Analysis of Delayed Sliding Mode Control Systems Under Zero-order Holder Discretization

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***Abstract*—Zero-order holder discretization effects in sliding mode control systems with an input delay are studied. Stability conditions are formulated. Conditions for the existence of periodic solutions are derived and the existence of periodic steady states is investigated. The influence of the discretization step and the delay on the period and the amplitude of steady state oscillations is discussed. Structure of basins of attraction of periodic orbits with different switching patterns is studied. Simulation results are presented to illustrate various behaviors.**

I. Introduction

SLIDING mode control (SMC) is a simple and robust control method which applies a discontinuous control signal, forcing the system to slide along a prescribed sliding surface [1], [2]. Sliding mode exists when the switching frequency is infinite. Nowadays, industrial control systems are implemented digitally which leads to the emergence of chattering — the most unwanted side effect of SMC. Discretization of the SMC designed in the continuous-time domain results in irregular behaviors [3], [4]. Discretization effects in single-input and multi-input SMC systems have been studied substantially; e.g. in [5], [6] the Euler-discretization and ZOH-discretization effects were studied; in [7], it has been shown that the implicit discretization generates less chattering than the explicit discretization scheme; and in [8], general discretization issues were discussed. In these studies, it is assumed that the control signal can be applied instantaneously — without a time-delay caused by control signal transmission delay.

Taking into account a time-delay associated with the application of the control signal leads to a delayed SMC system. The dynamics of the second order SMC systems in presence of input delays is studied in [9], [10] where it is shown that the presence of input delay causes chattering. To overcome the time-delay effect in SMC, some sophisticated control approaches can be used, for example, in [11], a delay-dependent sufficient condition is obtained to guarantee the asymptotic stability of a time-delayed system, and in [12], such methodology was extended to stochastic systems. However, in these papers, discretization effects are not considered.

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In this work, we study combined effects of both factors — the input time-delay and discretization — on the behaviors of SMC systems. This is a critical issue in implementation as digitization is common in industrial controllers and time-delay is also a problem when control signals are transmitted either through wired or wireless transmissions (or even the Internet). We present analysis of zero-order holder (ZOH) discretization effects in SMC systems with a time-delay. Preliminary results concerning this system were presented in [13], where various dynamical phenomena including the coexistence of periodic steady states with different periods have been reported. In this paper, the full stability analysis is given and various periodic behaviors are analyzed in details, accompanied by various simulations.

The layout of this paper is as follows. The discretized delayed SMC system is defined in Section II. In Section III, formulas for the stroboscopic map describing discretized delayed SMC systems are derived. Stability conditions are formulated in Section IV, and conditions for the existence of periodic orbits with specified switching patterns are given in Section V. Stability analysis of example systems is carried out in Section VI showing the usefulness of methods presented in previous sections. For different parameter values all short periodic steady states are found. The influence of the delay and the discretization step on the period and amplitude of steady state oscillations is studied. The structure of basins of attraction of different periodic solutions is discussed.

II. Discretized delayed SMC systems

Let us consider a single input n–dimensional linear system

$$\dot{x} = Ax + bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $u \in \mathbb{R}$ is the control signal. Without loss of generality we can assume that the system is in the controllable canonical form, i.e. $\dot{x}_i = x_{i+1}$ for $i = 1, 2, \ldots, n-1$, and $\dot{x}_n = -(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + u$.

The most popular SMC method is the equivalent control based SMC of the form

$$u(x) = -(c^\top b)^{-1} c^\top Ax - (c^\top b)^{-1}\alpha \operatorname{sgn}(c^\top x), \tag{2}$$

where $\alpha > 0$, $\operatorname{sgn}(x) = 1$ for $x \geq 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and the vector $c = (c_1, c_2, \ldots, c_n)^\top \in \mathbb{R}^n$ is chosen in such a way that the polynomial $c_n\lambda^{n-1} + c_{n-1}\lambda^{n-2} + \cdots + c_2\lambda + c_1$ is Hurwitz, $c_n > 0$, and $c^\top b \neq 0$. Note that system (1) under control (2) is asymptotically stable and the switching manifold $s = c^\top x = 0$ is reached in finite-time, which can be easily shown by using the Lyapunov function $V = \frac{1}{2}s^2$ such that $\dot{V} = s\dot{s} = -\alpha|s|$.

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We assume that the control signal (2) is applied with the constant and time-invariant delay τ. The resulting *delayed SMC system* is given by

$$\dot{x}(t) = Ax(t) - (c^\top b)^{-1}\left(bc^\top Ax(t-\tau) + \alpha b\,\mathrm{sgn}(c^\top x(t-\tau))\right). \tag{3}$$

To eliminate ambiguity what is the delayed state $x(t-\tau)$ before the control action is started at $t = 0$, we assume that for $t \in [-\tau, 0]$ the system evolves freely (with $u = 0$). We further assume that (3) is implemented by a zero-order holder at discrete moments $t_k = kh$, with the discretization step $h > 0$, i.e. that for $t \in [t_k, t_{k+1})$ the control signal u is constant

$$u_k = -(c^\top b)^{-1}\left(c^\top Ax(t_k-\tau) + \alpha\,\mathrm{sgn}\left(c^\top x(t_k-\tau)\right)\right). \tag{4}$$

It follows that for $t \in [t_k, t_{k+1})$ we have $\dot{x}(t) = Ax(t) + bu_k$.

