On zero-order holder discretization of delayed sliding mode control systems

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***Abstract*—Zero-order holder discretization effects in sliding mode control systems with an input delay are studied. Conditions for the existence of periodic solutions are derived and the existence of periodic steady states is investigated. The influence of the discretization step and the delay on the period and the amplitude of steady state oscillations is studied. Simulation results are presented to show the structure of basins of attraction of periodic orbits with different switching patterns.**

I. Introduction

Sliding mode control (SMC) is a simple and robust control method which applies a discontinuous control signal, forcing the system to slide along a prescribed sliding surface [1]. Sliding mode exists when the switching frequency is infinite. Nowadays, industrial control systems are implemented digitally which leads to the emergence of chattering — the most unwanted side effect of sliding mode control. Discretization of the SMC designed in the continuous-time domain results in irregular behaviors [2]–[4]. Discretization effects in single-input and multi-input SMC systems have been studied in [5]–[7]. In these studies, it is assumed that the control signal can be applied instantaneously — without a delay.

Taking into account a delay associated with the application of the control signal leads to a delayed SMC system. The dynamics of the SMC system in presence of input delays is studied in [8], [9]. It has been shown that the presence of input delay causes chattering. In these papers however, discretization effects are neglected.

In this work, we study combined effects of both factors — the input delay and discretization — on the behaviour of sliding mode control systems. We present analysis of zero-order holder discretization effects in sliding mode control systems with a delay. Results for the special case when the delay is a multiple of the discretization step were presented in [10], where various dynamical phenomena including the coexistence of periodic steady states with different periods have been reported. In this work, the general case, when the delay is not necessarily a multiple of the discretization step is investigated. We derive formulas for the stroboscopic map describing discretized delayed SMC systems. Conditions for the existence of periodic orbits are formulated. For different parameter values all short periodic steady states are found. The influence of the delay on the period and amplitude of steady state oscillations is studied. The structure of basins of attraction of different periodic solutions is discussed.

II. Discretized sliding mode control systems

Let us consider a single input n–dimensional linear system

$$\dot{x} = Ax + bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $u \in \mathbb{R}$ is the control signal. Without loss of generality we can assume that the system is in the controllable canonical form, i.e. $\dot{x}_k = x_{k+1}$ for $k = 1, 2, \dots, n-1$, and $\dot{x}_n = -(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + u$.

Let us consider the equivalent control based SMC:

$$u(x) = -(c^Tb)^{-1}c^TAx - (c^Tb)^{-1}\alpha\,\mathrm{sgn}(c^Tx), \tag{2}$$

where $\alpha > 0$, $\mathrm{sgn}(x) = 1$ for $x \geq 0$, $\mathrm{sgn}(x) = -1$ for $x < 0$, and the vector $c = (c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n$ is chosen in such a way that $c_n > 0$ and the polynomial $c_n\lambda^{n-1} + c_{n-1}\lambda^{n-2} + \cdots + c_2\lambda + c_1$ is Hurwitz.

We assume that the control signal (2) is applied with the delay τ. The resulting *delayed SMC system* is given by

$$\dot{x}(t) = Ax(t) - bc^TAx(t-\tau) - \alpha b\,\mathrm{sgn}(c^Tx(t-\tau)). \tag{3}$$

To eliminate ambiguity what is the delayed state $x(t-\tau)$ before the control action is started at $t = 0$, we assume that for $t \in [-\tau, 0]$ the system evolves freely (with $u = 0$). We further assume that (3) is implemented by a zero-order holder at discrete moments $t_k = kh$, with the discretization step $h > 0$, i.e. that for $t \in [t_k, t_{k+1})$ the control signal u_k is constant

$$u_k = -c^TAx(t_k-\tau) - \alpha\,\mathrm{sgn}\,(c^Tx(t_k-\tau))). \tag{4}$$

Let $x^{(k)}$ denote the solution of the control system (3) after time t_k, i.e., $x^{(k)} = x(t_k)$.

III. Stroboscopic map representation of discretized delayed SMC systems

In this section, we derive formulas for the stroboscopic map representation of discretized delayed SMC systems. The stroboscopic map is obtained from the continuous time system by sampling the phase plot at constant time intervals. Here, we will use the sampling interval equal to the discretization step h, and hence the stroboscopic map representation will allow us to compute $x^{(k)}$ based on previous samples $x^{(j)}$, $j < k$.

