INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS
Int. J. Circ. Theor. Appl., **27**: 589–604 (1999)

LOCAL TRANSVERSAL LYAPUNOV EXPONENTS FOR ANALYSIS OF SYNCHRONIZATION OF CHAOTIC SYSTEMS

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SUMMARY

In this paper we consider the problem of synchronization of coupled chaotic systems. Synchronization is studied by means of local transversal Lyapunov exponents. We show that they can be successfully used in investigations of synchronization properties. A criterion for synchronization based on this concept is developed and discussed. Using examples of coupled Hénon maps and coupled hyperchaotic electronic systems we compare this technique with other methods for investigation of synchronization properties. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: chaotic system; synchronization; transversal Lyapunov exponents

1. INTRODUCTION

It is well known that when chaotic systems are coupled, they may demonstrate identical oscillations associated with the onset of synchronization.[1–3] The source of this synchronization is additional dissipation introduced when the variables are not following the same trajectories. Recently, there has been a considerable interest in using the concept of synchronization of chaos to develop spread spectrum communication systems. In applications concerning extraction of information from transmitted chaotic signal, a response system must be synchronized with the signal. Therefore, the problem of synchronization of chaotic systems is of very high importance.

In this paper we study synchronization of unidirectionally coupled chaotic systems. We will consider coupled discrete-time systems

$$\mathbf{x}(t+1) = \mathbf{F}(\mathbf{x}(t)) \tag{1a}$$

$$\mathbf{y}(t+1) = \mathbf{F}(\mathbf{y}(t) + \mathbf{d}\cdot(\mathbf{x}(t) - \mathbf{y}(t))) \tag{1b}$$

and coupled continuous-time systems

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{2a}$$

$$\dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t)) + \mathbf{d}\cdot(\mathbf{x}(t) - \mathbf{y}(t)) \tag{2b}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ are the state variables of the drive and response systems and $\mathbf{d}$ is a diagonal matrix with diagonal elements $d_1, \ldots, d_n$ being the coupling coefficients.

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Contract/grant sponsor: University of Mining and Metallurgy, Kraków
Contract/grant number: 10.10.120.133

CCC 0098–9886/99/060589–16$17.50
Received September 1998
Revised April 1999

We say that the systems synchronize if $\|\mathbf{y}(t) - \mathbf{x}(t)\| \to 0$ as $t \to \infty$ (the trajectory of system (1) or (2) converges to the synchronization subspace $\mathbf{x} = \mathbf{y}$). It is clear that if the coupling coefficients are large enough the systems will synchronize. For communication tasks one usually chooses systems where only one of the coupling coefficients is non-zero (otherwise one needs to send more signals in order to extract the information).

There are several methods for investigating the synchronization problem. The first criterion for successful synchronization, introduced in Reference 3, is based on conditional Lyapunov exponents calculated along typical trajectory of the system (we will call them global transversal Lyapunov exponents). When all global transversal Lyapunov exponents of system (1b) driven by the signal $\mathbf{x}(t)$ are negative then one expects that the systems synchronize. This may not be true expecially in the presence of noise.[4–6] It may happen that in the neighbourhood of an unstable periodic orbit there exist a region where the trajectories are pushed away from the synchronization subspace. Such a situation occurs when one of the Lyapunov exponents associated with the measure supported by the periodic orbit is positive. In this case small noise inevitable in real systems could force the trajectory to enter such a region. This in turn could lead to a desynchronization burst. This is observed in many computer and laboratory experiments.[4–6] When there is no noise in the system (or the noise level is very low) one observes synchronization. But when the noise level is increased desynchronization bursts occur.

Using the above argument one could develop another criterion of successful synchronization based on transversal Lyapunov exponents computed along periodic orbits. In order to ensure synchronization one should compute transversal Lyapunov exponents for all periodic orbits and check whether they are negative. This is rather difficult and computationally expensive task. There is also another drawback of this method. Even if the periodic orbit attracts the trajectory to the synchronization space globally it is possible that it repels trajectories locally. Examples of such orbits are given in Reference 7. This may also cause desynchornization bursts.

In this paper we use a method based on the local transversal Lyapunov exponents. In Section 2 we recall the notion of local Lyapunov exponents and explain why local properties may be of some interest to synchronization. In Section 3 we observe behaviour of coupled Hénon maps in the presence of noise. In Section 4 we study synchronization of coupled Hénon maps using different techniques. First, we compute global transversal Lyapunov exponents. Then we find periodic orbits with period $n \leqslant 13$ and compute Lyapunov exponents associated with these periodic orbits. As the last method we use local transversal Lyapunov exponents and develop a new synchronization criterion. Finally, we compare the performance of these three methods and discuss their advantages. In Section 5 we investigate synchronization of hyperchaotic circuits in the presence of noise. In Section 6 we try to explain the synchronization behaviour using several methods. We use global Lyapunov function method, the method based on transversal Lyapunov exponents, transversal Lyapunov exponents along unstable periodic orbits and finally local transversal Lyapunov exponents.

2. LOCAL TRANSVERSAL LYAPUNOV EXPONENTS

First, we will briefly recall the notions of local Lyapunov exponents and transversal Lyapunov exponents.[3,8–10]

Lyapunov exponents $\lambda_i(\mathbf{x})$ of a trajectory based at $\mathbf{x}$ are the logarithms of the eigenvalues of the matrix:

$$\Lambda(\mathbf{x}) = \lim_{L \to \infty} ([\mathbf{T}^L(\mathbf{x})]^{\mathrm{T}} \mathbf{T}^L(\mathbf{x}))^{1/2L} \tag{3}$$

where for discrete-time systems

$$\mathbf{T}^L(\mathbf{x}) = \mathbf{DF}(\mathbf{F}^{L-1}(\mathbf{x})) \cdots \mathbf{DF}(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x}) \tag{4}$$

is the composition of L Jacobians and for continuous-time systems $\mathbf{T}^L(\mathbf{x})$ is the matrix of partial derivatives of the time-L map induced by the continuous-time system.