III. Stroboscopic map representation

In this section, formulas for the stroboscopic map representation of discretized delayed SMC systems are derived. The stroboscopic map is obtained from the continuous time system by sampling trajectories at constant time intervals. We will use the sampling interval equal to the discretization step h.

Let $x^{(k)}$ denote the solution of the control system (3) after time t_k, i.e., $x^{(k)} = x(t_k)$. Let us define $\hat{x}^{(k)} = ((x^{(k)})^\top, u_k)^\top \in \mathbb{R}^m$, where $m = n+1$. Given $\hat{x}^{(k)}$ one can compute the solution at the next discretization step using the formula

$$x^{(k+1)} = \Phi(h)x^{(k)} + \Gamma(h)u_k = \Psi(h)\hat{x}^{(k)}, \tag{5}$$

where $\Phi(h) = \mathrm{e}^{Ah}$, $\Gamma(h) = \int_0^h \mathrm{e}^{At}\,b\mathrm{d}t$, $\Psi(h) = (\Phi(h), \Gamma(h))$.

Let us denote by d the smallest integer number such that $\tau \le dh$, i.e. $d = -\lfloor -\tau/h \rfloor$ and let $\delta = dh - \tau$. Since $t_k - \tau = hk - (dh - \delta) = hk - \tau$ it follows that $x(t_k - \tau)$ is the solution obtained from the initial point $\hat{x}^{(k-d)} = ((x^{(k-d)})^\top, u_{k-d})^\top$ after time δ, i.e.

$$x(t_k - \tau) = \Phi(\delta)x^{(k-d)} + \Gamma(\delta)u_{k-d} = \Psi(\delta)\hat{x}^{(k-d)}. \tag{6}$$

From (4), (5), and (6) we obtain the update formula for the discrete system $\hat{x}^{(k)}$:

$$\hat{x}^{(k+1)} = \begin{pmatrix} \Psi(h)\hat{x}^{(k)} \\ -(c^\top b)^{-1}\left(c^\top A\Psi(\delta)\hat{x}^{(k-d+1)} + \alpha\,\mathrm{sgn}\left(c^\top\Psi(\delta)\hat{x}^{(k-d+1)}\right)\right)\end{pmatrix}. \tag{7}$$

Note that the state $\hat{x}^{(k+1)}$ depends on the previous state $\hat{x}^{(k)}$ and the delayed state $\hat{x}^{(k-d+1)}$. The extended set of coordinates $y^{(k)} = \left((\hat{x}^{(k-d+1)})^\top, \ldots, (\hat{x}^{(k-1)})^\top, (\hat{x}^{(k)})^\top\right)^\top$ is defined so that the next state of the system depends on the current state only.

The *stroboscopic map* $F\colon \mathbb{R}^{md} \mapsto \mathbb{R}^{md}$ defining behavior of the discretized delayed SMC system is given by

$$y^{(k+1)} = F(y^{(k)}) = Dy^{(k)} - \alpha s_k g, \tag{8}$$

where $s_k = \mathrm{sgn}(f^\top y^{(k)})$, $f = (c^\top\Psi(\delta), 0, \ldots, 0)^\top$, and $g = (0, \ldots, 0, (c^\top b)^{-1})^\top \in \mathbb{R}^{md}$. $s = (s_0, s_1, \ldots)$ is the *symbol sequence* corresponding to the initial condition $y^{(0)}$. $D \in \mathbb{R}^{(md)\times(md)}$ is a matrix given by

$$D = \begin{pmatrix} 0_{m\times m} & I_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times m} & 0_{m\times m} & I_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & I_{m\times m} \\ \Omega(\delta) & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & \Theta(h) \end{pmatrix},$$

$$\Omega(\delta) = \begin{pmatrix} 0_{n\times m} \\ -(c^\top b)^{-1}c^\top A\Psi(\delta)\end{pmatrix}, \quad \Theta(h) = \begin{pmatrix}\Psi(h) \\ 0_{1\times m}\end{pmatrix},$$

where $I_{k\times k} \in \mathbb{R}^{k\times k}$ denotes the identity matrix of dimension k, and $0_{k\times j} \in \mathbb{R}^{k\times j}$ denotes the zero matrix with k rows and j columns.

IV. Stability

First, we show that $\lambda = 1$ is an eigenvalue of D.

Lemma 1: Let us define $v = (\vartheta^\top, \vartheta^\top, \ldots, \vartheta^\top)^\top \in \mathbb{R}^{md}$ with $\vartheta = (\eta^\top, \xi)^\top \in \mathbb{R}^m$, where $0 \neq \eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ are such that

$$\left(A - (c^\top b)^{-1}bc^\top A\right)\eta = 0, \quad \xi = -(c^\top b)^{-1}c^\top A\eta. \tag{9}$$

Then v is an eigenvector of D corresponding to the eigenvalue $\lambda = 1$, i.e., $v \neq 0$, and $Dv = v$.

