:

$$\begin{aligned} -f'(x_1)h_1 + h_2 &= \hat{x}_2 - f(\hat{x}_1), \\ -f'(x_2)h_2 + h_3 &= \hat{x}_3 - f(\hat{x}_2), \\ &\dots \\ -f'(x_n)h_n + h_1 &= \hat{x}_1 - f(\hat{x}_n), \end{aligned} \tag{5}$$

Let $a_k = f'(x_k)$ and $g_k = f(\hat{x}_k) - \hat{x}_{k \bmod n+1}$. Assuming that $a_k \neq 0$ the equation (5) can be written as

$$\begin{aligned} h_1 &= a_1^{-1}(h_2 + g_1), \\ h_2 &= a_2^{-1}(h_3 + g_2), \\ &\dots \\ h_n &= a_n^{-1}(h_1 + g_n). \end{aligned} \tag{6}$$

In order to find h_1, we substitute h_i from the ith equation into the $(i-1)$th equation, starting from the last equation. After $n-1$ substitutions, we obtain

$$h_1 = a_1^{-1}(a_2^{-1}(a_3^{-1}(\dots a_{n-1}^{-1}(a_n^{-1}(h_1 + g_n) + g_{n-1}) \dots + g_3)) + g_2) + g_1). \tag{7}$$

Hence, h_1 can be computed as

$$h_1 = \left(1 - a_1^{-1}a_2^{-1}\dots a_n^{-1}\right)^{-1} \sum_{i=1}^{n} a_1^{-1}\dots a_i^{-1} g_i. \tag{8}$$

One can find $h_n, h_{n-1}, \dots, h_2$ by backward recursion using formula (6).

Before we proceed with the interval version of backward shooting, let us introduce the following notation. Let s be a continuous real–valued function of

m variables and let $\mathbf{y}_1, \mathbf{y}_1, \ldots, \mathbf{y}_m$ be intervals. In the sequel, the expression

$$s(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m) = \{s(y_1, y_2, \ldots, y_m) \colon y_k \in \mathbf{y}_k, 1 \le k \le m\},$$

will denote the interval of all values of s, when $y_k \in \mathbf{y}_k$, $1 \le k \le m$. For example using this notation the condition $f'(\mathbf{x}) \subset \mathbf{a}$ means that $f'(x) \in \mathbf{a}$ for all $x \in \mathbf{x}$.

The backward shooting method can be modified to find the enclosure $\mathbf{h}$ of $F'(\mathbf{z})^{-1}F(\hat{z})$ and for the rigorous evaluation of the interval Newton operator. This is stated in the following theorem.

Theorem 2. *Let* $\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, $\hat{z} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \mathbf{z}$. *Let us assume that the intervals* $\mathbf{a}_k$, $\mathbf{g}_k$, *and* $\mathbf{h}_k$, $k = 1, 2, \ldots, n$ *are such that*

$$0 \notin \mathbf{a}_k, \quad k = 1, \ldots, n, \tag{9}$$

$$f'(\mathbf{x}_k) \subset \mathbf{a}_k, \quad k = 1, \ldots, n, \tag{10}$$

$$f(\hat{x}_k) - \hat{x}_{k \bmod n+1} \in \mathbf{g}_k, \quad k = 1, \ldots, n, \tag{11}$$

$$\left(1 - \mathbf{a}_1^{-1}\mathbf{a}_2^{-1} \cdots \mathbf{a}_n^{-1}\right)^{-1} \sum_{i=1}^{n} \mathbf{a}_1^{-1} \cdots \mathbf{a}_i^{-1}\mathbf{g}_i \subset \mathbf{h}_1 \tag{12}$$

$$\mathbf{a}_k^{-1}(\mathbf{h}_{k \bmod n+1} + \mathbf{g}_k) \subset \mathbf{h}_k, \quad k = n, \ldots, 2. \tag{13}$$

Then $F'(\mathbf{z})^{-1}F(\hat{z}) \subset \mathbf{h}$ *and* $\hat{z} - \mathbf{h}$ *is the enclosure of* $\mathbf{N}(\mathbf{z})$.

Proof. Let us choose an arbitrary point $z = (x_1, x_2, \ldots, x_n) \in \mathbf{z}$. Since $f'(x_k) \in \mathbf{a}_k$ and $0 \notin \mathbf{a}_k$, it follows that $a_k = f'(x_k) \neq 0$ for $1 \le k \le n$. Hence $h = (h_1, h_2, \ldots, h_n)$ computed according to formulas (8) and (6) satisfy the equation $h = F'(z)^{-1}F(\hat{z})$.

