Rigorous investigations of piecewise linear circuits*

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ABSTRACT

In this work we study methods for rigorous investigations of piecewise linear systems. The methods are based on the concept of Poincaré map. We describe methods how to find regions, where the Poincare map is well defined and continuous and how to apply interval Newton method for locating all low-period cycles in this region.

Keywords: piecewise linear system, Poincaré map, periodic orbit, interval arithmetic,

1 INTRODUCTION

In this paper we describe methods for rigorous investigations of piecewise linear systems. Rigor is achieved by employing interval arithmetic. In interval analysis [1, 4] intervals are used instead of real numbers. On the set of intervals operations are defined in such a way that the result of operation on intervals contains results of the corresponding real operation for all combinations of values from these intervals. When interval arithmetic is implemented on a computer, the rounding of every elementary operation is directed outwards. In this way we are sure that the result obtained encloses the true solution (together with the rounding error). Thus interval arithmetic overcome the usual problem of computer calculations — the existence of rounding errors makes it difficult or even impossible to find the relation between true solutions and approximations obtained using standard computational methods.

In the first step of analysis of piecewise linear systems we reduce the continuous time system to the discrete time using the concept of the Poincaré map. The Poincaré map is defined in a natural way by planes separating the regions of linearity. We study problems associated with this reduction, caused by existence of degeneracies like points for which the Poincaré map is not defined or is not continuous. We also discuss implications of existence of such points on the results of rigorous investigations.

As an example we consider a third order circuit, with three linear regions. For the Poincaré map defined by planes separating these regions we find subsets where the Poincaré

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map can be rigorously evaluated. We also find the invariant part of this subset and all short periodic orbits enclosed in this set.

Using the concept of generalized Poincaré map the problem of existence of periodic orbits of continuous systems is reduced to that of existence of periodic point of the Poincaré map.

2 GENERALIZED POINCARÉ MAP

Let us assume that the m–dimensional nonlinear system is described by the following set of ordinary differential equations: $\dot{x} = F(x)$, where $x \in \mathbb{R}^m$ and F is a continuous piecewise linear vector field. Let us denote by $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$ the hyperplanes separating the linear regions of F, and by Σ the union of the planes Σ_i. Let us denote by $\varphi(t, x)$ the trajectory of the system starting at x.

Definition 1 *The generalized Poincaré map $H : \Sigma \mapsto \Sigma$ is defined by $H(x) = \varphi(\tau(x), x)$, where $\tau(x)$ is the time needed for the trajectory $\varphi(t, x)$ to reach Σ.*

For the rigorous study of the Poincaré map we need a method for computation of an enclosure of the image of a given set of points from Σ. The enclosure can be computed rigorously if the Poincaré map is continuous on this set. Usually the Poincaré map is not defined everywhere. It is not defined for points, trajectories of which never come back to the set Σ. The regions where P is not defined can be easily found by computing the intersection of $Sigma$ with stable manifolds of all fixed points. Even if the Poincaré map is well defined it does not have to be continuous. The Poincaré map is not continuous at points $x \in \Sigma$ for which the flow is parallel to the Poincaré plane at x or at $H(x)$.

In a close neighborhood of such points rigorous evaluation of the Poincaré map becomes very difficult. Closer to the point, the sets which have to be studied become smaller, and the computation time becomes larger. Practically in the regions close to the discontinuity points the rigorous evaluation of the Poincaré map is not feasible. Understanding this problem and knowing the regions, where the Poincaré map is not continuous is the starting point of the rigorous study.

For evaluation of P in regions where P is continuous we use the analytical formulas for solutions of linear systems. In order to evaluate the generalized Poincaré map on a box $\mathbf{x} \in \Sigma$ we first find the return time for all points in $\mathbf{x}$, i.e. the interval $\tau(\mathbf{x}) \supset \{\tau(x) \colon x \in \mathbf{x}\}$ and then use analytic solutions to compute $\varphi(\tau(\mathbf{x}), \mathbf{x})$. $P(\mathbf{x})$ is enclosed in the intersection of $\varphi(\tau(\mathbf{x}), \mathbf{x})$ with Σ. The Jacobian of P at $\mathbf{x}$ can be expressed in terms of the return time $\tau(\mathbf{x})$ the start box $\mathbf{x}$ and the image $P(\mathbf{x})$ (for the details see [2]).