Proof: It is clear that if $\eta \neq 0$ then also $v \neq 0$. The first $d-1$ equations in $Dv = v$ are satisfied automatically ($I\vartheta = \vartheta$). The last equation is $\Omega(\delta)\vartheta + \Theta(h)\vartheta = \vartheta$. It can be rewritten as

$$\begin{pmatrix} (\Phi(h) - I)\eta + \Gamma(h)\xi \\ -(c^\top b)^{-1}c^\top A\Phi(\delta)\eta - (c^\top b)^{-1}c^\top A\Gamma(\delta)\xi - \xi \end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{10}$$

The first equation above holds. Indeed, for any h we have

$$\begin{aligned}(\Phi(h) - I)\eta + \Gamma(h)\xi &= (\mathrm{e}^{Ah} - I)\eta + \xi\int_0^h \mathrm{e}^{At}\,b\mathrm{d}t \\ &= \left(\sum_{k=1}^\infty \frac{1}{k!}A^k h^k\right)\eta + \xi\sum_{k=0}^\infty \frac{1}{k!}A^k\int_0^h t^k b\mathrm{d}t \\ &= \left(\sum_{k=1}^\infty \frac{1}{k!}A^{k-1}h^k\right)(A\eta + \xi b) = 0. \end{aligned} \tag{11}$$

The last equality follows from $A\eta + \xi b = 0$, which is a consequence of (9). The second equation in (10) can be rewritten as $-(c^\top b)^{-1}c^\top A((\Phi(\delta) - I)\eta - \Gamma(\delta)\xi) - \left((c^\top b)^{-1}c^\top A\eta + \xi\right) = 0$. From (11) applied to $h = \delta$ it follows that the first component vanishes. The second component is zero based on (9). This completes the proof. ■

The following results states that there exist η satisfying the assumptions of Lemma 1.

Lemma 2: There exist a nonzero vector $\eta \in \mathbb{R}^n$ such that $\left(A - (c^\top b)^{-1}bc^\top A\right)\eta = 0$.

Proof: The determinant of a rank-one perturbation of the identity matrix can be computed as $\det(I + uv^\top) = 1 + v^\top u$ (compare [14]). Using this formula for $u = (c^\top b)^{-1}b$, $v = c$ we obtain $\det\left(A - (c^\top b)^{-1}bc^\top A\right) = \det\left(I - (c^\top b)^{-1}bc^\top\right)\det(A) = \left(1 - (c^\top b)^{-1}c^\top b\right)\det(A) = 0$. The assertion follows. ■

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From the above two lemmas, it follows that $\lambda = 1$ is an eigenvalue of D and v defined in Lemma 1 is the corresponding eigenvector. If the system (1) is in the controllable canonical form ($b = (0, 0, \ldots, 0, 1)^\top \in \mathbb{R}^n$ and $A_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$, $A_{n,i} = -a_i$ for $i = 1, 2, \ldots, n$, and $A_{i,j} = 0$ otherwise) then one may choose $\eta = (1, 0, \ldots, 0)^\top \in \mathbb{R}^n$, $\xi = -(c^\top b)^{-1} c^\top A\eta = a_1$, and $\vartheta = (1, 0, \ldots, 0, a_1)^\top$.

Let us select a basis in $\mathbb{R}^{md}$ in such a way that the first element in this basis is v. Let P be the change of basis matrix. The new vector of state variables is $\hat{y} = P^{-1}y$. In the new basis the dynamical system (8) has the form

$$\hat{y}^{(k+1)} = \hat{D}\hat{y}^{(k)} - \alpha s_k \hat{g}, \quad s_k = \operatorname{sgn}(\hat{f}^\top \hat{y}^{(k)}), \tag{12}$$

where $\hat{D} = P^{-1}DP$, $\hat{g} = P^{-1}g$, $\hat{f}^\top = f^\top P$. Since the first element of the new basis is the eigenvector v of D corresponding to the eigenvalue $\lambda = 1$, it follows that we can decompose the system in the following way

$$z^{(k+1)} = z^{(k)} + e^\top w^{(k)} - \alpha s_k \gamma_z, \tag{13a}$$

$$w^{(k+1)} = Ew^{(k)} - \alpha s_k \gamma_w, \tag{13b}$$

where $\hat{y}^{(k)} = (z^{(k)}, (w^{(k)})^\top)^\top$, $s_k = \operatorname{sgn}(f_z z^{(k)} + f_w^\top w^{(k)})$, $z^{(k)}, f_z \in \mathbb{R}$, $w^{(k)}, e, f_w \in \mathbb{R}^{dm-1}$. The following result formulates conditions for the stability of the system (13).

Theorem 1: If $\|E\| < 1$ (where $\|E\|$ denotes an induced matrix norm of E) then

$$\limsup_{k\to\infty} \|w^{(k)}\| \le w_{\text{bound}} = \frac{\alpha\|\gamma_w\|}{1 - \|E\|}, \tag{14}$$

Additionally, if $\gamma_z f_z > 0$, $\alpha|\gamma_z| > \|e\| w_{\text{bound}}$ then

$$\limsup_{k\to\infty} |z^{(k)}| \le z_{\text{bound}} = (\|f_w\|/|f_z| + \|e\|)w_{\text{bound}} + \alpha|\gamma_z| \tag{15}$$

Proof: Let us first consider the subsystem (13b). Since $\|w^{(k+1)}\| = \|Ew^{(k)} - \alpha s_k \gamma_w\| \le \|E\| \cdot \|w^{(k)}\| + \alpha\|\gamma_w\|$ it is clear that if $\|w^{(k)}\| \le w_{\text{bound}}$ then also $\|w^{(k+1)}\| \le w_{\text{bound}}$, i.e. the set $\{\|w\| \le w_{\text{bound}}\}$ is positively invariant. Now, we show that as long as w is outside this set the norm $\|w\|$ decreases, i.e.

$$\|w^{(k)}\| \ge w_{\text{bound}}(1 + \varepsilon) \Rightarrow \|w^{(k+1)}\| \le \|w^{(k)}\| - \varepsilon\alpha\|\gamma_w\|. \tag{16}$$

From the assumption $\|w^{(k)}\| \ge w_{\text{bound}}(1 + \varepsilon)$ it follows that $\|E\| \cdot \|w^{(k)}\| + \alpha\|\gamma_w\| \le \|w^{(k)}\| - \alpha\|\gamma_w\|\varepsilon$. Since $\|w^{(k+1)}\| \le \|E\| \cdot \|w^{(k)}\| + \alpha\|\gamma_w\|$, the assertion in (16) follows.

