BANG-BANG CONTROL OF CHAOTIC SYSTEMS

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Abstract

We describe a method for stabilisation of periodic orbits in chaotic systems for the case when the control parameter can assume two values only. We prove that the method can be successfully implemented if the unstable eigenvalue of the stabilized periodic orbit is smaller than 2 in absolute value. We present several simulation examples.

1 Introduction

There are several approaches to the problem of controlling chaotic dynamical systems [3]. In this paper we consider the approach, introduced in [4], based on the observation that the unstable periodic orbits fill densely the chaotic attractor. Ott, Grebogi and Yorke proposed a method (referred to as the OGY method) of controlling chaos by stabilizing one of unstable periodic orbits embedded in the chaotic attractor. For the method one needs one accessible system parameter, which can be perturbed within small interval around its nominal value.

In the first part we recall the original OGY control method. Then we describe the modification of this method, called the bang-bang control, for the case when the control parameter can assume two values only. Such type of control is much easier to implement in real systems than the standard method, where the control parameter is changed continuously over the interval. We prove a theorem determining the possibilities of the usage of the new method. Finally we show several simulation examples of application of this method.

2 Main results

We will describe our aproach for the case of stabilization of fixed points in two-dimensional discrete systems. The extension of the method for three-dimensional continuous-time systems and for periodic orbits is straightforward [1, 4].

Let $\mathbf{P}$ be a differentiable map of the real plane $\mathbb{R}^2$ into itself depending on the parameter p:

$$\mathbf{P} : \mathbb{R}^2 \times \mathbb{R} \ni (\xi, p) \longrightarrow \mathbf{P}(\xi, p) \in \mathbb{R}^2. \tag{1}$$

We assume that p can be changed around its nominal value p_0. Let us assume that ξ_F is a fixed point of the map $\mathbf{P}$, for $p = p_0$ (i.e. $\mathbf{P}(\xi_F, p_0) = \xi_F$). Let us assume that the first-order approximation of $\mathbf{P}$ in the neighbourhood of the point (ξ_F, p_0) is of the form:

$$\mathbf{P}(\xi, p) \approx \mathbf{P}(\xi_F, p_0) + \mathbf{A} \cdot (\xi - \xi_F) + \mathbf{w} \cdot (p - p_0), \tag{2}$$

where $\mathbf{A}$ is the Jacobian of $\mathbf{P}$ and $\mathbf{w} = \frac{\partial \mathbf{P}}{\partial p}(\xi_F, p_0)$.

Let $\mathbf{e}_s, \mathbf{e}_u$ be the stable and unstable eigenvectors of matrix $\mathbf{A}$, and λ_s, λ_u the corresponding eigenvalues. Let $\mathbf{f}_s$, $\mathbf{f}_u$ denote the contravariant eigenvectors defined by: $\mathbf{f}_s^T \mathbf{e}_u = \mathbf{f}_u^T \mathbf{e}_s = 0$ and $\mathbf{f}_s^T \mathbf{e}_s = \mathbf{f}_u^T \mathbf{e}_u = 1$.

Let us first briefly recall the OGY control method. In order to stabilize the fixed point ξ_F we monitor the system trajectory. If the distance between the trajectory and the stablized fixed point is small we modify the control parameter in order to push the trajectory in the next step onto the stable manifold of the fixed point. The control formula is given by:

$$p = p_0 - \frac{\lambda_u}{\mathbf{f}_u^T \mathbf{w}} \mathbf{f}_u^T (\xi - \xi_F). \tag{3}$$

We will modify the above method to make it usable also in the case when the control parameter can take two values only. Let us assume that the acceptable values of the control parameters are p_1 and p_2, $p_1 < p_2$. Let us define $p_0 = (p_1 + p_2)/2$, and $\Delta_p = (p_2 - p_1)/2$.

