Study of periodic solutions in discretized two-dimensional sliding-mode control systems

Zbigniew Galias, *Member, IEEE,* Xinghuo Yu, *Fellow, IEEE*

***Abstract*—The existence of periodic solutions in discretized two-dimensional equivalent control based sliding mode control systems is studied. Admissibility conditions for the existence of periodic solutions with specific symbol sequences are derived, and admissibility regions for short periodic sequences are found. It is shown that for certain parameter values there exist arbitrarily long periodic orbits for arbitrarily small discretization steps. Theoretical results are illustrated with simulation examples.**

***Index Terms*—sliding mode control, discretization, periodic solutions, chattering.**

I. Introduction

SLIDING mode control (SMC) modifies dynamics of a system by applying a high-frequency switching control [1]. A discontinuous control law is designed in such a way that in a vicinity of the prescribed switching manifold trajectories move toward the manifold. In the sliding mode a trajectory remains on the sliding surface for all time. This requires in general an infinite switching frequency. In practical applications, as a result of inherent delays of applying the discontinuous control signal a trajectory after intersecting the switching manifold leaves this manifold. This can lead to chattering, loss of energy and other unwanted phenomena like excitation of undesired dynamics. Therefore, study of discretization effect of SMC systems is of high practical importance. It helps to evaluate quantitatively discretization behaviors and develop preventive measures of unwanted phenomena.

It has been reported that discretization of SMC may cause irregular behaviors such as periodic trajectories and strange attractors [2], [3], [4], [5]. In [6], it was shown that for the Euler's discretization, for sufficiently small h, each trajectory approaches the sliding manifold and converges to a period–2 cycle with an amplitude proportional to h. In [2], interesting phenomena including the existence of periodic solutions with different periods were observed in an equivalent control based SMC system discretized using zero-order holder (ZOH).

Dynamical properties of discrete systems arising by ZOH implementation of the SMC control strategy for two-dimensional systems have been analysed in [7]. Accurate bounds for the chattering amplitude were derived, and it was shown that for sufficiently small discretization steps there cannot be more than two iterations without a switching taking place.

In this work we analyse this discrete system in terms of the existence of periodic solutions. We show that periodic orbits are fully characterized by their switching patterns. Admissibility conditions for the existence of periodic orbits with a given symbol sequence are formulated. This result is used to locate in the parameter space regions of existence of periodic orbits with specific switching patterns. Certain classes of periodic orbits with arbitrary length are considered. It is shown that under certain assumptions some of them exist for arbitrarily small discretization steps. The existence of complex switching patterns is confirmed in simulations.

II. ZOH discretization of the 2D equivalent control SMC system

Let us consider a single input two-dimensional linear system in the controllable canonical form

$$\dot{x} = Ax + bu = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad (1)$$

and the equivalent control based SMC: $u(x) = -(c^T b)^{-1} c^T Ax(t) - (c^T b)^{-1} \alpha \operatorname{sgn}(c^T x)$, where $c = (c_1, 1)^T$, $c_1 > 0$, $\alpha > 0$, $\operatorname{sgn}(x) = 1$ for $x \geq 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. The *switching function* $c^T x$ defines the *sliding manifold* $L = \{x \colon c^T x = 0\}$.

The sliding mode control system is described by

$$\dot{x} = Ax + b(-c^T Ax - \alpha \operatorname{sgn} c^T x). \qquad (2)$$

Assume that the control system (2) is implemented by a zero-order holder at $t_k = kh$, with the discretization step $h > 0$. The control signal $u_k = -c^T Ax^{(k)} - \alpha \operatorname{sgn}\left(cx^{(k)}\right)$ is constant for $t \in [t_k, t_{k+1}]$. Let us denote $x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}\right)^T = x(t_k)$.

It can be shown [7] that the update equation is

$$\left(x_1^{(k+1)}, x_2^{(k+1)}\right) = \left(x_1^{(k)} + vx_2^{(k)} - \alpha\gamma_1 r_k, dx_2^{(k)} - \alpha\gamma_2 r_k\right),$$

where $(\gamma_1, \gamma_2)^T = \int_0^h e^{A\tau}\, b d\tau$, $v = \gamma_2 - \gamma_1(c_1 - a_2)$, $d = 1 - a_1\gamma_1 - c_1\gamma_2$, and the symbol sequence $r = (r_0, r_1, r_2, \ldots)$ is defined by $r_k = \operatorname{sgn}\left(cx^{(k)}\right)$.