Let us assume that the bound (14) is not true. From this hypothesis and (16) with $\varepsilon = 0$ it follows that there exists $\varepsilon > 0$ such that $\|w^{(k)}\| \ge w_{\text{bound}}(1 + \varepsilon)$ for all $k \ge 0$. From (16) it follows that $\|w^{(k)}\| \le \|w^{(0)}\| - \alpha\|\gamma_w\|\varepsilon k$ for all $k \ge 0$, and that there exists $k \ge 0$ such that $\|w^{(k)}\| < 0$ which contradicts properties of the norm. Thus, we have proved the bound (14).

Let us now consider the variable z of (13). First, we will show that in the steady state

$$|f_z z^{(k)}| \ge \|f_w\| w_{\text{bound}} \Rightarrow \operatorname{sgn}(z^{(k+1)} - z^{(k)}) = -\operatorname{sgn}(z^{(k)}). \tag{17}$$

Since $\|w^{(k)}\| \le w_{\text{bound}}$ (steady state assumption) and $\alpha|\gamma_z| > \|e\| w_{\text{bound}}$ we obtain $\|e^\top w^{(k)}\| \le \|e\| \cdot \|w^{(k)}\| \le \|e\| w_{\text{bound}} < \alpha|\gamma_z|$. It follows that the term $-\alpha s_k \gamma_z$ dominates in $e^\top w^{(k)} - \alpha s_k \gamma_z$ and $\operatorname{sgn}(z^{(k+1)} - z^{(k)}) = \operatorname{sgn}(e^\top w^{(k)} - \alpha s_k \gamma_z) = \operatorname{sgn}(-\alpha s_k \gamma_z)$. From $|f_z z^{(k)}| \ge \|f_w\| w_{\text{bound}}$ it follows that $s_k = \operatorname{sgn}(f_z z^{(k)} + f_w^\top w^{(k)}) = \operatorname{sgn}(f_z z^{(k)})$, and hence $\operatorname{sgn}(-\alpha s_k \gamma_z) = \operatorname{sgn}(-\alpha f_z^\top z^{(k)} \gamma_z) = -\operatorname{sgn}(z^{(k)})$, which completes the proof of (17). Property (17) means that if $|z^{(k)}|$ is sufficiently large then change in z is towards zero. In the steady state $|z^{(k+1)}| \le \|f_w\| w_{\text{bound}} |f_z|^{-1} + \|e^\top w^{(k)} - \alpha s_k \gamma_z\|$, and (15) follows. ■

The above theorem states that the steady state of the subsystem (13b) is bounded provided that the matrix norm of E is less than one. For sufficiently small discretization step all eigenvalues of E lie within the unit circle, and hence there exist a matrix norm with $\|E\| < 1$. Note that the solution is bounded no matter what is the symbol sequence s_k. Depending on the transformation P we may obtain different estimates on the steady states. If matrix E is (partially) diagonalized one may use (14) to obtain independent bounds for parts of w, which may lead to lower bounds for the original variables y. An example is given in Section VI.

The assumption $\alpha|\gamma_z| > \|e\| w_{\text{bound}}$ is technical and is needed to obtain bound (15) for a specified e. To avoid this assumption one may (partially) diagonalize matrix D in such a way that the vector e is zero, and then the only requirement is that γ_z is nonzero. Such a partial diagonalization is always possible if $\lambda = 1$ is an eigenvalue of D with multiplicity one, which is true for h sufficiently small. The assumption that f_z and γ_z are of the same sign is natural. In the opposite case for large $|z^{(k)}|$ we have $\operatorname{sgn}(-\alpha s_k \gamma_z) = \operatorname{sgn}(-\alpha f_z^\top z^{(k)} \gamma_z) = \operatorname{sgn}(z^{(k)})$. It follows that the term $-\alpha s_k \gamma_z$ in (13a) is of the same sign as $z^{(k)}$ and in consequence z diverges. Here, we neglect the term $e^\top w^{(k)}$ which has to be smaller than $|\alpha\gamma_z|$ if the control action is supposed to work properly.

V. Periodic orbits

In this section, conditions for the existence of periodic orbits with specific switching patterns are derived. We say that $y^{(0)}$ is a *period–p point* if $y^{(0)} = F^p(y^{(0)})$, and $p > 0$. We say that the periodic symbol sequence $s = (s_0, s_1, \ldots, s_{p-1})$ is *admissible* if there exist a period–p point $y^{(0)}$ with the corresponding sequence s. Note, that not all periodic orbits of (8) correspond to trajectories of (7). This is a consequence of the fact that when the control action is initiated at the moment $t = 0$ past symbols are not properly defined. We say that the periodic orbit $(y^{(0)}, y^{(1)}, \ldots, y^{(p-1)})$ is *reachable* if there exist an initial point x such that the trajectory $F^k(y)$ based at $y = (x^\top, 0, (\Phi(h)x)^\top, 0, (\Phi(h)^2 x)^\top, 0, \ldots, (\Phi(h)^{d-1}x)^\top, 0)^\top$, i.e. that the system evolves with zero control action for $t \in [-dh, 0]$, converges to this orbit.