3 GLOBAL INTERVAL NEWTON METHOD

The next problem which is adressed in this study if that of existence of periodic orbits. Periodic orbits may be rigorously studied by means of interval Newton method. In the interval Newton method [4] in order to investigate the existence of zeros of a function $\mathbb{R}^m \ni x \mapsto f(x) \in \mathbb{R}^m$ in an m-dimensional interval $\mathbf{x}$ one evaluates the interval Newton operator

$$\mathbf{N}(\mathbf{x}) = x_0 - (\mathrm{D}f(\mathbf{x}))^{-1} f(x_0), \tag{1}$$

where $\mathrm{D}f(\mathbf{x})$ is the interval matrix containing all Jacobian matrices of f for $x \in \mathbf{x}$ and x_0 is an arbitrary point belonging to $\mathbf{x}$. If $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ then there exists exactly one zero of f in $\mathbf{x}$. If $\mathbf{N}(\mathbf{x}) \cap \mathbf{x} = \emptyset$ then there are no zeros of f in $\mathbf{x}$. Hence the interval Newton's method can be used to prove the existence and uniqueness of zeros.

In order to apply this method for proving the existence of periodic solutions for continuous–time systems one considers the Poincaré map P associated with the continuous–time flow. To prove the existence of a period–n orbit of P one applies the interval Newton method to the map $G : (\mathbb{R}^m)^n \mapsto (\mathbb{R}^m)^n$ defined by

$$[G(z)]_k = x_{(k+1) \bmod n} - P(x_k) \quad \text{for } 0 \le k < n,$$

where $z = (x_0, \ldots, x_{n-1})$. See that $G(z) = 0$ if and only if x_0 is a fixed point of H^n.

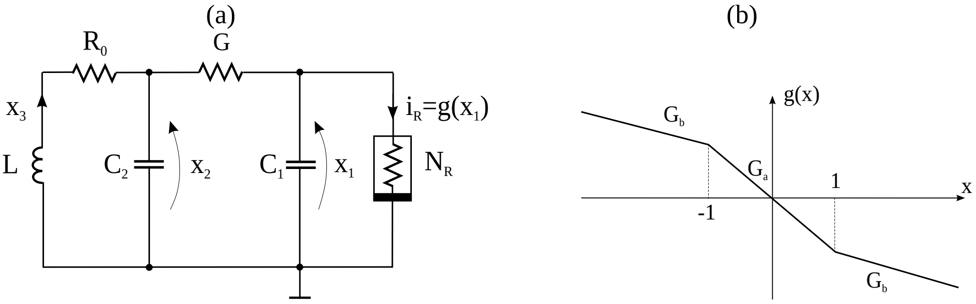

Figure 1: Electronic circuit (a) and characteristics of the nonlinear resistor (b)

In order to find all low period cycles in a given region one can use the combination of the interval Newton method and the generalized bisection [3]. Obviously this technique is limited to the subsets of Σ where the Poincaré map can be effectively evaluated. In the next section we show an example how to find all short periodic orbits satisfying theis condition.

4 PIECEWISE LINEAR CIRCUIT

As an example we consider a simple third–order electronic circuit (see Fig. 1(a)) — called the Chua's circuit — defined by the following set of ordinary differential equations:

$$\begin{aligned} C_1\dot{x}_1 &= G(x_2 - x_1) - g(x_1), \\ C_2\dot{x}_2 &= G(x_1 - x_2) + x_3, \\ L\dot{x}_3 &= -x_2 - R_0 x_3, \end{aligned} \tag{2}$$

where $g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)$ is a three-segment piecewise-linear function (see Fig. 1(b)). For parameter values $C_1 = 1$, $C_2 = 9.3515$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$ the system (3) exhibits chaotic behavior.

The state space $\mathbb{R}^3$ can be divided into three open regions $U_1 = \{x \in \mathbb{R}^3 \colon x_1 < -1\}$, $U_2 = \{x \colon |x_1| < 1\}$ and $U_3 = \{x \colon x_1 > 1\}$ separated by planes $\Sigma_1 = \{x \colon x_1 = -1\}$ and

$\Sigma_2 = \{x\colon x_1 = 1\}$. These planes define the Poincaré map. In the regions U_i the system is linear, the state equation can be written as: $\dot{x} = A_i(x - p_i)$, where A_i are matrices with real coefficients, and p_i are vectors, and the solution has the form $\varphi(t, x) = \mathrm{e}^{A_i t}(x - p_i) + p_i$.