From the assumptions (10–13) it follows that $h \in \mathbf{h} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$. It is clear that $F'(\mathbf{z})^{-1}F(\hat{z}) \subset \mathbf{h}$. In consequence $\mathbf{N}(\mathbf{z}) = \hat{z} - F'(\mathbf{z})^{-1}F(\hat{z}) \subset \hat{z} - \mathbf{h}$. □

The above theorem can be used for implementation of the interval Newton method based on backward shooting. First, we find intervals $\mathbf{a}_k$, $\mathbf{g}_k$, and $\mathbf{h}_k$ such that the assumption of the theorem are satisfied. If the condition $\hat{z} - \mathbf{h} \subset \mathbf{z}$ holds then since $\hat{z} - \mathbf{h}$ encloses $\mathbf{N}(\mathbf{z})$ it follows from the Theorem 1 that there exist a unique period–n orbit in $\mathbf{z}$.

In practice the intervals $\mathbf{a}_k$, $\mathbf{g}_k$ and $\mathbf{h}_k$ are found by a computer using interval evaluations of the following formulas:

$$\mathbf{a}_k = f'(\mathbf{x}_k), \quad k = 1, \ldots, n, \tag{14}$$

$$\mathbf{g}_k = f(\hat{x}_k) - \hat{x}_{k \bmod n+1}, \quad k = 1, \ldots, n, \tag{15}$$

$$\mathbf{h}_1 = \left(1 - \mathbf{a}_1^{-1}\mathbf{a}_2^{-1} \cdots \mathbf{a}_n^{-1}\right)^{-1} \sum_{i=1}^{n} \mathbf{a}_1^{-1} \cdots \mathbf{a}_i^{-1}\mathbf{g}_i \tag{16}$$

$$\mathbf{h}_k = \mathbf{a}_k^{-1}(\mathbf{h}_{k \bmod n+1} + \mathbf{g}_k), \quad k = n, \ldots, 2. \tag{17}$$

The interval evaluation of an arithmetic expression is obtained from the given expression by replacing all operands by intervals and all operations by corresponding interval operations. From the inclusion property of interval evaluations (Alefeld and Herzberger, 1983) it follows that interval evaluation of a function s over the intervals $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$ includes the set $\{s(y_1, \ldots, y_m) \colon y_k \in \mathbf{y}_k, 1 \leq k \leq m\}$. Hence the conditions (10–13) are automatically satisfied.

Computation of the interval Newton operator based on the shooting technique is very efficient. For a chaotic map and an unstable periodic orbit, we expect that $a_k = f'(x_k)$ has on average the absolute value larger than 1. In this case multiplication by $\mathbf{a}_k^{-1}$ reduces the diameter of the product and we avoid the "wrapping effect". Hence by using formula (16), we obtain a very narrow enclosure of $\mathbf{h}_1$. Also the backward recursion does not produce wide intervals due to the existence of the factor $\mathbf{a}_k^{-1}$ in (17).

The amount of memory needed to implement the method and the computation time depend linearly on the length of the orbit. Hence periodic orbits of length of order 1000000 are easily handled.

2.3 Finding a candidate for $\mathbf{z}$

Backward shooting is very efficient for evaluation of the interval Newton operator. Another problem is how to obtain a good candidate for $\mathbf{z}$. In this section we describe how to improve the pseudo-periodic orbit found by a computer, so that its position can be used as the center of $\mathbf{z}$.

Before we proceed, let us recall the definition of a δ–pseudo–periodic orbit. One says that a sequence $(x_1, x_2, \ldots, x_n)$ is a *δ–pseudo–orbit* for f if $|f(x_i) - x_{i+1}| \leq \delta$ for $1 \leq i < n$. A δ–pseudo–orbit $(x_1, x_2, \ldots, x_n)$ is called a *δ–pseudo–periodic orbit* if additionally $|f(x_n) - x_1| \leq \delta$.