Let us consider symbol sequences of length $p > 0$ for which the sum of symbols is zero, i.e. $\sum_{j=0}^{p-1} s_j = 0$. This is the only type of admissible sequences observed in simulations. The following result provides admissibility conditions for such sequences.

Theorem 2: Let us assume that $\sum_{j=0}^{p-1} s_j = 0$ and that the matrix $\sum_{j=0}^{p-1} D^j$ is invertible. The symbol sequence $s = (s_0, s_1, \ldots, s_{p-1})$ is admissible if and only if

$$\mu_{\min} = \max_{k:\, s_k=1} -\frac{f^\top y^{(k)}}{f^\top v} < \min_{k:\, s_k=-1} -\frac{f^\top y^{(k)}}{f^\top v} = \mu_{\max}, \tag{18}$$

where v is an eigenvector of D corresponding to the eigenvalue $\lambda = 1$, i.e. $Dv = v$, $f^\top v > 0$, and

$$y^{(0)} =\alpha\left(\sum_{j=0}^{p-1} D^j\right)^{-1}\left(\sum_{j=0}^{p-2} D^{p-2-j}\sum_{i=0}^{j} s_i\right)g, \tag{19}$$

$$y^{(k)} =F^k(y^{(0)}, s_0,\dots,, s_{k-1}) = D^k y^{(0)} - \alpha\left(\sum_{j=0}^{k-1} D^{k-1-j} s_j\right)g. \tag{20}$$

The corresponding periodic points are given by $y^{(0)}+\mu v$, where $\mu \in [\mu_{\min}, \mu_{\max})$.

Proof: Given the symbol sequence $s = (s_0, s_1, \dots)$ corresponding to the initial condition $y^{(0)}$, the iterate $y^{(k)}$ can be computed using (20). If $y^{(0)}$ is a period–p point then it satisfies

$$(I - D^p)y^{(0)} = -\alpha\left(\sum_{j=0}^{p-1} D^{p-1-j} s_j\right)g. \tag{21}$$

From Lemma 1 it follows that $\lambda = 1$ is an eigenvalue of D, and v defined in Lemma 1 is the corresponding eigenvector. Without loss of generality we may assume that $f^\top v > 0$. Let us note that $f^\top v = c^\top(\Phi(\delta), \Gamma(\delta))\vartheta = c^\top(\Phi(\delta)\eta + \Gamma(\delta)\xi) = c^\top\eta$, where the last equality follows from (11). If $f^\top v = c^\top\eta < 0$ we may choose $-\eta$ to define the eigenvector v). It follows that 0 is an eigenvalue of $(I - D^p)$. We will show that (21) has infinitely many solutions of the form $y^{(0)} + \mu v$, $\mu \in \mathbb{R}$. From the assumption $\sum_{j=0}^{p-1} s_j = 0$ it follows that $\sum_{j=0}^{p-1} D^{p-1-j}s_j = \sum_{j=0}^{p-1} D^{p-1-j}s_j - \sum_{j=0}^{p-1} I s_j = (D-I)\sum_{j=0}^{p-2}\left(D^{p-2-j}\sum_{i=0}^{j} s_i\right)$. Therefore, the term $I-D$ can be extracted from both sides of (21). Skipping this term yields $\left(\sum_{j=0}^{p-1} D^j\right)y^{(0)} = \alpha\left(\sum_{j=0}^{p-2} D^{p-2-j}\sum_{i=0}^{j} s_i\right)g$, which gives the solution $y^{(0)}$ defined in (19). The above solution is one of the solutions of (21). Other solutions can be expressed as the sum of (19) and an element of the null space of $I - D$, i.e. the solution set is $\{y^{(0)} + \mu v \colon \mu \in \mathbb{R}\}$.

Let us select $y = y^{(0)} + \mu v$. It is clear that $F(y, s_0) = F(y^{(0)} + \mu v, s_0) = Dy^{(0)} + D\mu v - \alpha s_0 g = y^{(1)} + \mu Dv$. Similarly, one can show that $F^k(y, s_0,\dots,, s_{k-1}) = y^{(k)} + \mu D^k v$. Since $Dv = v$ it follows that $F^k(y, s_0,\dots,, s_{k-1}) = y^{(k)} + \mu v$. The solution y is a periodic point of (8) if $s_k = \operatorname{sgn}(f^\top F^k(y)) = \operatorname{sgn}(f^\top(y^{(k)} + \mu v))$ for each $k = 0, 1, \dots p-1$, i.e. $\mu f^\top v \geq -f^\top y^{(k)}$ for $s_k = 1$ and $\mu f^\top v < -f^\top y^{(k)}$ for $s_k = -1$. Since $f^\top v > 0$, these conditions are equivalent to $\mu \in [\mu_{\min}, \mu_{\max})$, where $\mu_{\min}$ and $\mu_{\max}$ are defined in (18). ■

From this theorem it is clear that opposite to what is observed for non-discretized delayed SMC systems, none of the periodic orbits is asymptotically stable. Instead, periodic orbits are not isolated and initial conditions of orbits with the same symbol sequence form intervals.