Local Lyapunov exponents $\lambda_i(\mathbf{x}, L)$ are the logarithms of the eigenvalues of the matrix:

$$\Lambda(\mathbf{x}, L) = ([\mathbf{T}^L(\mathbf{x})]^{\mathrm{T}} \mathbf{T}^L(\mathbf{x}))^{1/2L} \tag{5}$$

Local Lyapunov exponents say how rapidly perturbations of the initial point $\mathbf{x}$ changes in L steps from the moment of perturbation. From multiplicative ergodic theorem of Oseledec[8] it follows that local Lyapunov exponents tend to global exponents as L goes to infinity (for a typical point on the chaotic attractor, i.e. for almost all points with respect to the natural measure, global Lyapunov exponents do not depend on the initial point).

Local transversal Lyapunov exponents are the local Lyapunov exponents corresponding to eigenvectors transversal to the synchronization subspace. It turns out that they are very useful in studies of synchronization of chaotic systems, especially in the presence of noise.

Global transversal Lyapunov exponents, which are frequently used for investigation of synchronization give us stability information which is averaged over the whole attractor. On the other hand, local transversal Lyapunov exponents tell us how trajectories of the coupled system are repelled or attracted to the synchronization subspace in time L. By investigating the average value of maximum local transversal exponent we are able to find out how on an average, trajectories are repelled or attracted to the synchronization subspace in time L. Obviously, local transversal Lyapunov exponents may vary significantly with the point on the synchronization subspace, especially for small L. Hence using local exponents we are able to find regions, where synchronization behaviour is less robust and where noise could easily destroy the demanded behaviour. For synchronization problem we are mainly interested whether local transversal Lyapunov exponents are negative. If for a given point on a synchronization subspace and a given value of L all local transversal Lyapunov exponents are negative then in the neighbourhood of the considered point trajectories are attract to the synchronization subspace in time $t \in [0, L]$. Using this observation we will develop a criterion for robust synchronization based on local transversal Lyapunov exponents.

In the subsequent sections we consider two examples of coupled systems and show how synchronization can be studied using this technique.

3. SYNCHRONIZATION OF HÉNON MAPS

As the first example we consider uni-directionally coupled Hénon maps. The drive system is the Hénon map defined by

$$h(x, y) = (1 + y - ax^2, bx) \tag{6a}$$

where $a = 1{\cdot}4$ and $b = 0{\cdot}3$ are standard parameter values. A typical trajectory of the Hénon map is shown in Figure 3(a).

The response system is

$$h(x', y') = h(x' + d_1(x - x'), y' + d_2(y - y')) \tag{6b}$$

where d_1 and d_2 are the coupling coefficients. We will consider the case when only d_1 is non-zero.

In the first experiment we test synchronization of system (6) in a noise free environment. We have considered four values of coupling coefficient: $d_1 = 0{\cdot}4$, $0{\cdot}5$, $0{\cdot}6$, $0{\cdot}8$. For all cases the system eventually synchronizes.

In the second experiment we have checked the behaviour of the system in the case when the driving signal is corrupted by random noise of amplitude 0·1. The results are shown in Figure 1. For $d_1 = 0{\cdot}4$ one observes frequent desynchronization bursts. For $d_1 = 0{\cdot}5$, $0{\cdot}6$ one can see only very short desynchronization pulses and for $d_1 = 0{\cdot}8$ the error signal is of the noise level.

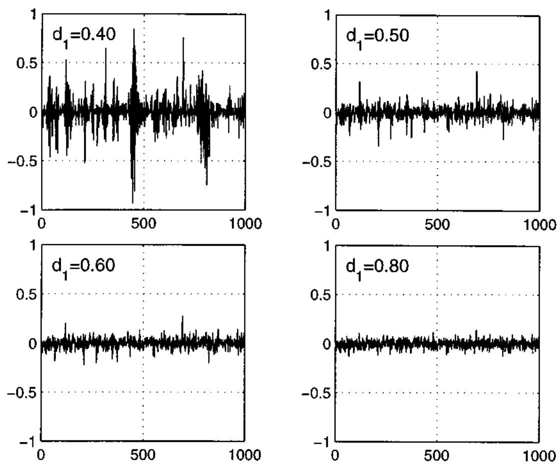

Figure 1. Synchronization error for different coupling coefficient d_1, driving signal perturbed by additive noise

4. STUDY OF SYNCHRONIZATION

In this section we explain the behaviour of coupled Hénon maps observed in the previous section.

4.1. Global transversal Lyapunov exponents

First, we investigate synchronization in terms of global transversal Lyapunov exponents.

According to the criterion for successful synchronization based on global transversal Lyapunov exponents[3] systems (6a) and (6b) synchronize if all transversal Lyapunov exponents are negative. We have computed transversal Lyapunov exponents for $d_1 \in [0, 2]$. During the computations we have used the algorithms proposed in Reference 8. The maximum transversal Lyapunov exponents λ_{max} versus d_1 is plotted in Figure 2. One can see that λ_{max} is negative for $d_1 \in [0{\cdot}34, 1{\cdot}68]$. This explains synchronization of Hénon maps in a noise free environment for $d_1 = 0{\cdot}4, 0{\cdot}5, 0{\cdot}6, 0{\cdot}8$. Global transversal Lyapunov exponents cannot explain the phenomena of loss of synchronization observed when the driving signal is perturbed by noise.

4.2. Transversal Lyapunov exponents associated with periodic orbits

As mentioned before desynchronization bursts can be explained by the existence of positive transversal Lyapunov exponents computed along periodic trajectories embedded within chaotic attractor.