Let us denote by d the smallest integer number such that $\tau \leq dh$, i.e. $d = -\lfloor -\tau/h \rfloor$. The case when τ is a multiple of h

(i.e., τ/h is integer) was studied in [10]. Since for $t \in [t_k, t_{k+1})$ the control signal u_k is constant, it follows that

$$\dot{x}(t) = Ax(t) + bu_k, \qquad \text{for } t \in [t_k, t_{k+1}).$$

Let us define the state of the system as $z^{(k)} = ((x^{(k)})^T, u_k)^T \in \mathbb{R}^m$, where $m = n+1$. Given the state $z^{(k)}$ one can compute the solution at the next discretization step using the formula

$$x^{(k+1)} = \Phi(h)x^{(k)} + \Gamma(h)u_k = \Psi(h)z^{(k)}, \tag{5}$$

where $\Phi(h) = \mathrm{e}^{Ah}$, $\Gamma(h) = \int_0^h \mathrm{e}^{At}\, b\mathrm{d}t$, $\Psi(h) = (\Phi(h), \Gamma(h))$.

Let us denote $\delta = dh - \tau$. Since $t_k - \tau = hk - (dh-\delta) = hk - \tau$ it follows that $x(t_k - \tau)$ is the solution obtained from the initial point $z^{(k-d)} = ((x^{(k-d)})^T, u_{k-d})^T$ after time δ, i.e.

$$x(t_k - \tau) = \Phi(\delta)x^{(k-d)} + \Gamma(\delta)u_{k-d} = \Psi(\delta)z^{(k-d)}. \tag{6}$$

From (4), (5), and (6) we obtain the update formula for the discrete system $z^{(k)} = ((x^{(k)})^T, u_k)^T$:

$$x^{(k+1)} = \Psi(h)z^{(k)}, \tag{7a}$$

$$u_{k+1} = -c^T A\Psi(\delta)z^{(k-d+1)} - \alpha\, \mathrm{sgn}\,(c^T\Psi(\delta)z^{(k-d+1)}). \tag{7b}$$

A. Extended set of coordinates

Note that in (7) the state $z^{(k+1)} = (x^{(k+1)}, u_{k+1})$ depends on the previous state $z^{(k)}$ and the delayed state $z^{(k-d+1)}$. The extended set of coordinates $y^{(k)} = \left((z^{(k-d+1)})^T, \ldots, (z^{(k-1)})^T, (z^{(k)})^T\right)^T$ is defined so that the next state state of the system depends on the current state only.

The *stroboscopic map* $F\colon \mathbb{R}^{md} \mapsto \mathbb{R}^{md}$ defining behavior of the discretized delayed SMC system is given by

$$y^{(k+1)} = F(y^{(k)}) = Dy^{(k)} - \alpha s_k e, \tag{8}$$

where $s_k = \mathrm{sgn}(f^T y^{(k)})$, $f = (c^T\Psi(\delta), 0, \ldots, 0)^T$, $e = (0, \ldots, 0, 1)^T \in \mathbb{R}^{md}$. $s = (s_0, s_1, \ldots)$ is the *symbol sequence* corresponding to the initial condition $y^{(0)}$. $D \in \mathbb{R}^{(md)\times(md)}$ is a matrix given by

$$D = \begin{pmatrix} 0_{m\times m} & I_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times m} & 0_{m\times m} & I_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & I_{m\times m} \\ \Omega(\delta) & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & \Theta(h) \end{pmatrix},$$

where $I_{k\times k} \in \mathbb{R}^{k\times k}$ denotes the identity matrix, and $0_{k\times j} \in \mathbb{R}^{k\times j}$ denotes the zero matrix with k rows and j columns,

$$\Omega(\delta) = \begin{pmatrix} 0_{n\times m} \\ -c^T A\Psi(\delta) \end{pmatrix}, \quad \Theta(h) = \begin{pmatrix} \Psi(h) \\ 0_{1\times m} \end{pmatrix}.$$

B. Periodic orbits

We say that $y^{(0)}$ is a *period–p point* if $y^{(0)} = F^p(y^{(0)})$, and $p > 0$. We say that the periodic symbol sequence $s = (s_0, s_1, \ldots, s_{p-1})$ is *admissible* if there exist a period–p point $y^{(0)}$ with the corresponding sequence s. Note, that not all periodic orbits of (8) correspond to trajectories of (7). This is a consequence of the fact that when the control action is initiated at the moment $t = 0$ past symbols are not properly defined. We say that the periodic orbit $(y^{(0)}, y^{(1)}, \ldots, y^{(p-1)})$ is *reachable* if there exist an initial point x such that the trajectory $F^k(y)$ based at $y = (x^T, 0, (\Phi(h)x)^T, 0, (\Phi(h)^2x)^T, 0, \ldots, (\Phi(h)^{d-1}x)^T, 0)^T$ converges to this orbit.