Computer generated trajectories are pseudo–orbits with the error $|f(x_i) - x_{i+1}|$ of evaluating the single iteration close to the machine precision. In order to find a δ–pseudo–periodic orbit we monitor the trajectory until the condition $|f(x_n) - x_1| < \delta$ holds. The pseudo–periodic orbit $z = (x_1, x_2, \ldots, x_n)$ obtained in this way usually cannot serve as a center of $\mathbf{z}$ because the distance between

$f(x_n)$ and x_1 is typically much larger than the machine precision. In order to obtain a better pseudo–periodic orbit with a smaller global error

$$\delta = \max_{i=1,\ldots,n} |f(x_i) - x_{i \bmod n+1}| \tag{18}$$

we evaluate the interval Newton operator over the point interval vector $z = ([x_1, x_1], \ldots, [x_n, x_n])$. Since we do not want to prove the existence at this stage we may use the standard Newton operator or equivalently the interval Newton operator with a point interval as an input. If the initial approximation z is good enough then $N(z)$ is a better approximation of the position of a true periodic orbit. We may repeat this procedure several times in order to further improve the approximation. Usually after 3 to 5 iterations, we do not observe further improvements. Finally as a candidate, we choose the interval vector $\mathbf{z} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) = ([x_1 - \varepsilon, x_1 + \varepsilon], \ldots, [x_n - \varepsilon, x_n + \varepsilon])$ centered around the position of the pseudo–periodic orbit we obtained from the interval Newton operator. We use the uniform radius ε of the intervals $\mathbf{x}_i$ along the orbit. The radius ε must be chosen in such a way that the condition $0 \notin f'(\mathbf{x}_i)$ holds for all i. In case of the logistic map with one maximum at $c = 0.5$ we have to choose

$$\varepsilon < \varepsilon_{\max} = \min_{i=1,\ldots,n} |x_i - c|. \tag{19}$$

If there are no clues for the value of ε we may find the proper value by trial-and-error.

3 Computation of the Lyapunov exponents of periodic orbits

The behavior of solutions in a small neighborhood of a periodic orbit is determined by the Lyapunov (characteristic) exponents of the orbit (Guckenheimer and Holmes, 1983). If one of the Lyapunov exponents is positive then the periodic orbit is unstable and a typical trajectory is repelled from the orbit. The Lyapunov exponent of the periodic orbit $(x_1, x_2, \ldots, x_k)$ for the one–dimensional map f is defined by

$$\lambda = \frac{1}{n} \log |(f^n)'(x_1)| \tag{20}$$

$$= \frac{1}{n} \log |f'(x_n) \ldots f'(x_2) f'(x_1)| \tag{21}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log |f'(x_i)|. \tag{22}$$

If we know that the point x_i belongs to the interval $\mathbf{x}_i$, we may easily find the interval enclosing the Lyapunov exponent of the orbit using the last equation. Observe that although formulas (21) and (22) are mathematically equivalent the latter one is much more useful for computation of the accurate bound for the Lyapunov exponent of the orbit. The reason is that the first formula contains the product of n intervals, which when evaluated in interval arithmetic may produce very large errors.