First using the interval Newton method[11] we have found for the Hénon map all periodic orbits with period $n \leqslant 13$ in the trapping region Q shown in Figure 3(a). For the details see Reference 12. The number $Q(n)$ of periodic orbits with principal period n is shown in Table I. There are 97 periodic orbits (and hence 1055 periodic points) with period $n \leqslant 13$ embedded within Hénon attractor. All these orbits lie within the attractor observed numerically. They are shown in Figure 3(b).

For all these periodic orbits we have computed transversal Lyapunov exponents for several $d_1 \in [0, 2]$. The maximum transversal Lyapunov exponents are plotted in Figure 4.

For $d_1 \in [0{\cdot}51, 1{\cdot}45]$ transversal Lyapunov exponents corresponding to these periodic orbits are negative. One can see that the largest transversal Lyapunov exponents are associated with two shortest periodic orbits, namely period-1 and period-2 orbits. Period-1 orbit gives the strongest condition for $d_1 < 1$ while period-2 orbit is the most restrictive for $d_1 > 1$. Now, we can explain very frequent desynchronization bursts occurring for $d_1 = 0{\cdot}4$ (cf. Figure 1). For $d_1 = 0{\cdot}4, 0{\cdot}5$ the maximum transversal exponents of period-1 orbit is positive

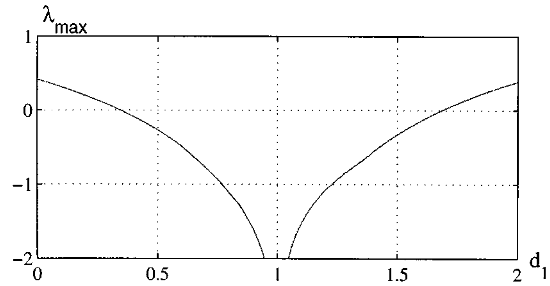

Figure 2. Maximum global transversal Lyapunov exponent

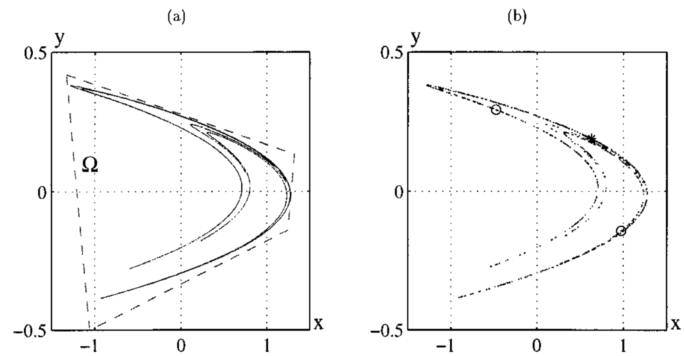

Figure 3. (a) Trajectory of the Hénon map consisting of 10,000 points, the quadrangle Ω—a trapping region for the Hénon map, (b) cycles with period $n \leqslant 13$ for the Hénon map, period-1 and period-2 points are plotted with a star and a circle symbols, respectively

Table I. Periodic orbits for the Hénon map in the trapping region Ω. $Q(n)$ is the number of cycles with principal period n, $P(n)$ is the number of fixed points of h^n

n	$Q(n)$	$P(n)$
1	1	1
2	1	3
3	—	1
4	1	7
5	—	1
6	2	15
7	4	29
8	7	63
9	6	55
10	10	103
11	14	155
12	19	247
13	32	417

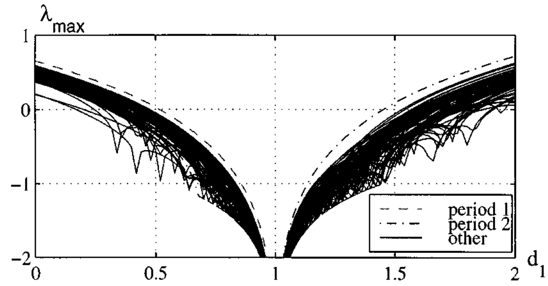

Figure 4. Maximum transversal Lyapunov exponent for cycles with period $n \leqslant 13$

(for $d_1 = 0{\cdot}4$ we have $\lambda_{max} = 0{\cdot}19$). It is clear that criterion based on periodic orbits is stronger than condition obtained using global transversal Lyapunov exponents. Usually, in order to obtain a reliable condition for synchronization it is sufficient to compute transversal Lyapunov exponents for low-period orbits. Lyapunov exponents of longer periodic orbits are usually close to global Lyapunov exponents. However, one is never sure whether a very long periodic orbits with a positive transversal Lyapunov exponents does not exist.

The main drawback of this method is the necessity of finding periodic orbits, which may be difficult for more complex systems. Another problem is that low-period orbits do not fill the attractor completely (compare Figures 3(a) and 3(b)). Hence using the method we do not obtain any information about the behavior of the system in parts of attractor not visited by low-period cycles.

In the next subsection we discuss a method which is free of the problems mentioned above.

4.3. Local transversal Lyapunov exponents

Here, we use local transversal Lyapunov exponents for the analysis of the synchronization properties.

First, let us discuss how local Lyapunov exponents $\lambda_i(\mathbf{x}, L)$ depend on L. In Figure 5 we show histograms of maximum local transversal Lyapunov exponents $\lambda_1(\mathbf{x}, L)$ for different number of steps L. The eigenvalues of matrix (5) have been computed for 50,000 points on the attractor.

In the construction of histograms we have chosen bins of the length 0·05 covering the interval $[-2, 1]$. For each bin the number of points for which the maximum Lyapunov exponents belongs to the bin is plotted. For small L the spectrum is rather wide, while for greater L it becomes narrower and much higher. In the limit $L \to \infty$ there should be a very narrow peak at the value of the maximum global transversal Lyapunov exponent.

One can also see that for $L = 1$ there are number of points on the attractor with $\lambda_1(\mathbf{x}, 1) > 0$. This is an explanation for the existence of desynchronization bursts observed in the presence of noise.