Let us now consider symbol sequences for which the sum of symbols is zero, i.e. $\sum_{j=0}^{p-1} s_j = 0$. This is the only type of admissible sequences observed in simulations. The following result provides admissibility conditions for such sequences.

Theorem 1: Let us assume that $\sum_{j=0}^{p-1} s_j = 0$ and that the matrix $\sum_{j=0}^{p-1} D^j$ is invertible. The symbol sequence $s = (s_0, s_1, \ldots, s_{p-1})$ is admissible if and only if

$$\beta_{\min} = \max_{k\colon s_k=1} -\frac{f^T y^{(k)}}{f^T D^k v} < \min_{k\colon s_k=-1} -\frac{f^T y^{(k)}}{f^T D^k v} = \beta_{\max}, \tag{9}$$

where v is an eigenvector of D corresponding to the eigenvalue $\lambda = 1$, i.e. $Dv = v$,

$$y^{(0)} = \alpha\left(\sum_{j=0}^{p-1} D^j\right)^{-1}\left(\sum_{j=0}^{p-2} D^{p-2-j}\sum_{i=0}^{j} s_i\right)e, \tag{10}$$

$$y^{(k)} = F^k(y^{(0)}, s_0, \ldots, , s_{k-1}) = D^k y^{(0)} - \alpha\left(\sum_{j=0}^{k-1} D^{k-1-j} s_j\right)e. \tag{11}$$

The corresponding periodic points are given by $y^{(0)} + \beta v$, where $\beta \in [\beta_{\min}, \beta_{\max})$.

Proof: Given the symbol sequence $s = (s_0, s_1, \ldots)$ corresponding to the initial condition $y^{(0)}$, $y^{(k)}$ can be computed using (11). If $y^{(0)}$ is a period–p point then it satisfies

$$(I - D^p)y^{(0)} = -\alpha\left(\sum_{j=0}^{p-1} D^{p-1-j} s_j\right)e. \tag{12}$$

It can be shown that 1 is an eigenvalue of D. Let v be a corresponding eigenvector. It follows that 0 is an eigenvalue of $(I - D^p)$. We will show that (12) has infinitely many solutions of the form $y^{(0)} + \beta v$, $\beta \in \mathbb{R}$. From the assumption $\sum_{j=0}^{p-1} s_j = 0$ it follows that $\sum_{j=0}^{p-1} D^{p-1-j} s_j = \sum_{j=0}^{p-1} D^{p-1-j} s_j - \sum_{j=0}^{p-1} I s_j = (D - I)\sum_{j=0}^{p-2}\left(D^{p-2-j}\sum_{i=0}^{j} s_i\right)$. Therefore, the term $I - D$ can be extracted from both sides of (12). Skipping this term yields

$$\left(\sum_{j=0}^{p-1} D^j\right)y^{(0)} = \alpha\left(\sum_{j=0}^{p-2} D^{p-2-j}\sum_{i=0}^{j} s_i\right)e, \tag{13}$$

which gives the solution $y^{(0)}$ defined in (10). The above solution is one of the solutions of (12). Other solutions can be expressed as the sum of (10) and an arbitrary element of the null space of $I - D$, i.e. the solution set is $\{y^{(0)} + \beta v\colon \beta \in \mathbb{R}\}$.