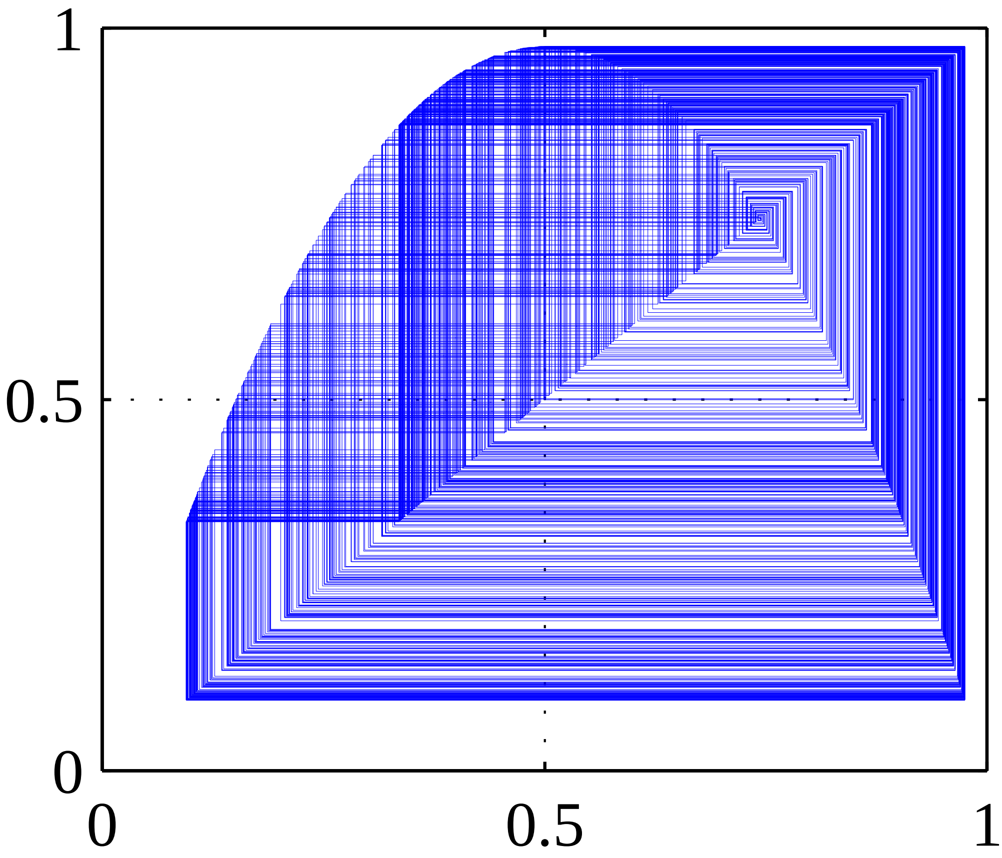

Fig. 1. Trajectory of the logistic map composed of 1000 points.

4 Long periodic orbits for the logistic map

In this section, we apply the method described previously to the logistic map

$$f(x) = ax(1 - x), \tag{23}$$

with the parameter value $a = 3.9$. For this value of the parameter a chaotic behavior is observed. A trajectory of the logistic map consisting of 1000 points is plotted in Fig. 1.

First let us illustrate the method for a short periodic orbit. For this case all assumptions of theorem 2 can be checked without help of a computer.

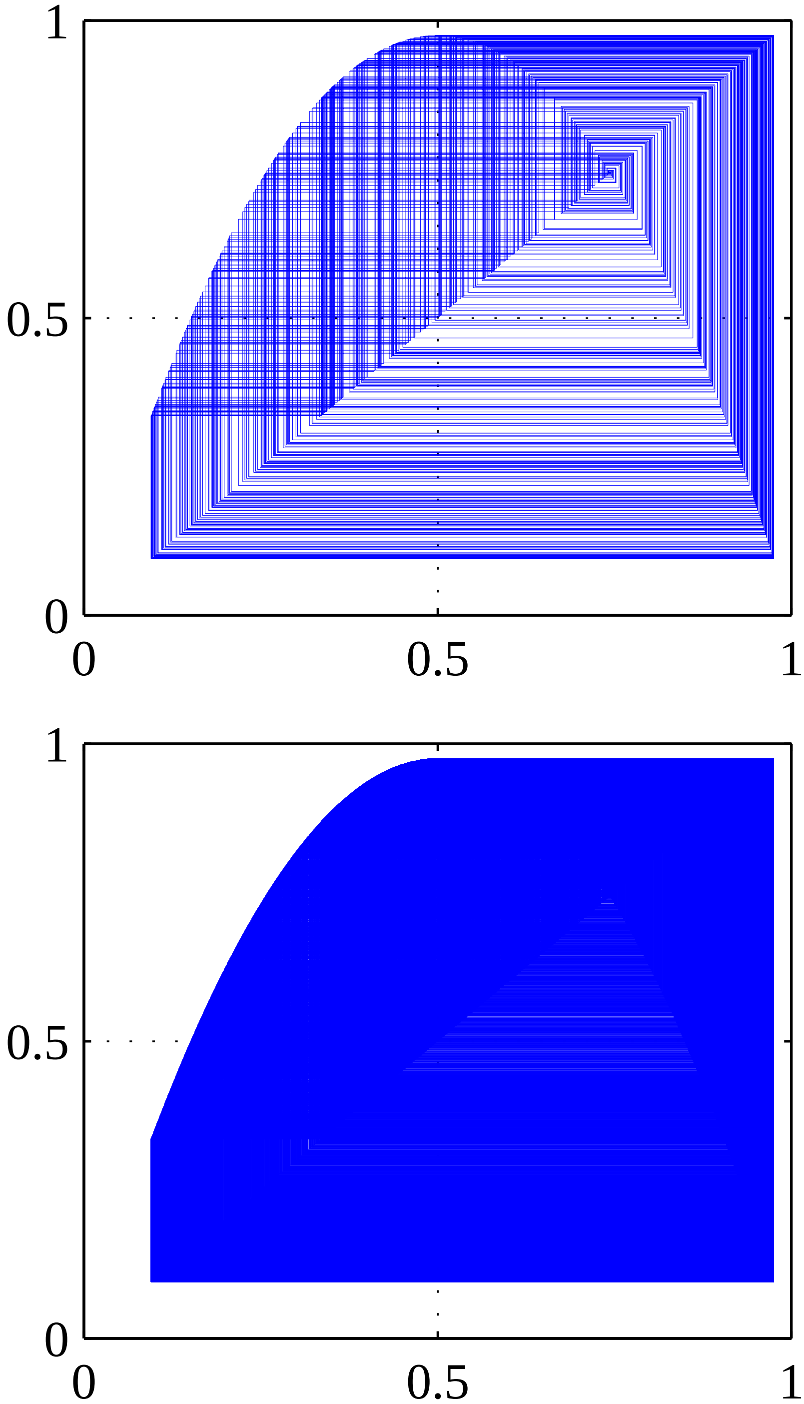

Fig. 2. Periodic orbits for the logistic map, (a) period 701 orbit, (b) period 11348 orbit.