In Figure 6 we show how the histogram changes when the coupling coefficient d_1 is modified. For $d_1 = 0{\cdot}6$ large part of the histogram lies above zero. For stronger coupling $d_1 = 0{\cdot}70$ the part of the histogram above zero is smaller and for coupling $d_1 = 0{\cdot}75$ the whole histogram lies below zero (cf. Figure 6).

In order to find the value of d_1 for which all local transversal Lyapunov exponents are negative we have computed local transversal Lyapunov exponents at 50,000 points along the attractor (for $d_1 \in [0, 2]$ with the step 0·01). At each point we have chosen the maximum local transversal Lyapunov exponent. For each d_1 we computed the minimum, average and maximum of these maximum exponents. The results are plotted in Figure 7.

One can clearly see the continuous change of these values with the change of the coupling coefficient. The average value (solid line) is negative for $d_1 \in [0{\cdot}38, 1{\cdot}62]$ while the maximum value (dashed line) is negative for $d_1 \in [0{\cdot}73, 1{\cdot}27]$. It is clear that when there is no noise in the system one can achieve synchronization with $d_1 \in [0{\cdot}38, 1{\cdot}62]$ (the average value is very close to maximum global transversal Lyapunov exponent).

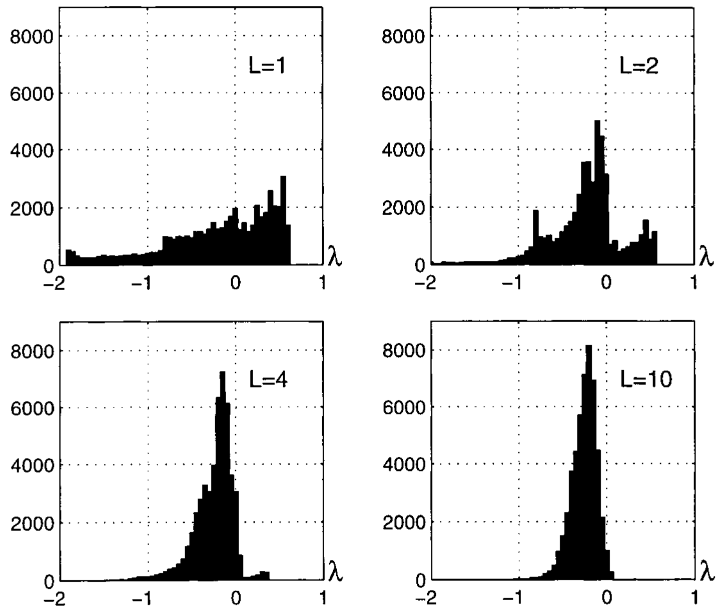

Figure 5. Histograms of maximum local transversal Lyapunov exponent for different time L (local exponents computed at 50,000 points, $d_1 = 0{\cdot}5$)

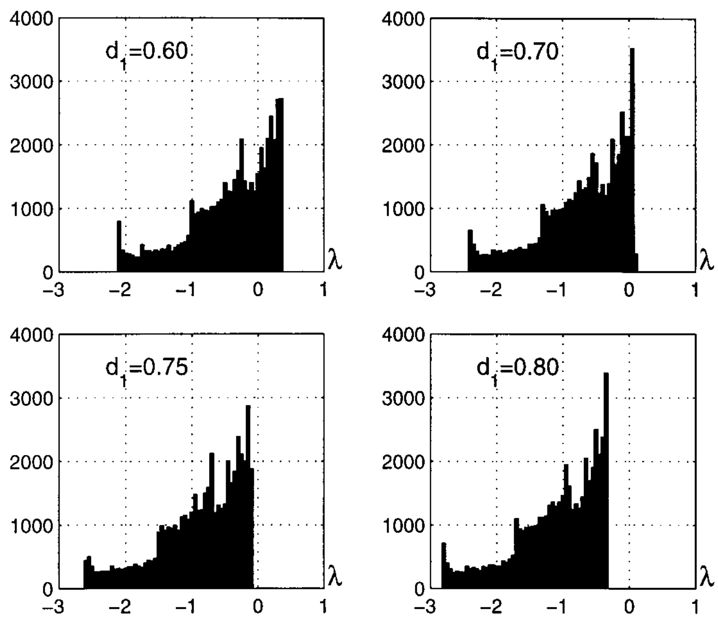

Figure 6. Histograms of maximum local transversal Lyapunov exponent for different coupling coefficient d_1 (local exponents computed at 50,000 points, $L = 1$)

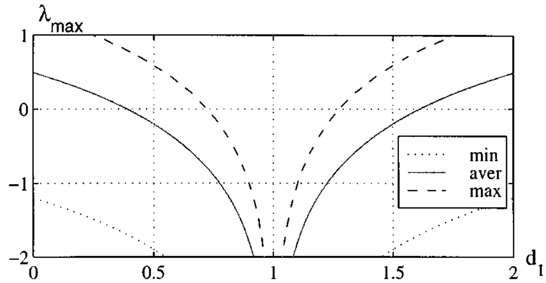

Figure 7. Maximum local transversal Lyapunov exponent. Average, maximum and minimum values are plotted as solid, dashed and dotted lines, respectively. The number of steps $L = 1$

Table II. Conditions for synchronization obtained using different methods

Method	Condition
Global Lyapunov exponents	$d_1 \in [0{\cdot}34, 1{\cdot}68]$
Lyapunov exponents for cycles	$d_1 \in [0{\cdot}51, 1{\cdot}45]$
Local Lyapunov exponents	$d_1 \in [0{\cdot}73, 1{\cdot}27]$

However if due to some noise the trajectory is pushed away from the synchronization subspace and this happens in the region where the maximum local transversal Lyapunov exponents is positive then the trajectory will be repelled form the synchronization subspace and synchronization burst will occur. In order to avoid this possibility we must ensure that local transversal Lyapunov exponents are negative everywhere on the attractor. This condition is true for $d_1 \in [0{\cdot}73, 1{\cdot}27]$.