To find all periodic orbits of a given length p with the sum of symbols equal to zero, one should consider all symbol sequences of length p of this type, for each sequence find $y^{(k)}$ using (19) and (20), evaluate $\mu_{\min}$ and $\mu_{\max}$ using formula (18), and verify whether $\mu_{\min} < \mu_{\max}$.

For symbol sequences of type $((+1)^m(-1)^m)$ with $m \geq 1$, which are most frequently observed in simulations, (19) can be simplified to $y^{(0)} = \alpha(I+D^m)^{-1}\left(I + D + D^2 + \cdots + D^{m-1}\right)g$. Let us consider the symbol sequence $s = (+1, -1)$. For discretized SMC systems without a delay, periodic orbits with this symbol sequence are the simplest and most desired steady state behaviours (compare [15]). In this case we have $y^{(0)} = \alpha(I+D)^{-1}g$, or in the original variables $\hat{x}^{(k-1)} = -\hat{x}^{(k)}$ for $-d+2 \leq k \leq 0$, $\Omega(\delta)\hat{x}^{(-d+1)} + (I + \Theta(h))\hat{x}^{(0)} = \alpha\beta$, where $\beta = (0,\dots,0,(c^\top b)^{-1})^\top \in \mathbb{R}^m$. Eliminating other variables yields $((-1)^{d+1}\Omega(\delta) + I + \Theta(h))\hat{x}^{(0)} = \alpha\beta$. For fixed h and δ the solution depends on the parity of d only. One can show that if the sequence $(+1, -1)$ is admissible for a given delay τ, it is also admissible for the delay $\tau + 2hk$, with $k = 1, 2, \dots$. However, orbits with $k > 0$ are not reachable. Similar results can be obtained for other symbol sequences of type $((+1)^m(-1)^m)$. Examples will be given in the next section.

VI. Simulation examples

As an example let us consider sliding mode control of a permanent-magnet synchronous motor (PMSM). The dynamics of PMSM can be represented as a two dimensional linear system of the form (1) with $a_1 = 0$, $a_2 > 0$, where x_1 is the angle, x_2 is the angle velocity and u represents the control torque (for details see [16]). We select $a_2 = 0.242$ and the sliding surface defined by $c = (c_1, c_2) = (1, 1)$. Trajectory of the sliding mode control system without a delay with the initial point $(x_1, x_2) = (0.01, 0.1)$ is plotted as a thick red line in Fig. 1(a). The sliding surface $c_1x_1 + c_2x_2 = 0$ is plotted as a dashed line. In this case the system trajectory reaches the sliding surface and then approaches the origin along the sliding surface. Trajectory of the delayed SMC system with the delay $\tau = 0.1$ is plotted as a narrow blue line in Fig. 1(a). In the case of infinite switching frequency the system trajectory converges to a stable periodic orbit with length 0.4148 and amplitude 0.108. Steady state does not depend on initial conditions. When the system is discretized various behaviors are observed. Two example trajectories observed for the discretization step $h = \tau = 0.1$ are shown in Fig. 1(b,c). Depending on the initial condition trajectories converge to an orbit with period $T = 0.6$ with amplitude $\|x\|_{\max} \approx 0.1644$ or to an orbit with period $T = 0.8$ and amplitude $\|x\|_{\max} \approx 0.2157$ which is approximately two times larger than in the non-discretized case. From this example we conclude that discretization of the delayed SMC system may further deteriorate performance of the control system. Discretization increases chattering amplitudes and the uniqueness property of the steady state is lost. This is also a consequence of Theorem 2 which states that none of periodic orbits is asymptotically stable.

Let us now study a more demanding control example with parameters $a = (a_1, a_2) = (-2, 2)$, $c = (1, 1)^\top$, $\alpha = 1$, $h = \tau = 0.02$ giving rise to more complicated dynamical phenomena. First, we study stability of the system. For the selected parameter values the extended system has the form (8) with

$$D = \begin{pmatrix} 1.0004 & 0.019608 & 0.00019737\\ 0.039216 & 0.96118 & 0.019608\\ -2 & 1 & 0 \end{pmatrix}, \tag{22}$$

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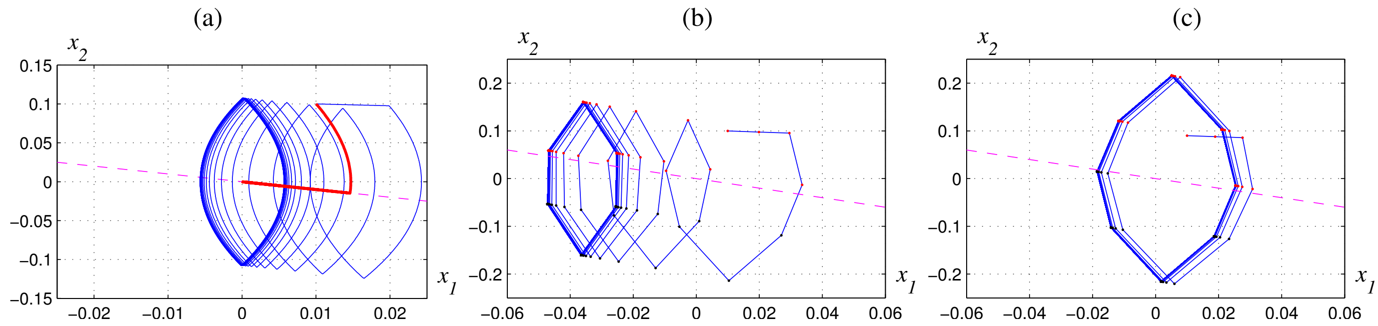

Fig. 1. Examples of steady states for the delayed SMC system with $a = (0, 0.242)$, $c = (1, 1)^\top$, $\alpha = 1$, $\tau = 0.1$, (a) infinite switching frequency with and without delay, $(x_1, x_2) = (0.01, 0.1)$, (b) $h = 0.1$, period–6 orbit, $(x_1, x_2) = (0.01, 0.1)$, (c) $h = 0.1$, period–8 orbit, $(x_1, x_2) = (0.01, 0.09)$.