Let us consider an arbitrary solution $y = y^{(0)} + \beta v$. It is clear that $F(y, s_0) = F(y^{(0)} + \beta v, s_0) = Dy^{(0)} + D\beta v - \alpha s_0 e = y^{(0)} + \beta Dv$. Similarly, one can show that $F^k(y, s_0, \ldots, , s_{k-1}) = y^{(0)} + \beta D^k v$. The solution y is a periodic point of (8) if $s_k = \mathrm{sgn}(f^T F^k(y)) = \mathrm{sgn}(f^T(y^{(k)} + \beta D^k v))$ for each $k = 0, 1, \ldots p-1$, i.e. $\beta f^T D^k v \geq -f^T y^{(k)}$ for $s_k = 1$ and $\beta f^T D^k v < -f^T y^{(k)}$ for $s_k = -1$, which is equivalent to the condition $\beta \in [\beta_{\min}, \beta_{\max})$. ■

Let us note that for symbol sequences of type $((+1)^m(-1)^m)$ with $m \geq 1$, which are most frequently observed in simulations, (10) can be simplified to (compare also [10])

$$y^{(0)} = \alpha(I + D^m)^{-1}\left(I + D + D^2 + \cdots + D^{m-1}\right)e. \tag{14}$$

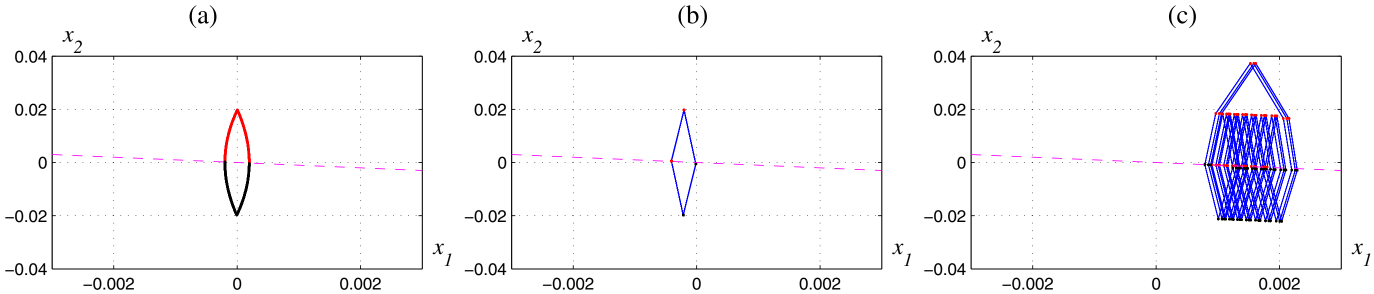

Fig. 1. Examples of steady states for the delayed SMC system with $a = (-2, 2)$, $c = (1, 1)^T$, $\alpha = 1$, $\tau = 0.02$, (a) infinite switching frequency, (b) $h = 0.02$, period–4 orbit, (c) $h = 0.02$, period–94 orbit

As a special case, let us consider the symbol sequence $s = (+1, -1)$. For discretized SMC systems without a delay, these are the most frequently observed steady state behaviours (compare [7]). From (14) for $m = 1$ we have $y^{(0)} = \alpha(I+D)^{-1}e$, or in the original variables $z^{(-d+1)}+z^{(-d+2)} = 0, \ldots z^{(-1)}+z^{(0)} = 0$, $\Omega(\delta)z^{(-d+1)} + (I + \Theta(h))z^{(0)} = \alpha\xi$, where $\xi = (0, \ldots, 0, 1)^T \in \mathbb{R}^m$. Eliminating other variables yields

$$((-1)^{d+1}\Omega(\delta) + I + \Theta(h))z^{(0)} = \alpha\xi.$$

For fixed h and δ the solution depends on the parity of d only. One can show that if the sequence $(+1, -1)$ is admissible for a given delay τ, it is also admissible for the delay $\tau + 2hk$, with $k = 1, 2, \ldots$. However, for $k > 0$ this orbit is not reachable. Similar results can be obtained for other symbol sequences of type $((+1)^m(-1)^m)$. Examples will be given in the next section.

IV. Simulation examples

As an example let us consider a two-dimensional delayed SMC system with the following parameters: $a = (a_1, a_2) = (-2, 2)$, $c = (1, 1)^T$, $\alpha = 1$, $\tau = 0.02$.

Steady state of the trajectory of the non-discretized (infinite switching frequency) delayed SMC system is shown in Fig. 1(a). This is a periodic orbit with period $T \approx 0.0793$ and amplitude $\|x\|_{\max} \approx 0.01962$. Steady state does not depend on initial conditions. When the system is discretized various behaviors are observed. Two examples of steady states existing for the discretization step $h = \tau = 0.02$ are shown in Fig. 1(b,c). Depending on the initial condition trajectories converge to an orbit with period $T = 0.08$ (period–4 of the discrete system) with amplitude $\|x\|_{\max} \approx 0.01977$ close to the amplitude observed for non-discretized system or to a much longer orbit with period $T = 1.88$ (period–94 of the discrete system) and amplitude $\|x\|_{\max} \approx 0.03733$ which is approximately two times larger than in the previous case.