Based on the above discussion we propose to use the following criterion.

Criterion 1 (*synchronization of discrete-time systems*). *If for a long trajectory* (*covering densely the attractor*) *all of the local transversal Lyapunov exponents* $\lambda(\mathbf{x}, 1)$ *are smaller than zero then the chaotic systems will synchronize. In this case noise of a small amplitude will not influence the synchronization behaviour* (*synchronization is robust*).

In Table II we collect conditions for synchronization obtained using three different techniques. It is evident that the method based on local transversal Lyapunov exponents gives the condition stronger than the two other methods. Its main advantage is that local Lyapunov exponents can be easily computed and we do not need to find periodic orbits, which is necessary for the second method. It may seem that the condition obtained using the last technique is too restrictive. What we gain is the robustness of synchronization. If local transversal Lyapunov exponents are everywhere negative then we are sure that in the neighbourhood of the synchronization subspace the trajectory is attracted to this subspace.

5. SYNCHRONIZATION OF HYPERCHAOTIC CIRCUITS IN THE PRESENCE OF NOISE

As a second example let us consider a four-dimensional electronic circuit[13] defined by the following state equation:

$$C_1 \dot{v}_1 = f(v_2 - v_1) - i_1$$

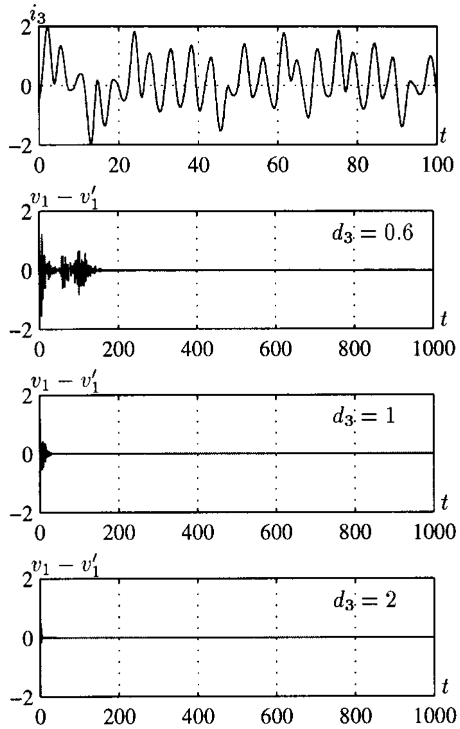

Figure 8. Synchronization of hyperchaotic circuits: (a) transmitted signal, The synchronization error $v_1 - v_1'$ for $d_3 = 0{\cdot}6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

$$C_2\dot{v}_2 = -f(v_2 - v_1) - i_2$$

$$L_1\dot{i}_1 = v_1 + Ri_1$$

$$L_2\dot{i}_2 = v_2 \tag{7a}$$

where f is given by

$$f(x) = m_0x + 0{\cdot}5(m_1 - m_0)(|x+1| - |x-1|) \tag{7b}$$

For parameters $C_1 = 0{\cdot}5$, $C_2 = 0{\cdot}05$, $L_1 = 1$, $L_2 = 2/3$, $R = 1$, $m_0 = 3$ and $m_1 = -0{\cdot}2$ system (7) has a hyperchaotic attractor with two positive Lyapunov exponents: $\lambda_1 \approx 0{\cdot}25$, $\lambda_2 \approx 0{\cdot}07$, $\lambda_3 = 0$ and $\lambda_4 \approx -53{\cdot}2$.

Two identical systems are connected by means of unidirectional coupling. Let us denote the state variables of the response system by v_1', v_2', i_1', i_2' and the “error” variables by $e_1 = v_1 - v_1'$, $e_2 = v_2 - v_2'$, $e_3 = i_1 - i_1'$ and $e_4 = i_2 - i_2'$. The unidirectional coupling is obtained by introducing the error feedback term d_ie_i in the ith equation. We consider the case when only $d_3 \neq 0$ and $d_1 = d_2 = d_4 = 0$.

In this section we investigate the influence of noise added to the transmitted signal on the synchronization behaviour.

In the first experiment we drive the response system with the driving signal not corrupted by noise (see Figure 8(a)). Synchronization error $v_1 - v_1'$ for different d_3 is shown in Figures 8(b)–8(d). For all values of coupling constant ($d_3 = 0{\cdot}6, 1, 2$) the synchronization takes place eventually. However for $d_3 = 0{\cdot}6$ the time necessary to obtain the synchronization is rather long ($t > 150$).

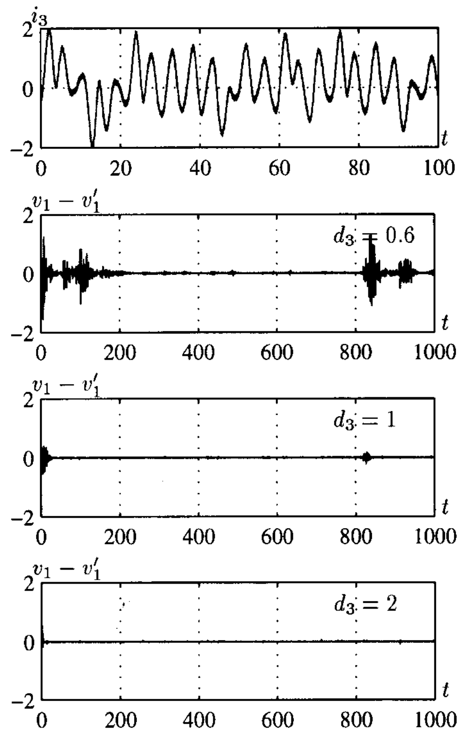

Figure 9. Synchronization of hyperchaotic circuits in the presence of channel noise: (a) transmitted signal with additive noise of amplitude 0·1, The synchronization error $v_1 - v_1'$ for $d_3 = 0{\cdot}6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

Next, we consider synchronization behaviour in a more realistic situation when additive noise with amplitude 0·1 is present in the channel. The driving signal (containing the noise) is shown in Figure 9(a). In Figures 9(b)–9(d) the synchronization error for three different values of d_3 is plotted. For $d_3 = 0{\cdot}6$ large desynchronization bursts are observed (see Figure 9(b)). For $d_3 = 1$ we observe almost perfect synchronization. Only one desynchronization burst with a small amplitude is visible. For $d_3 = 2$ the synchronization error remains small for the whole experiment.