The bang-bang control idea is straightforward. If we want to stabilize the fixed point we compute p using formula (3) and we apply the parameter:

$$p' = p_0 + \Delta_p sgn(p - p_0) = \begin{cases} p_2 & if \quad p \geq p_0 \\ p_1 & if \quad p < p_0 \end{cases} .$$

Now we will discuss the problem, when is it possible to stabilize the fixed point using the described method. We will need the following Lemma:

Lemma 1 *Let* $f(\xi) = \mathbf{A}\xi + \mathbf{w} \cdot \Delta_p \cdot sgn\left(-\frac{\lambda_u \mathbf{f}_u^T}{\mathbf{f}_u^T \mathbf{w}} \cdot \xi\right)$*, where* $\mathbf{A}$ *is a diagonal matrix defined by* $\mathbf{A} = \begin{pmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{pmatrix}$*, and* $\mathbf{w} = (w_1, w_2)^T$*. Let us assume that* $w_1, w_2 \neq 0$*. Let* δ *be a positive real number. Let* C_δ*,* D_δ *be vertical and horizontal stripes defined by:* $C_\delta = \{\xi = (\xi_1, \xi_2)^T : |\xi_1| < \delta\}$*,* $D_\delta = \{\xi = (\xi_1, \xi_2)^T : |\xi_2| < \delta\}$*. Then:*

$$\frac{\delta(|\lambda_u| - 1)}{|w_1|} < \Delta_p < \frac{\delta}{|w_1|} \Leftrightarrow (f(C_\delta) \subset C_\delta) \tag{4}$$

$$\Delta_p < \frac{\delta(1 - |\lambda_s|)}{|w_2|} \Leftrightarrow (f(D_\delta) \subset D_\delta) \tag{5}$$

As an important consequence of the above Lemma we obtain the following theorem.

Theorem 2 *Let* $\mathbf{P}$ *be a map defined by (1). Let us assume that* $|\lambda_u| < 2$ *and* $\mathbf{f}^T\mathbf{w} \neq 0$*. Then for every* $\varepsilon > 0$ *there exists a neighbourhood* U *of the fixed point* ξ_F *included inside the ball* $B(\xi_F, \varepsilon)$ *and* Δ_p *such that*

$$\forall \xi \in U \qquad \mathbf{P}\left(\xi, p_0 + \Delta_p sgn\left(-\frac{\lambda_u \mathbf{f}_u^T}{\mathbf{f}_u^T \mathbf{w}} \cdot (\xi - \xi_F)\right)\right) \in U. \tag{6}$$

To explain the statement of the above theorem let us choose a positive real value ε and let us assume that the unstable eigenvalue of the fixed point is smaller than 2 in absolute value. Then we can find an arbitrary small neighbourhood U of ξ_F such that every trajectory which enters U will stay in it for ever due to the control action. As the chaotic system is ergodic and the fixed point belongs to the attractor the trajectory will fall into the neighbourhood U in finite time and will never escape from it. It follows from Lemma 1 that in such case for smaller Δ_p the stabilized trajectory remains closer to the fixed point. It is possible to keep the trajectory arbitrarily close to the fixed point. Hence the described method will work properly.

It is easy to extend this result for continuous-time systems. If we consider a transversal section intersecting the periodic orbit we can use Theorem 2 for the Poincaré map associated with the continuous flow.

We want to stress that these results are valid for ideal systems only. In the case of system with noise it is not possible to guarantee the condition (6) for arbitrarily small neighbourhoods. Because the control signal $\pm\Delta_p$ must exceed the level of noise the size δ of the neighbourhood U cannot be very small (δ depends linearly on Δ_p).

3 Simulation results

First we present the stabilization of a fixed point of Hénon map [2] defined by:

$$\mathbf{h}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}, \tag{7}$$

where $a = 1.4$, $b = 0.3$. The parameter a with the initial value $a_0 = 1.4$ is chosen as the control parameter.