In computer simulation one observes the following pseudo–periodic orbit: $z = (0.44871, 0.96474, 0.13265)$. Let us choose an interval vector $\mathbf{z}$ and a point $\hat{z} \in \mathbf{z}$ for which we apply the interval Newton method to prove the existence of period–3 orbit in a neighborhood of z:

$$\begin{aligned}
\mathbf{z} &= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = ([0.44, 0.46], [0.95.97], [0.12, 0.14]),\\
\hat{z} &= (\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0.45, 0.96, 0.13).
\end{aligned}$$

Next we find intervals $\mathbf{a}_k$, $\mathbf{g}_k$, and $\mathbf{h}_k$ satisfying the assumptions of the Theorem 2.

$$\begin{aligned}
&f'(\mathbf{x}_1) = a(1 - 2\mathbf{x}_1) = [0.312, 0.468] \subset \mathbf{a}_1 = [0.311, 0.469],\\
&f'(\mathbf{x}_2) = a(1 - 2\mathbf{x}_2) = [-3.666, -3.510] \subset \mathbf{a}_2 = [-3.667, -3.509],\\
&f'(\mathbf{x}_3) = a(1 - 2\mathbf{x}_3) = [2.808, 2.964] \subset \mathbf{a}_3 = [2.807, 2.965],\\
&f(\hat{x}_1) - \hat{x}_2 = 0.00525 \in \mathbf{g}_1 = [0.00524, 0.00526],\\
&f(\hat{x}_2) - \hat{x}_3 = 0.01976 \in \mathbf{g}_2 = [0.0195, 0.01977],\\
&f(\hat{x}_3) - \hat{x}_1 = -0.00891 \in \mathbf{g}_3 = [-0.00892, -0.00890],\\
&\mathbf{a}_1^{-1}\mathbf{a}_2^{-1}\mathbf{a}_3^{-1} \subset \mathbf{a} = [-0.327, -0.196],\\
&\mathbf{a}_1^{-1}\mathbf{g}_1 + \mathbf{a}_1^{-1}\mathbf{a}_2^{-1}\mathbf{g}_2 + \mathbf{a}_1^{-1}\mathbf{a}_2^{-1}\mathbf{a}_3^{-1}\mathbf{g}_3 \subset \mathbf{b} = [-0.01, 0.01]],\\
&(1 - \mathbf{a})^{-1}\mathbf{b} \subset \mathbf{h}_1 = [-0.009, 0.009],\\
&\mathbf{a}_3^{-1}(\mathbf{h}_1 + \mathbf{g}_3) \subset \mathbf{h}_3 = [-0.008, 0.001],\\
&\mathbf{a}_2^{-1}(\mathbf{h}_3 + \mathbf{g}_2) \subset \mathbf{h}_2 = [-0.006, -0.003],\\
&\hat{z} - \mathbf{h} \subset \mathbf{R} = ([0.441, 0.459]; [0.963, 0.966]; [0.129, 0.138]).
\end{aligned}$$

It can be easily checked that $\mathbf{R} \subset \mathbf{z}$. From the Theorem 2 it follows that $\mathbf{N}(\mathbf{z}) \subset \mathbf{z}$ and in consequence there exist a unique periodic orbit within $\mathbf{N}(\mathbf{z})$. Observe that all conditions for existence of periodic orbit are of the form of inequalities, and can be checked rigorously using interval arithmetic implemented on a computer.