Finally, we consider synchronization with the driving signal contaminated by additive noise of amplitude 1 (cf. Figure 10(a)). For weak coupling $d_3 = 0{\cdot}6$ one can see frequent desynchronization bursts of large amplitude (Figure 10(b)). For $d_3 = 1$ amplitude of bursts is lower and bursts are less frequent. For strong coupling $d_3 = 2$ non-coherent behaviour is quickly damped, although small bursts are still visible.

6. ANALYSIS OF SYNCHRONIZATION

In this section we analyse the synchronization using a Lyapunov function, global transversal Lyapunov exponents, transversal exponents of periodic orbits and local transversal Lyapunov exponents.

6.1. Global Lyapunov function method

Using the method of a global Lyapunov function one can show[14] that if $d_1 > -2m_1$, $d_2 > -2m_1$, $d_3 > R$ and $d_4 > 0$ then for all initial conditions the drive and response systems synchronize. As a Lyapunov function one may choose $V(e_1, e_2, e_3, e_4) = \frac{1}{2}(C_1 e_1^2 + C_2 e_2^2 + L_1 e_3^2 + L_2 e_4^2)$. Observe that in order to obtain negative

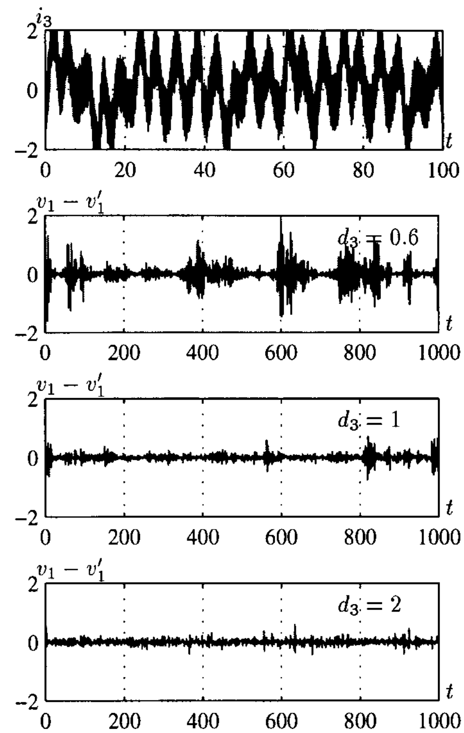

Figure 10. Synchronization of hyperchaotic circuits in the presence of channel noise: (a) transmitted signal with additive noise of amplitude 1, The synchronization error $v_1 - v_1'$ for $d_3 = 0{\cdot}6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

derivative of the Lyapunov function along the trajectory we have to use four positive coupling coefficients. We were not able to find a global Lyapunov function for the case of a single driving variable.

6.2. Global transversal Lyapunov exponents

In order to analyse the coupled system when only one coupling coefficient is positive we have computed global transversal Lyapunov exponents as a function of $d_3 \in [0, 2]$ ($d_1 = d_2 = d_4 = 0$). The results are shown in Figure 11. For the values of the coupling strength d_3 considered in the previous section global transversal Lyapunov exponents are all negative. For $d_3 = 0{\cdot}6$ the largest transversal Lyapunov exponents λ_{max} is only slightly negative $\lambda_{max} \approx -0{\cdot}02$. For $d_3 = 1$ and $d_3 = 2$ the largest transversal Lyapunov exponent is $\lambda_{max} \approx -0{\cdot}15$ and $-0{\cdot}37$, respectively. Thus one could expect synchronization. We have already seen that this is true but only when the driving signal is not contaminated by noise (cf. Figure 8). In order to explain the behaviour of the coupled systems in the presence of noise we will use two other methods.

6.3. Transversal Lyapunov exponents of unstable periodic orbits

In order to find short periodic orbits we have generated a trajectory of the uncoupled system on the plane $v_1 = 0$ consisting of 500,000 points. Then using the method of close returns[15] we have located several periodic orbits of the Poincaré map. Positions of these orbits were then sharpened by means of the Newton method. Their projections onto the plane (v_2, i_2) are shown in Figure 12(b). In this way, we have found approximate positions of 52 short periodic orbits for system (7) with length $T < 50$. See that the periodic orbits found do not visit some parts of the attractor (cf. Figure 12(a)).

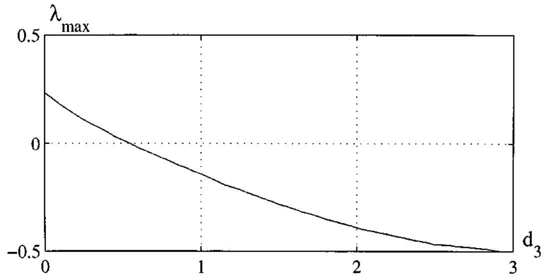

Figure 11. The maximum global transversal Lyapunov exponent of coupled circuits for different values of coupling coefficient d_3 ($d_1 = d_2 = d_4 = 0$)