The linearization of the Hénon map in the neighborhood of $(x_F, y_F)^T$, a_0 is of the form:

$$\mathbf{h}\left((x,y)^T\right) \approx (x_F, y_F)^T + \mathbf{A}(x - x_F, y - y_F)^T + \mathbf{w}(a - a_0), \tag{8}$$

where the fixed point $x_F = y_F = 0.5\left[(b-1) + \sqrt{(b-1)^2 + 4a_0}\right] \approx 0.8839$, Jacobian $\mathbf{A} = \begin{pmatrix} -2x_F & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1.7678 & 0.3 \\ 1 & 0 \end{pmatrix}$, vector $\mathbf{w} = (1,0)^T$ are computed analitically. Vector $\mathbf{c} = (c_1, c_2)^T$ necessary for the calculation of the control signal can be computed as:

$$\mathbf{c} = (c_1, c_2)^T = -\frac{\lambda_u}{\mathbf{f}_u^T \mathbf{w}} \mathbf{f}_u \approx (1.92374, -0.3)^T, \tag{9}$$

where the unstable eigenvalue $\lambda_u = -x_F - \sqrt{x_F^2 + b} \approx -1.92374$ and the contravariant vector $\mathbf{f}_u = (1 + \lambda_u^2)^{1/2}(\lambda_u - \lambda_s)^{-1}(1, -\lambda_s)^T$. In order to stabilize the fixed point (x_F, y_F) we apply the control parameter a:

$$a = a_0 + \Delta_p \mathrm{sgn}((x - x_F, y - y_F)\mathbf{c}). \tag{10}$$

According to Theorem 2 the successful control should be possible (absolute value of the unstable eigenvalue is smaller than 2). In Fig. 1 we show the results of stabilization of the fixed point of the Hénon map using the bang-bang control method. With this kind of control we cannot obtain the convergence of the trajectory towards the stabilized periodic orbit, but only remaining of the trajectory in a small neighbourhood of it. In the first experiment we have used $\Delta_p = 0.015$, and in the second one $\Delta_p = 0.005$. One can easily see, that in the second example, when the control signal is three times smaller, the trajectory finally stays closer to the stabilized fixed point. But in this case the transient time is usually longer. In our examples the trajectory starting from the same initial point has been stabilized in the first case after 250 iterations, and in the second one after approximately 1650 iterations.

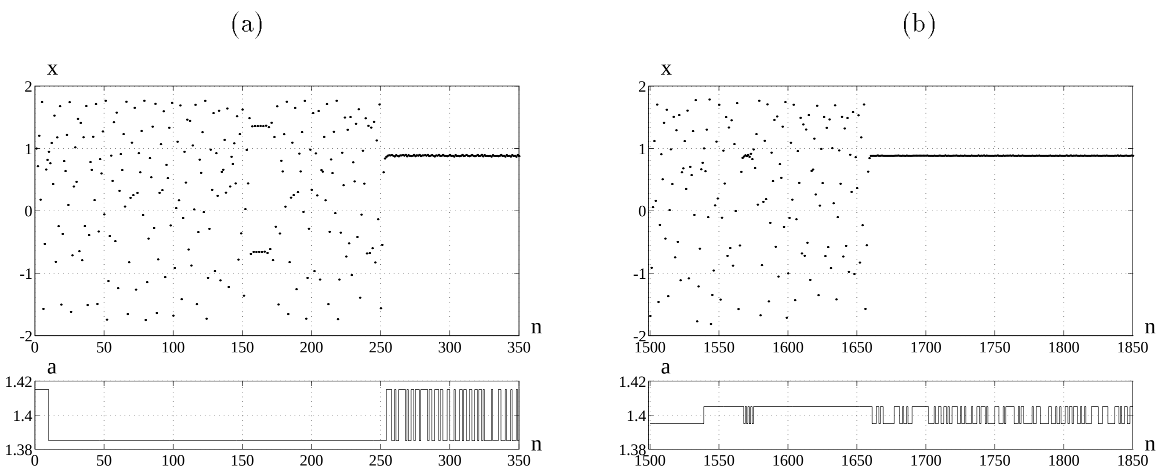

Figure 1: Stabilization of the fixed point of Hénon map using the bang-bang control method, (a) $\Delta_p = 0.015$, (b) $\Delta_p = 0.005$