When we iterate the interval Newton operator starting with $\mathbf{z}_0 = \mathbf{z}$ we obtain the following sequence of interval vectors $\mathbf{z}_{i+1} = \mathbf{N}(\mathbf{z}_i)$ containing the solution:

$$
\begin{aligned}
\mathbf{z}_0 =&([0.44, 0.46], [0.95.97], [0.12, 0.14]),\\
\mathbf{z}_1 =&(0.4^{543}_{431}, 0.96^{544}_{411}, 0.13^{469}_{069}),\\
\mathbf{z}_2 =&(0.448^{740}_{697}, 0.96474^{556}_{134}, 0.1326^{603}_{452}),\\
\mathbf{z}_3 =&(0.44871775^{356}_{282}, 0.964743511^{571}_{498}, 0.13265252^{723}_{696}),\\
\mathbf{z}_4 =&(0.4487177531952^{604}_{597}, 0.9647435115343658^{328}_{216}, 0.132652527098158^{585}_{307}),\\
\mathbf{z}_5 =&(0.4487177531952^{604}_{597}, 0.9647435115343658^{328}_{216}, 0.132652527098158^{585}_{307}).
\end{aligned}
$$

After five iterations the sequence stabilizes, and we obtain a very narrow enclosure of the true periodic orbit. The center of the last interval vector is the approximation of the position of the true periodic orbit with the error smaller than $3 \cdot 10^{-16}$.

The strength of the method lies in the fact that all assumptions of the existence theorem can be rigorously checked by a computer. Now, we show an example how to find and prove the existence of a very long periodic orbit. First, we need to find a pseudo–periodic orbit. In this particular example, we choose $x_1 = 0.66$ as the initial point. After $n = 5480633$ iterations, the computer generated trajectory returns to the small neighborhood of the initial point

$$\delta_1 = |x_{5480634} - x_1| < 2.21 \cdot 10^{-7}.$$

After two iterations of the interval Newton operator with point interval as an input, we obtain a better approximation $(x_1, x_2, \ldots, x_n)$ of the periodic orbit with the global error near the machine precision

$$\delta_2 = \max_{i=1,\ldots,n} |f(x_i) - x_{i \bmod n+1}| < 4.16 \cdot 10^{-16}.$$

The distance between the maximum point $c = 0.5$ and the pseudo–periodic orbit is

$$\varepsilon_{\max} = \min_{i=1,\ldots,n} |x_i - c| = 4.9 \cdot 10^{-8}.$$

The radius of the interval vector $\mathbf{z}$ have to be smaller than $4.9 \cdot 10^{-8}$. Next, we create the interval vector $\mathbf{z} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ where $\mathbf{x}_i = [x_i - \varepsilon, x_i + \varepsilon]$ with $\varepsilon = 10^{-8}$. Finally, using the methods described in the previous section the interval Newton operator is evaluated on $\mathbf{z}$, and we check that $\mathbf{N}(\mathbf{z}) \subset \mathbf{z}$. In this way, we prove that there exist a unique periodic orbit within $\mathbf{z}$. By iterating the interval Newton operator twice, we obtain the approximate position of the true periodic orbit with uncertainty smaller than $\delta_3 = 2.08 \cdot 10^{-10}$ at every point along the orbit. As 5480633 is a prime number, we are sure that this is the principal period of the orbit. In this way, we have shown that there exists a true periodic trajectory near the one generated by a computer. In particular, we know that the interval

$$\mathbf{x}_1 = [0.6599999999999998, 0.66000000000000026]$$

contains a periodic point with period 5480633. Time necessary to complete the existence proof is 87.14 seconds. Its main part is the time needed to evaluate the interval Newton operator. Computations were performed on a Pentium III computer with 500MHz processor. Program was written in C++ (gcc version egcs-2.91.66). In the program, we used the Bias/Profil packages for interval computations (Knüppel, 1993).

	n	x_1	δ_1	p	δ_2	ε	δ_3	t [s]
1	701	0.70	$4.20 \cdot 10^{-5}$	3	$3.05 \cdot 10^{-16}$	10^{-3}	$9.50 \cdot 10^{-15}$	0.01
2	11348	0.70	$3.45 \cdot 10^{-6}$	2	$3.89 \cdot 10^{-16}$	10^{-6}	$3.38 \cdot 10^{-12}$	0.12
3	723543	0.70	$2.23 \cdot 10^{-7}$	2	$4.02 \cdot 10^{-16}$	10^{-6}	$1.15 \cdot 10^{-11}$	7.96
4	1921687	0.60	$1.56 \cdot 10^{-7}$	2	$4.16 \cdot 10^{-16}$	10^{-7}	$1.61 \cdot 10^{-11}$	20.95
5	2444017	0.64	$2.56 \cdot 10^{-7}$	3	$4.16 \cdot 10^{-16}$	10^{-8}	$2.08 \cdot 10^{-10}$	27.29
6	5480633	0.66	$2.21 \cdot 10^{-7}$	2	$4.16 \cdot 10^{-16}$	10^{-8}	$2.08 \cdot 10^{-10}$	87.14
7	8076157	0.62	$8.18 \cdot 10^{-8}$	2	$4.16 \cdot 10^{-16}$	10^{-8}	$7.03 \cdot 10^{-10}$	166.00

Table 1
Examples of long periodic orbits for the logistic map, n is the period of the orbit, x_1 is a start point of the computer generated δ_1–pseudo–periodic orbit, t is the computation time necessary to prove the existence (see complete explanation in the text).