To further study the phenomenon of coexistence of periodic orbits with different switching patterns and investigate the influence of the value of the discretization step on the amplitude of chattering we have carried out the following computations. For 1500 discretization steps uniformly filling the interval $[0, 3\tau]$ using random initial conditions we have found the amplitude of the steady state. The results are shown in Fig. 2(a). Random changes in initial conditions introduce certain level of randomness into the plot, but several structures are clearly visible. Note that in all cases the amplitude of the steady state is not smaller than the amplitude of the steady state for non-discretized case ($h/\tau = 0$).

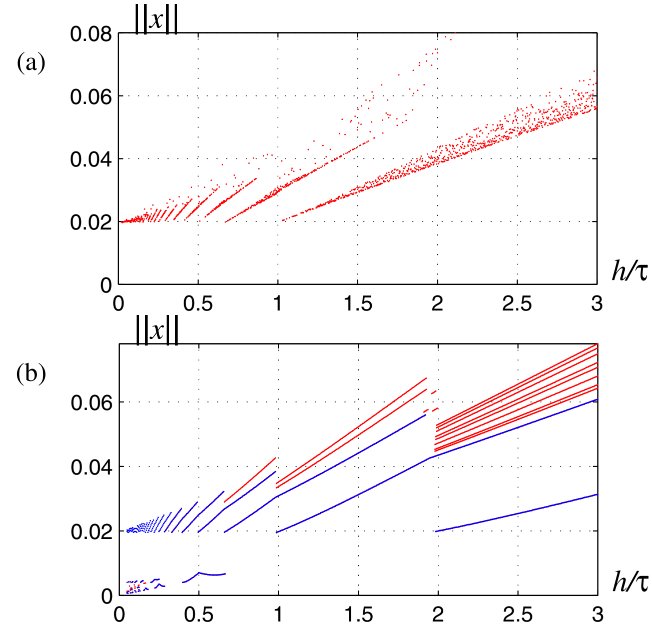

Fig. 2. Discretized delayed SMC system with $a = (-2, 2)$, $c = (1, 1)^T$, $\alpha = 1$, $\tau = 0.02$; (a) amplitudes of steady state oscillations obtained from random initial conditions versus h, (b) Amplitudes of short periodic orbits

To better understand these results let us use Theorem 1 to find all short periodic orbits. For the same values of discretization steps as in the previous computations, we have found all periodic orbits with periods $p \leq 14$ and additionally all periodic orbits with symbol sequences of type $((+1)^m(-1)^m)$ with $m \leq 50$ (plotted in Fig. 2(b) using the blue color). One can see that dense patterns (lines) in Fig. 2(a) correspond to periodic orbits of type $((+1)^m(-1)^m)$, while clouds of points correspond to periodic orbits of different type. It follows that orbits of type $((+1)^m(-1)^m)$ are easier to find — they have larger basins of attraction. Also note that for small h ($h < 0.7\tau$) there are admissible periodic orbits with amplitudes smaller than the amplitude of the non-discretized system. Below, we show that these orbits are not reachable.

Fig. 3 shows amplitudes of admissible periodic orbits when the discretization step is fixed at $h = 0.02$ and the delay is varied. Note that the pattern corresponding to the shortest periodic orbit (with symbol sequence $(+1,-1)$) visible for $\tau/h < 0.5$ is repeated for τ/h close to $2, 4, 6, \ldots$. This is in agreement with a theoretical prediction that systems with the delays τ and $\tau + 2h$ have the same admissibility conditions for the symbol sequence $(+1,-1)$. Similar effects are observed for sequences of type $((+1)^m(-1)^m)$ with larger m. The difference is that the pattern repeats after $2mh$.