As the second example we consider the Chua's circuit, with the dynamics described by the following state equation:

$$\begin{aligned} C_1 \frac{dx}{dt} &= -g(x) + z \\ C_2 \frac{dy}{dt} &= -Gy + z \\ L \frac{dz}{dt} &= -x - y - Rz \end{aligned} \tag{11}$$

where $g(\cdot)$ is a piece-wise linear function: $g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|)$. As the control parameter we used parameter C_1. We have tried to stabilize two periodic orbits, namely the short orbit

$\gamma_{1,0}$ with one winding around the nonzero equilibrium of the system and the longer symmetric one $\gamma_{2,2}$ with two windings around each of the nonzero equilibria of the Chua's system. This time periodic orbits and their Jacobians were found without knowledge of the state equation using the three dimensional time series obtained by numerical integration of the equation (11). We were not able to stabilize the orbit $\gamma_{1,0}$ using single-point control method (we computed the unstable eigenvalue of the Jacobian of the periodic orbit $\gamma_{1,0}$ to be approximately -2.73, which is greater than 2 in absolute value). The successful control is possible when the control parameter is modified three times per period. In Fig. 2a we present the result of stabilization of the orbit $\gamma_{1,0}$. In Fig. 2b one can see the successful stabilization of the orbit $\gamma_{2,2}$ using eight-point method.

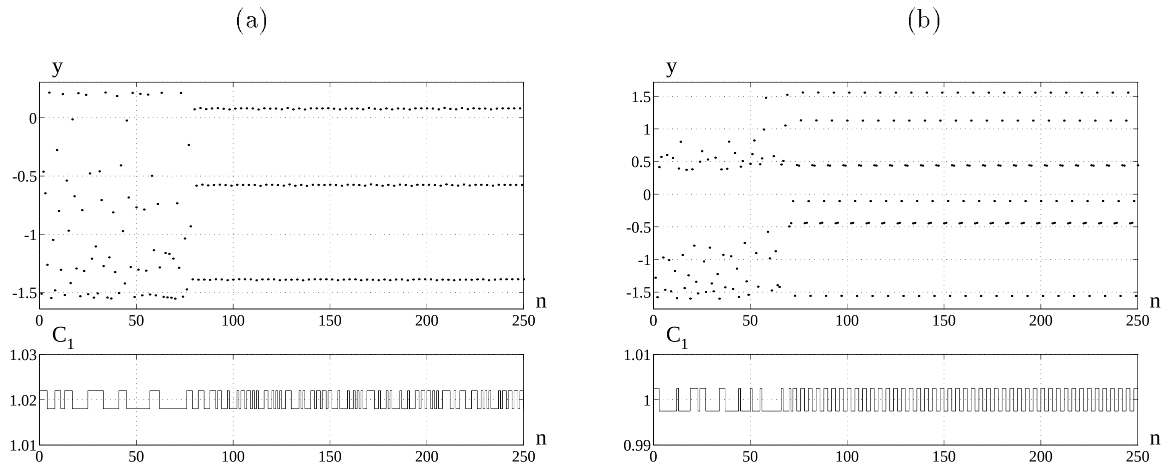

Figure 2: Stabilization of periodic orbits of Chua's circuit using the bang-bang control method, (a) periodic orbit $\gamma_{1,0}$, $n = 3$, $\Delta_p = 0.002$, (b) periodic orbit $\gamma_{2,2}$, $n = 8$, $\Delta_p = 0.0025$

4 Conclusions

We have presented the bang-bang control method, which can be used for the stabilization of periodic orbits in chaotic systems in the case when the control parameter can take two values only. We have proved the theorem, stating conditions under which the successful control is guaranteed. We tested the method presented in this paper on several examples. First we have considered the Hénon map. We have stabilized the fixed point using different values of the control signal Δ_p. We confirmed that for smaller Δ_p the trajectory remains closer to the periodic orbit, but usually the transient time before we can start the control is longer. The second example considered was the Chua's circuit. We were able to stabilize several periodic orbits in the double-scroll attractor using multipoint bang-bang method.

Acknowledgement

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