$g = (0, 0, 1)^\top$, $f = (1, 1, 0)^\top$. The transformation matrix

$$P = \begin{pmatrix} 0.44721 & -0.30684 & -0.00019048 \\ 0 & 0.27965 & 0.01998 \\ -0.89443 & 0.90975 & -0.9998 \end{pmatrix} \quad (23)$$

diagonalizes D and for the new variable $\hat{y} = P^{-1}y$ we obtain the system (12) with $\hat{D} = \mathrm{diag}(1, 0.98194, -0.020365)$. The decomposed system has the form (13) with $E = \mathrm{diag}(0.98194, -0.020365)$, $\gamma_z = 0.04758$, $\gamma_w = (0.06996, -0.9791)^\top$, $f_z = 0.4472$, $f_w = (-0.02719, 0.0198)^\top$. Note that due to full diagonalization of D we have $e = 0$. Since $\gamma_z \neq 0$ the assumption $\alpha|\gamma_z| > \|e\| w_{\text{bound}}$ is automatically satisfied for arbitrary bound w_{bound}. Using Theorem 1 we obtain bounds $w_{\text{bound}} = 54.35$, $z_{\text{bound}} = 4.134$. It follows that for the variable $\hat{y}$ we have the bound $\|\hat{y}\| \leq \hat{y}_{\text{bound}} = \sqrt{w^2_{\text{bound}} + z^2_{\text{bound}}} = 54.6$, and for the original variables $\|y\| = \|P\hat{y}\| \leq \|P\| \cdot \|\hat{y}\| \leq \|P\| \hat{y}_{\text{bound}} = y_{\text{bound}} = 91.8$.

As it was mentioned before one may improve the bounds using the fact that the matrix $\hat{D}$ is strictly diagonal. Applying (14) separately to each variable w_k we obtain: $|w_1| \leq w_{1,\text{bound}} = \alpha\|\gamma_{w1}\|/(1 - \|E_{1,1}\|) = 3.8732$, $|w_2| \leq w_{2,\text{bound}} = \alpha\|\gamma_{w2}\|/(1 - \|E_{2,2}\|) = 0.99946$. The improved bound for z can be computed using the formula $z_{\text{bound}} = (|f_{w1}|/|f_z| + |e_1|)w_{1,\text{bound}} + (|f_{w2}|/|f_z| + |e_2|)w_{2,\text{bound}} + \alpha|\gamma_z| = 0.32726$. Having individual bounds $(\hat{y}_{1,\text{bound}}, \hat{y}_{2,\text{bound}}, \hat{y}_{3,\text{bound}}) = (0.32726, 3.8732, 0.99946)$ for elements of $\hat{y}$, we obtain bounds $y_{1,bound} = 1.335$, $y_{2,bound} = 1.1031$, and $y_{3,bound} = 4.8157$ using the formula $y_{k,bound} = |P_{k,1}|\hat{y}_{1,\text{bound}} + |P_{k,2}|\hat{y}_{2,\text{bound}} + |P_{k,2}|\hat{y}_{3,\text{bound}}$. $y_{1,bound}$ and $y_{2,bound}$ are bounds for the original variables $x_{1,2}$, while $y_{3,bound}$ is the bound for the control signal u.

To study the phenomenon of coexistence of periodic orbits with different switching patterns and investigate the influence of the value of the discretization step on the amplitude of chattering we have carried out the following computations. For 1500 discretization steps uniformly filling the interval $[0, 3\tau]$ using random initial conditions we have found the amplitude of the steady state. The results are shown in Fig. 2(a). Random changes in initial conditions introduce certain level of randomness into the plot, but several structures are clearly visible. Note that in all cases the amplitude of the steady state is not smaller than the amplitude of the steady state for non-discretized case ($h/\tau = 0$).

To further understand these results let us use Theorem 2 to find all short periodic orbits. For the same values of discretization steps as in the previous computations, we have found all periodic orbits with periods $p \leq 14$ and additionally all periodic orbits with symbol sequences of type $((+1)^m(-1)^m)$ with $m \leq 50$ (plotted in Fig. 2(b) using the blue color). One can see that dense patterns (lines) in Fig. 2(a) correspond to periodic orbits of type $((+1)^m(-1)^m)$, while clouds of points correspond to periodic orbits of different type. It follows that orbits of type $((+1)^m(-1)^m)$ are easier to find — they have larger basins of attraction. Also note that for small h ($h < 0.7\tau$) there are admissible periodic orbits with amplitudes smaller than the amplitude of the non-discretized system. Below, we show that these orbits are not reachable.