In a similar way, we have found several other long periodic orbits for the logistic map. Two of these orbits (with periods 701 and 11348) are plotted in Fig 2. Computation details are collected in Table 1, where n denotes the period of the orbit, x_1 is the initial point of the computer generated δ_1–pseudo–periodic orbit $(x_k)_{k=1}^n$ ($|f(x_n) - x_1| < \delta_1$). After p iterations of the Newton operator, we obtain a very good approximation of periodic orbit with errors $|f(x_{k \bmod n+1}) - x_k| < \delta_2$ for $k = 1, \ldots, n$. ε is the uniform radius of the interval vector $\mathbf{z}$ ($\mathbf{x}_i = [x_i - \varepsilon, x_i + \varepsilon]$) used as an input to the interval Newton operator and δ_3 is the maximum radius of the intervals for which the existence of the periodic orbit is proved — it is the accuracy of the position of the periodic orbit found. In the last column, we give the time necessary to complete the existence proof.

In Table 2 several parameters of the orbits are given. Using equation (22), we have computed the interval $\boldsymbol{\lambda}$ containing the Lyapunov exponent of the orbit (for example 0.496142_{49}^{51} denotes the interval $[0.49614249, 0.49614251]$).

In order to see how densely the orbit fills the attractor, we compute two numbers. The value of $d_{\max}$ which is the maximum gap between the points belonging to the orbit gives us the global information about the system. If there exists an attracting periodic orbit for the map then basin of its attraction cannot contain an interval with diameter larger than $d_{\max}$. $d_{\min}$ is the minimum distance between the centers of intervals $\mathbf{x}_i$. One can clearly see that the distance between the centers is much less than the maximum radius of these intervals (δ_3 in the Table 1). This means that the intervals $\mathbf{x}_i$ overlap.

	$\boldsymbol{\lambda}$	$d_{\max}$	$d_{\min}$
1	0.5030428303868^{9}_{7}	0.014	$2.83 \cdot 10^{-7}$
2	0.4983757464^{7}_{4}	0.0012	$2.08 \cdot 10^{-9}$
3	0.496148838^{6}_{4}	$2.75 \cdot 10^{-5}$	$2.02 \cdot 10^{-13}$
4	0.495602678^{6}_{4}	$1.11 \cdot 10^{-5}$	$7.92 \cdot 10^{-14}$
5	0.49553530^{9}_{5}	$9.11 \cdot 10^{-6}$	$1.40 \cdot 10^{-14}$
6	0.49589456^{9}_{5}	$4.70 \cdot 10^{-6}$	$1.75 \cdot 10^{-14}$
7	0.496142^{51}_{49}	$3.15 \cdot 10^{-6}$	$2.78 \cdot 10^{-15}$

Table 2
Properties of long periodic orbits for the logistic map, $\boldsymbol{\lambda}$ is the enclosure of the Lyapunov exponent of the orbit, $d_{\max}$ is the maximum gap between the points belonging to the orbit, $d_{\min}$ is the minimum distance between the points in the orbit.

5 Performance of the method for short orbits

Interval methods for proving the existence of periodic orbits are capable of finding all short period cycles. The method for finding all period–n cycles is a combination of the interval Newton method and generalized bisection technique (Galias, 2001). First the region under investigation is covered by a finite number of boxes (intervals in the one–dimensional case). For each of these boxes the interval Newton operator is evaluated. If the existence condition is fulfilled the box contains exactly one periodic orbit. If the non-existence condition holds there are no period–n orbits within the box considered. If none of these two conditions is true the box is divided into smaller parts and the computations are repeated. This method allows to find very good approximations of the positions of all period–n cycles and to find the exact number of period–n cycles contained in the given region. As an interval operator one may use the interval Newton operator, the Krawczyk operator or the Hansen–Sengupta operator (Neumaier, 1990) in its standard or global version. In the standard version one evaluates the interval operator for the map $g(x) = f^n(x) - x$, while in the global version the method is applied to the global map (2). It appears that the global version is more suitable for investigation of longer cycles (Galias, 2001).