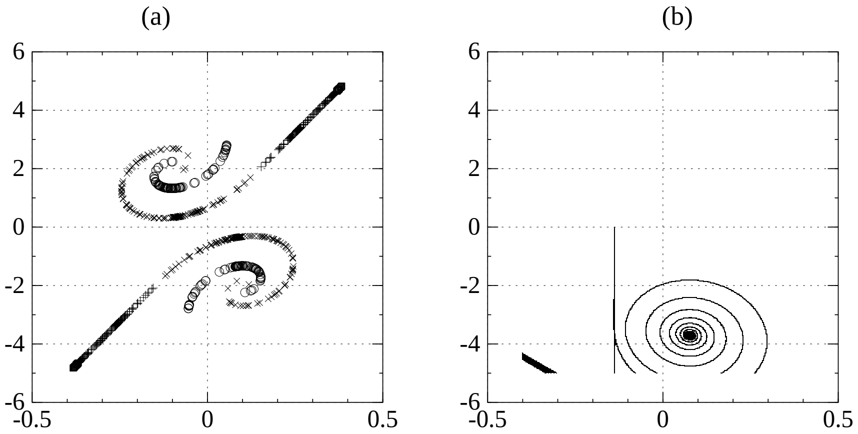

Figure 2: (a) trajectory of the Poincaré map, $x_3 > 0 \Rightarrow x \in \Sigma_1$, $x_3 < 0 \Rightarrow x \in \Sigma_2$ (b) regions for which the Poincaré map cannot be computed rigorously

In Fig. 2 we show the computer generated trajectory of P. The intersection of the attractor with the plane Σ_2 is contained in the halfplane $\{x_3 < 0\}$ and symmetrically intersection with the plane Σ_1 is contained in the halfplane $\{x_3 > 0\}$. Now we briefly describe the behavior of the Poincaré map on the attractor for $x \in \Sigma_2$ (the lower part of the Fig. 2). The left part of the plot (contained in the region $\{x_2 < -1.4, x_3 < 0\}$) is an almost straight line. Trajectories starting there enter the central linear region U_2. Points in the lower left corner (plotted with a $\Box$ symbol) reach the plane Σ_1 and their image forms a smaller spiral ($\bigcirc$) in the upper halfplane. The other points return back to the Σ_2 plane and their image forms the larger spiral ($\times$) in the lower halfplane. The right part of the plot is composed of two spirals ($\bigcirc$ and $\times$). Trajectories starting there enter the linear region U_3 and return back to Σ_2. Their images form the left part of the plot ($+$ or $\Box$).

For the Chua's circuit we start the analysis by finding the subsets of Σ, where the Poincaré map can be rigorously evaluated. The rectangle $\{1\} \times [-0.4, 0.3] \times [-5, 0] \subset \Sigma_2$ containing the numerically observed attractor is covered by boxes of the form $\{1\} \times [i/400, (i+1)/400] \times [j/40, (j+1)/40]$. The boxes for which we were not able to compute the image under the Poincaré map are plotted in Fig. 2(b).

There are three important parts of this set. Vertical line of boxes contains a set of points in the state space where the vector field is parallel to the plane Σ_2 ($\dot{x}_1 = 0$ and $x_1 = 1$, i.e., $x_2 = 1 + G_a/G \approx -0.1383$). The part in the lower left corner contains a curve in Σ_2 of points x for which the intersection of the trajectory with the plane Σ_1 is not transversal. This curve separates points for which $P(x) \in \Sigma_2$, from points for which $P(x) \in \Sigma_1$. The spiral on the right hand side contains points for which the intersection of the trajectory with the plane Σ_2 at $P(x)$ is not transversal (the spiral is the preimage of the vertival line in Σ_2 for

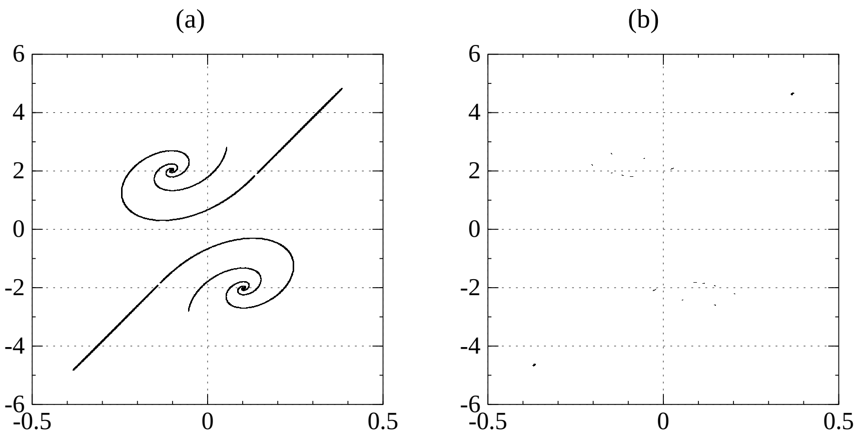

Figure 3: (a) covering of the computer generated trajectory of the Poincaré map by boxes, (b) boxes for which we were not able to evaluate the Poincaré map

which $\dot{x}_1 = 0$). The region in the center of the spiral contains the intersection of the stable manifold of the equilibrium enclosed in the region U_3 with the plane Σ_2. The Poincaré map is not defined at this point.