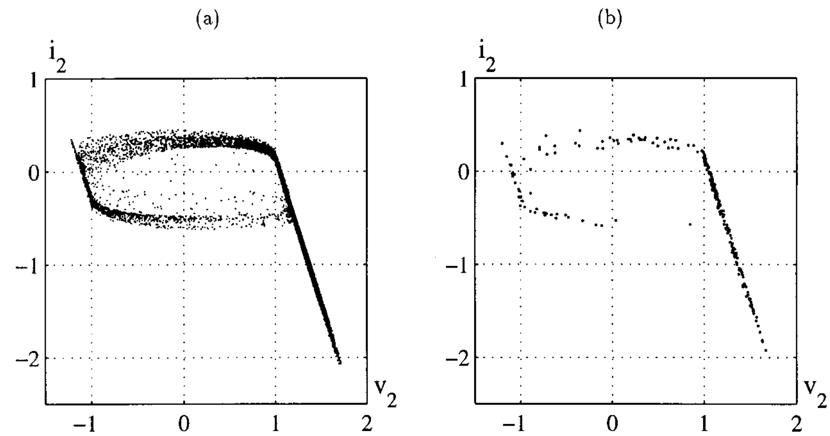

Figure 12. (a) projection onto the plane (v_2, i_2) of a trajectory of a Poincaré map with the transversal plane $v_1 = 0$ associated with flow (7), (b) positions of 52 short periodic orbits (248 points) of the system (7)

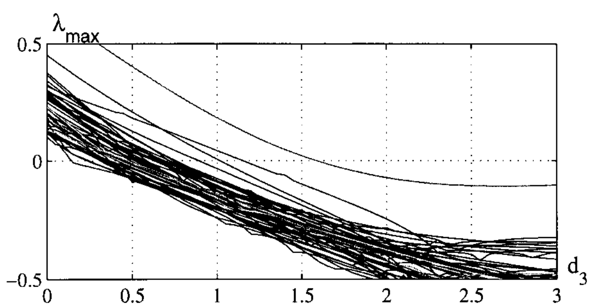

Figure 13. The maximum transversal global Lyapunov exponent for short cycles as a function of d_3 ($d_1 = d_2 = d_4 = 0$)

For each of these orbits transversal Lyapunov exponents of the coupled system have been computed for different values of coupling d_3. The results are shown in Figure 13, where for each orbit the maximum transversal Lyapunov exponent λ_{max} versus d_3 is plotted. The largest maximum transversal Lyapunov exponent is observed for the shortest orbit with period $T \approx 4{\cdot}86$.

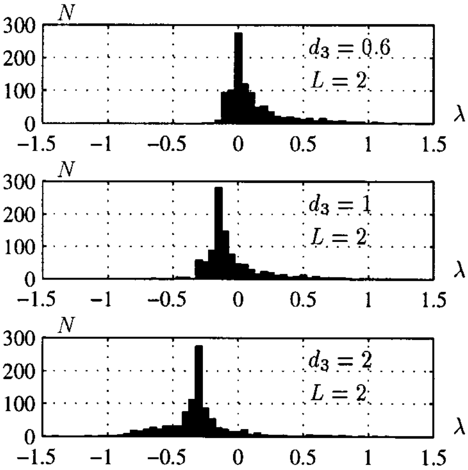

Figure 14. Histogram of maximum local transversal Lyapunov exponent for $L = 2$

6.4. Local transversal Lyapunov exponents

Now, we present the results of computation of local transversal Lyapunov exponents for three different values of L. Local Lyapunov exponents have been computed using the method proposed in Reference 10. First we choose the time $L = 2$. Local transversal Lyapunov exponents have been computed at 1000 points on the attractor. Then we have constructed histograms of the largest of them. In the construction of histograms we have used bins of the length 0·05. The results are presented in Figure 14. One can clearly see that the spectrum moves to negative values as the coupling coefficient is increased. For $d_3 = 0{\cdot}6$ the average value is positive $\lambda_{\text{aver}} \approx 0{\cdot}11$. For only 39% points local transversal Lyapunov exponents are all negative. This explains desynchronization bursts in Figure 9(b). For $d_3 = 2$ the average value is negative $\lambda_{\text{aver}} \approx -0{\cdot}31$ and more than 90% of the spectrum lies below zero. This is an indication that for this coupling the synchronized behaviour is more robust (cf. Figure 9(b)).

In Figure 15 we show histograms of maximum local transversal Lyapunov exponents computed for time $L = 6$. Spectrum is narrower (in comparison with $L = 2$) and shifted slightly towards negative values. For $d_3 = 0{\cdot}6$ the average value is still positive $\lambda_{\text{aver}} \approx 0{\cdot}012$. Now for approximately 57% points on the attractor all local transversal Lyapunov exponents are negative. For $d_3 = 2$ the average value is $\lambda_{\text{aver}} \approx -0{\cdot}35$ and 97% of the spectrum lies below zero.

Finally, we have computed local transversal Lyapunov exponents for $L = 20$ (see Figure 16). For $d_3 = 0{\cdot}6$ the average value is still slightly positive $\lambda_{\text{aver}} \approx 0{\cdot}004$. For strong coupling $d_3 = 2$ the whole spectrum is situated below zero and the average value is $\lambda_{\text{aver}} \approx -0{\cdot}37$. If we further increase the value of L we will observe a narrow peak at the value of the maximum global transversal Lyapunov exponent.

For investigations of synchronization one should consult the histogram of local transversal Lyapunov exponents. The size of the part of the histogram lying above zero tells us how frequently (with respect to the natural measure on the attractor) trajectories are repelled from the synchronization subspace.

In the last experiment we have computed maximum transversal local Lyapunov exponent over a long trajectory as a function of the coupling strength d_3 for different time L. The results are shown in Figure 17. See that the for $L = 0$ the maximum transversal local Lyapunov exponent practically does not depend on the coupling strength. When we increase L the maximum transversal local Lyapunov exponent decreases and approaches the maximum global transversal Lyapunov exponent ($L = \infty$).