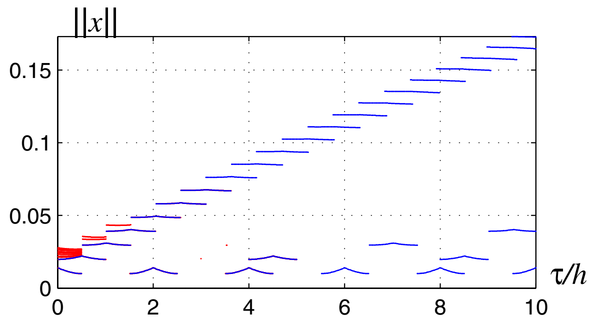

Fig. 3. Discretized delayed SMC system with $a = (-2, 2)$, $c = (1, 1)^T$, $\alpha = 1$, $h = 0.02$, amplitudes of admissible periodic steady states versus the delay τ

Let us now consider the case $h = 0.02$, $\tau = 4.1 \cdot h = 0.082$. For this case there are four admissible symbol sequences in Fig. 3: $(+1 - 1)$, $((+1)^2(-1)^2)$, $((+1)^8(-1)^8)$, $((+1)^9(-1)^9)$. Positions of the corresponding periodic orbits are shown in Fig. 4(a). Their amplitudes are 0.012713, 0.019889, 0.075984, and 0.085373, respectively. Fig. 4 shows basins of attraction of different periodic orbits. Basins were constructed by choosing 500×500 initial points in the area $[-0.02, 0.02] \times [-0.1, 0.1]$ and plotting with different colors initial points for which trajectories in a steady state have different symbol patterns. Basin of attraction of the period–16 and period–18 orbits are plotted using blue and reds color, respectively. Period–2 and period–4 orbits have empty basins of attraction — they are not reachable. Note that there are other periodic solutions. Points converging to them are plotted using the cyan color. These periodic solutions have long complex switching patterns and were not detected by the search procedure.

We have shown that when h is small then there exist short unreachable periodic orbits with amplitudes considerably smaller than amplitudes of reachable orbits and what is equivalent smaller than amplitudes of non-discretized SMC systems (compare Fig. 2(b) and Fig 3). This observation suggests that it may be possible to reduce chattering in the delayed SMC system with sufficiently small h by redesigning the control in such a way that trajectories reach one of small amplitude orbits. This may be equivalent to the predictor based control approach.

V. Conclusion

Discretization effects in delayed SMC systems have been studied. Analytical formula for the stroboscopic map describing the discretized system have been derived. Conditions for the existence of periodic orbits have been formulated. The influence of the discretization step and delay on the period and amplitude of steady state oscillations have been discussed.

Acknowledgment

This work was supported in part by the AGH University of Science and Technology, grant no. 11.11.120.343, and under the Australian Research Councils Discovery Grant Scheme, project no. DP130104765.

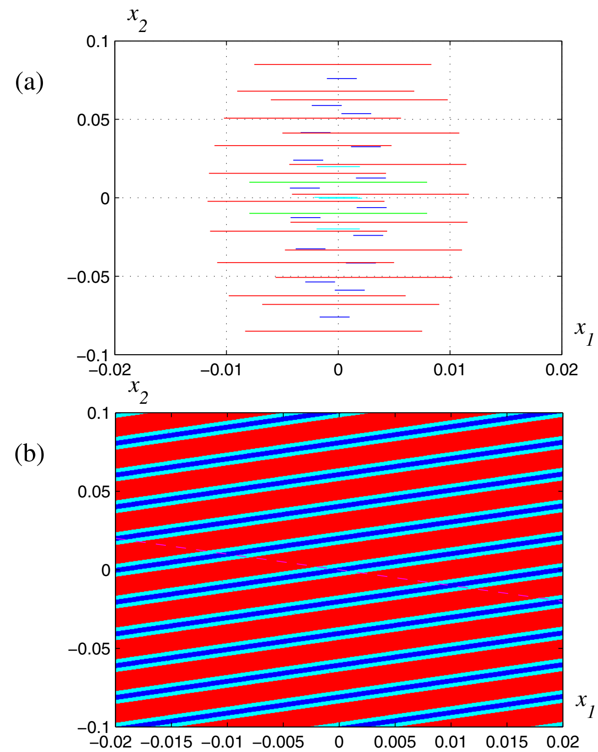

Fig. 4. Periodic steady states for delayed SMC systems with $a = (-2, 2)$, $c = (1, 1)^T$, $h = 0.02$, $\tau = 4.1h$, $\alpha = 1$, (a) positions of period–16 orbits (blue), period–18 orbits (red), unreachable period–2 orbits (green), and unreachable period–4 orbits (cyan), (b) basins of attraction of period–16 (blue) and period–18 (red) orbits

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