One can see that for the Chua's circuit all types of the phenomena leading to problems with rigorous evaluation of the Poincaré map occur. They limit the completness of the results which can be obtained by studying the system rigorously.

In the second step of our study we would like to find all short cycles of the Poincaré map. It is well known that interval Newton method and bisection technique can be successfully used for finding all low period cycles of discrete time systems. We cannot find all periodic orbits of the Poincaré map. It should however be possible to find all periodic orbits enclosed in the region for which the Poincaré map can be evaluated.

In order to reduce the time complexity of the problem we limit our investigations to the region containing the numerically observed attractor. To this end we cover the numerically generated trajectory of the Poincaré map by 15346 boxes of size 0.001×0.01 (see Fig. 3(a)). For 204 boxes in 16 connected components the computation of the Poincaré map was unsuccessful (see Fig. 3(b)). These boxes are located close to the intersection of the computer generated attractor with the set of points on Σ where the Poincaré map is not defined or is not continuous. The set V is defined as the union of the remaining boxes.

In Fig. 4(a) we show the image of the set V under the Poincaré map and in Fig. 4(b) we show the invariant part of the set V. The invariant part of V is found be removing boxes which has empty intersection with $P(V)$ and boxes whose image has empty intersection with V. The procedure is continued until no boxes can be removed.

Using the generalized bisection and the interval Newton method we have found all period-2 and period-4 orbits of the Poincaré map enclosed withing the invariant part found in the previous section. There is one period-2 orbit and five period-4 orbits found. They correspond to the periodic orbits of the continuous time system shown in Fig. 5.

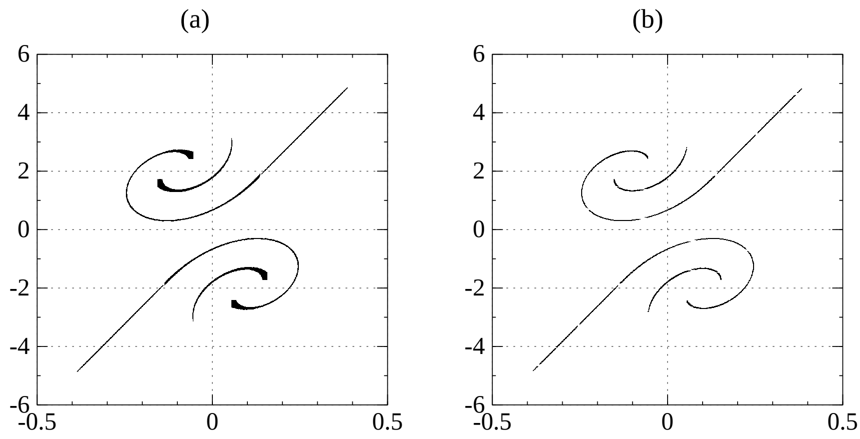

Figure 4: (a) image of the covering boxes under the Poincaré map, (b) invariant part

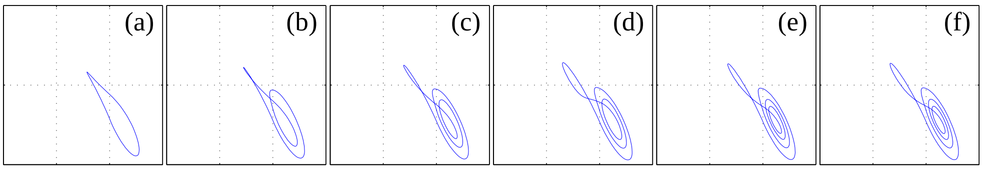

Figure 5: Periodic orbits for the Chua's circuit, projection to the plane (x_1, x_2), (a) period-2 orbit, (b)–(f) period-4 orbits of the Poincaré map

5 Conclusions

In this paper we have performed a rigorous study of the Poincaré map associated with the continuous time piecewise linear system. We have found the regions, where the Poincaré map is defined and continuous. We have also found all short cycles enclosed in this region.

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