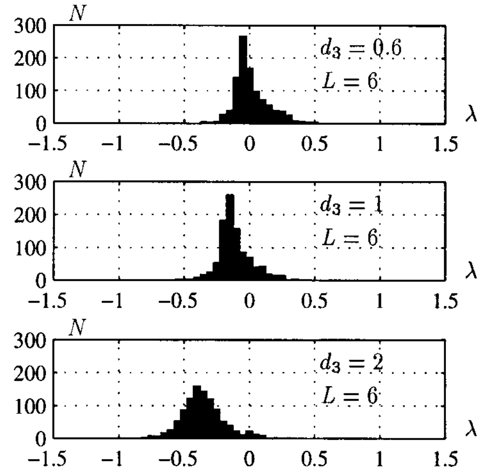

Figure 15. Histogram of maximum local transversal Lyapunov exponent for $L = 6$

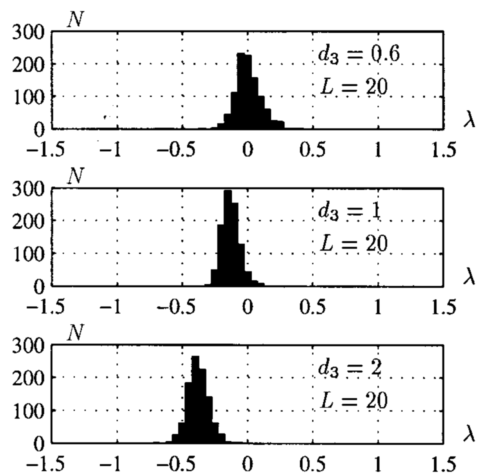

Figure 16. Histogram of maximum local transversal Lyapunov exponent for $L = 20$

In order to use local Lyapunov exponents for analysis of synchronization one should choose the time L properly. The time L cannot be too large as this would protect us from obtaining any information about the systems behaviour in short time (for $L \to \infty$ local exponents converge to global exponents).

This leads us in a natural way to the following synchronization criterion:[5]

Criterion 2 (*Synchronization of continuous-time systems*). *Synchronization is ensured if all local transversal Lyapunov exponents sampled over the entire attractor are negative in the limit* $L \to 0$.

Using the above criterion we have found the coupling values ensuring synchronization behaviour. For our system due to its piecewise linearity it is possible to derive these conditions by computing the eigenvalues of the response system in all linear regions.[5]

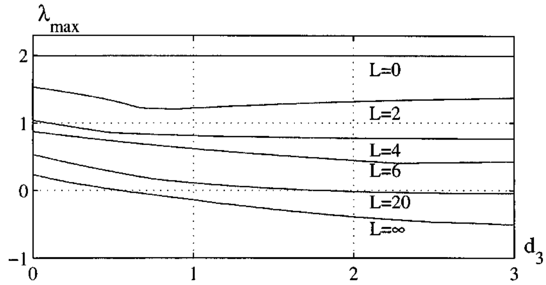

Figure 17. The maximum transversal local Lyapunov exponent of the four-dimensional synchronized circuit for different values of d_3, and different time delay L

With a single driving variable (one non-zero coupling coefficient) one cannot choose coupling values to fulfil the condition in Criterion 2. It becomes possible when we allow two non-zero coupling coefficients. The examples are: $d_1 = 0$, $d_2 = 4{\cdot}2$, $d_3 = 1{\cdot}7$, $d_4 = 0$ or $d_1 = 0$, $d_2 = 0$, $d_3 = 2{\cdot}5$, $d_4 = 4$. With equal coupling coefficients the condition in Criterion 2 holds for $d_1 = d_2 = d_3 = d_4 > 1{\cdot}995$.

We believe that Criterion 2 is too restrictive. One can obtain synchronization behaviour not sensitive to the channel noise for systems not fulfilling conditions in criterion 1. In particular criterion 1 does not hold for the coupling coefficients $d_1 = 0{\cdot}5$, $d_2 = 0{\cdot}5$, $d_3 = 1{\cdot}1$, $d_4 = 0{\cdot}1$ for which synchronization is ensured according to the existence of a global Lyapunov function. We suggest using local transversal Lyapunov exponents with time L separated from zero. In our opinion it is usually a good choice to use the following:

Criterion 3 (*Synchronization of continuous-time systems*) *Synchronization is robust if all local transversal Lyapunov exponents samples over the entire attractor are negative for L being approximately the 'natural' period of the system or the length of the shortest unstable periodic orbit embedded within the chaotic attractor.*

Obviously, the above heuristic criterion is not strictly a sufficient condition for robust synchronization. See that in our case none of the two above criteria is fulfilled for a single driving variable. Local transversal Lyapunov exponents are all negative for $L > 20$ and $d_3 > 2$ (cf. Figure 17) while the period of the shortest orbit is approximately $T = 4{\cdot}86$.

7. CONCLUSIONS

In this paper we have discussed the possibility of using local transversal Lyapunov exponents for characterization of synchronization of coupled chaotic systems. We have shown that there is a strong correlation between local transversal Lyapunov exponents and behaviour of coupled chaotic systems. We have shown that local transversal Lyapunov exponents could be effectively used for prediction of synchronization behaviour, especially in the presence of noise. Their main advantage is that they could be easily computed. It is not necessary to find periodic orbits and compute their transversal Lyapunov exponents to investigate synchronization properties. We have developed a criterion for synchronization based on local transversal Lyapunov exponents and confirmed that it is useful in analysis of synchronization of both discrete-time and continuous-time systems. For continuous-time systems we have also discussed the problem of choosing the time L for which local Lyapunov exponents are evaluated. It is clear that this technique is not limited to uni-directionally coupled systems considered in this paper and can be easily generalized for other coupling types.

ACKNOWLEDGEMENTS

This research was sponsored by the University of Mining and Metallurgy, Kraków, grant no.: 10.10.120.133.

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