n	Q_n	P_n	Global Newton method		with backward shooting	
			I_n	time [s]	I_n	time [s]
1	2	2	9	0.01	15	0.00
2	1	4	47	0.03	41	0.01
3	2	8	137	0.09	129	0.02
4	1	8	381	0.23	269	0.03
5	2	12	1043	0.69	759	0.07
6	3	28	3121	2.49	2143	0.25
7	6	44	10197	9.03	6589	0.79
8	9	80	36301	37.03	22789	3.04
9	14	134	119519	141.71	72447	10.55
10	21	224	433467	578.35	255247	40.93
11	34	376	1588405	2442.86	940733	163.47
12	52	656	5970917	10619.64	3519917	652.01
13	86	1120	23321301	46500.98	13599583	2670.59
14	133	1908	—	—	48137901	10001.56

Table 3
Comparison of performance of the global Newton method and its modification using backward shooting for finding all low–period cycles for the logistic map for $a = 3.9$. Q_n — number of periodic orbits with period n, P_n — number of fixed points of f^n, I_n is the number of intervals into which the domain of the map is divided in order to find all period–n cycles.

In this section, we compare the performance of the global Newton method and its version based on backward shooting for finding all low period cycles of the logistic map.

The results are summarized in Table 3. Q_n and P_n denote the number of periodic orbits with period n and the number of fixed points of f^n, respectively. In the table, we report the number of intervals into which the domain of the map is divided in order to find all period–n cycles and the computation

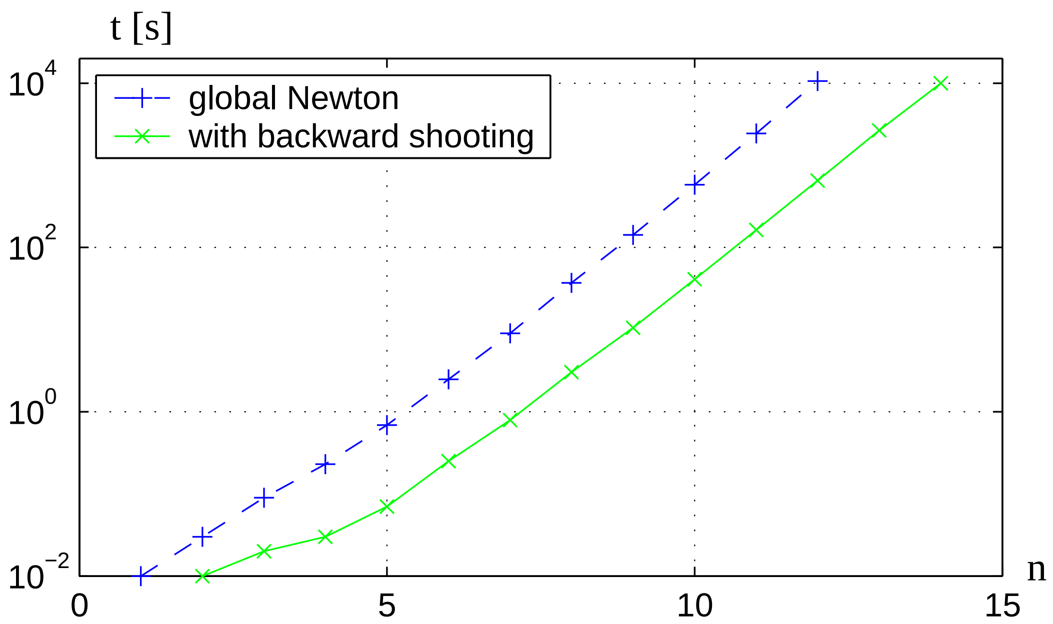

Fig. 3. Time necessary to find all period–n cycles with global Newton method and the version using backward shooting.

time (also plotted in Fig. 3). One can see that the number of rectangles is smaller (almost twice for larger n) for the method using backward shooting. The difference in the computation time is even more significant (the improved version is approximately 17 times faster for $n = 13$). The evaluation of the backward shooting version of the Newton operator is much faster, and the difference grows with n. This is due to the fact that the improved method does not use $n \times n$ matrices, which are extensively used by the unmodified version. In consequence, using the method with backward shooting, we are able in the same time to find all periodic orbits for n larger by 2.

6 Conclusions

In this paper, we have described how to prove the existence of a true periodic orbit in a neighborhood of the one generated by a computer using the interval Newton method and the backward shooting technique. We have applied this method to show the existence of very long periodic orbits for the logistic map. We have also shown that the modification using the shooting technique is superior to the unmodified version when they are applied to the problem of finding all low–period cycles.

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