Fig. 2(c) shows amplitudes of admissible periodic orbits when the discretization step is fixed at $h = 0.02$ and the delay is varied. Note that the pattern corresponding to the shortest periodic orbit (with symbol sequence $(+1, -1)$) visible for $\tau/h < 0.5$ is repeated for τ/h close to $2, 4, 6, \ldots$. This is in agreement with a theoretical prediction that systems with delays τ and $\tau + 2kh$ have the same admissibility conditions for the symbol sequence $(+1, -1)$. Similar effects are observed for sequences of type $((+1)^m(-1)^m)$ with larger m. The difference is that the pattern repeats after $2mh$.

Let us now consider the case $h = 0.02$, $\tau = 4.1 \cdot h = 0.082$. For this case there are four admissible symbol sequences in Fig. 2(c): $(+1 - 1)$, $((+1)^2(-1)^2)$, $((+1)^8(-1)^8)$, $((+1)^9(-1)^9)$. Positions of the corresponding periodic orbits are shown in Fig. 3(a). Their amplitudes are 0.012713, 0.019889, 0.075984, and 0.085373, respectively. Fig. 3(b) shows basins of attraction of different periodic orbits. Basins were constructed by choosing 500×500 initial points in the area $[-0.02, 0.02] \times [-0.1, 0.1]$ and plotting with different colors initial points for which trajectories in the steady state have different symbol patterns. Basins of attraction of the period–16 and period–18 orbits are plotted in blue and red, respectively. Period–2 and period–4 orbits have empty basins of attraction — they are not reachable. Points converging to other periodic solutions that have long complex switching patterns and were not detected by the search procedure are plotted using the cyan color.

We have shown that if h is small then there exist short unreachable periodic orbits with amplitudes considerably smaller than amplitudes of reachable orbits and what is equivalent

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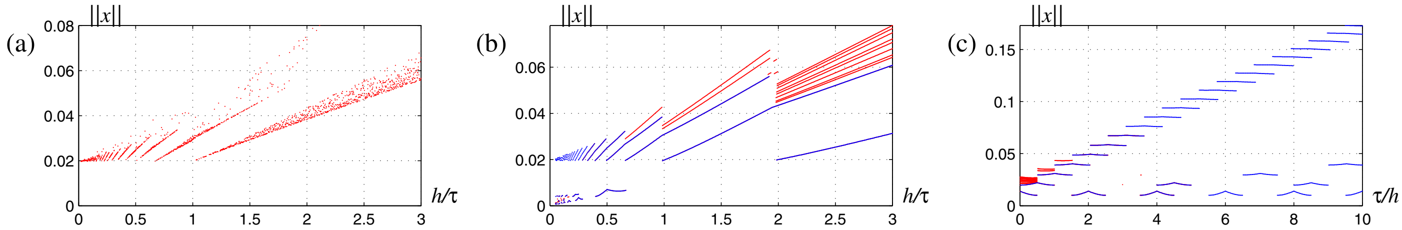

Fig. 2. Discretized delayed SMC system with $a = (-2, 2)$, $c = (1, 1)^\top$, $\alpha = 1$; (a) $\tau = 0.02$, amplitudes of steady state oscillations obtained from random initial conditions versus h/τ, (b) $\tau = 0.02$, amplitudes of short periodic orbits, (c) $h = 0.02$, amplitudes of admissible periodic steady states versus the delay τ/h.

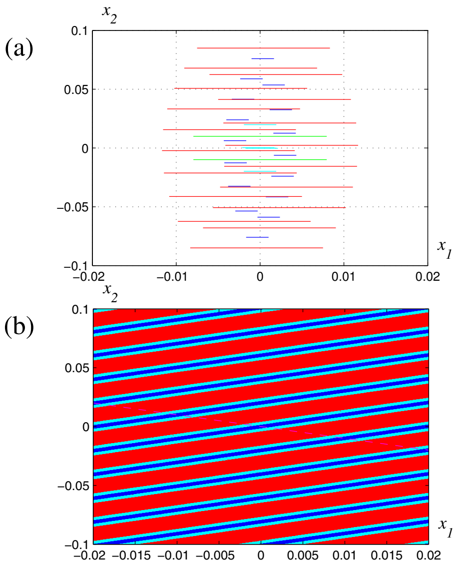

Fig. 3. Periodic steady states for delayed SMC systems with $a = (-2, 2)$, $c = (1, 1)^\top$, $h = 0.02$, $\tau = 4.1h$, $\alpha = 1$, (a) positions of period–16 orbits (blue), period–18 orbits (red), unreachable period–2 orbits (green), and unreachable period–4 orbits (cyan), (b) basins of attraction of period–16 (blue) and period–18 (red) orbits.

smaller than amplitudes of non-discretized SMC systems (compare Fig. 2(b,c)). This observation suggests that it may be possible to reduce chattering in the delayed SMC system with sufficiently small h by redesigning the control in such a way that trajectories reach one of small amplitude orbits.

VII. Conclusions

Discretization effects in delayed SMC systems have been studied. Analytic formula for the stroboscopic map describing the discretized system has been derived. Stability conditions have been given. Conditions for the existence of periodic orbits have been formulated. The influence of the discretization step and delay on the period and amplitude of steady state oscillations have been discussed. In future work, we plan to exploit the idea of redesigning the control strategy so that shorter orbits are reachable with the goal to reduce chattering amplitudes of the controlled system. We will also consider extending the results to multi-input systems. The challenge lies in applying the idea of a stroboscopic mapping to multi-input systems where multiple symbol sequences for multiple inputs (or sequences of vectors of symbols) need to be considered. Stability conditions and periodic orbits conditions would involve intertwined periodic multiple symbol sequences, which require new mathematical tools to analyze.

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