III. Analysis of the discretized system

Let us change coordinates in such a way that the symbol r_k depends on a single variable: $y_1^{(k)} = c_1 x_1^{(k)} + x_2^{(k)}$, $y_2^{(k)} = x_2^{(k)}$. In these coordinates the update equation reads

$$(y_1^{(k+1)}, y_2^{(k+1)}) = (y_1^{(k)} + ay_2^{(k)} - er_k, dy_2^{(k)} - fr_k), \qquad (3)$$

Manuscript received Month 01, 2010; revised Month 01, 2010. This work was supported in part by the AGH University of Science and Technology, grant no. 11.11.120.611 and under the Australian Research Council's Discovery Grant Scheme, project no. DP0986376.

Z. Galias is with the Department of Electrical Engineering, AGH University of Science and Technology, 30–059 Krakow, Poland (e-mail: galias@agh.edu.pl).

X. Yu is with the School of Electrical and Computer Engineering, RMIT University, Melbourne, VIC 3001, Australia (e-mail: x.yu@rmit.edu.au).

where $r_k = \operatorname{sgn}\left(y_1^{(k)}\right)$, $a = (c_1a_2 - c_1^2 - a_1)\gamma_1$, $f = \alpha\gamma_2$, $e = (c_1\gamma_1 + \gamma_2)\alpha$, and $d = 1 - a_1\gamma_1 - c_1\gamma_2$.

Let us recall main results presented in [7]. Given the symbol sequence $r = (r_0, r_1, \ldots)$ the formula for $y^{(k)}$ is

$$y_1^{(k)} = y_1^{(0)} + ay_2^{(0)} \sum_{j=0}^{k-1} d^j - af \sum_{j=0}^{k-2} r_j \sum_{i=0}^{k-2-j} d^i - e \sum_{j=0}^{k-1} r_j, \tag{4a}$$

$$y_2^{(k)} = d^k y_2^{(0)} - f \sum_{j=0}^{k-1} d^{k-1-j} r_j. \tag{4b}$$

By differential analysis one can easily obtain the following estimates useful for small h: $\gamma_1(h) = 0.5h^2 + o(h^3)$, $\gamma_2(h) = h + o(h^2)$, $a(h) = 0.5(a_2c_1 - c_1^2 - a_1)h^2 + o(h^3)$, $e(h) = \alpha h + o(h^2)$, $f(h) = \alpha h + o(h^2)$, and $d(h) = 1 - c_1h + o(h^2)$.

Since $\alpha > 0$ and $c_1 > 0$ it follows that for sufficiently small h the following conditions are satisfied: $\gamma_1(h) > 0$, $\gamma_2(h) > 0$, $e(h) > 0$, $f(h) > 0$, $d(h) < 1$. Observe that when h goes to 0, the parameters e, f and $1 - d$ converge to zero as h, while a converges to zero as h^2.

The following result provides conditions under which γ_1, γ_2 and in consequence also e and f are positive.

Lemma 1: If $a_2^2 \geq 4a_1$ and $h > 0$ then $\gamma_1 > 0$ and $\gamma_2 > 0$. If $a_2^2 < 4a_1$ and $h \in (0, \pi/\omega)$, where $\omega = \sqrt{a_1 - a_2^2/4}$ then $\gamma_1 > 0$ and $\gamma_2 > 0$.

The next lemma states that $\Delta = e(1-d) + af \geq 0$.

Lemma 2: If $a_1 > 0$, $a_2 = 0$ and $\cos\sqrt{a_1}h = 1$ then $d = 1$ and $\Delta = e(1 - d) + af = 0$. Otherwise $\Delta > 0$.

For $|d| > 1$ the system is unstable. The case $d = 1$ for the system (3) is analysed in [9]. In this work, we focus on the case $|d| < 1$. In the remaining part of this paper, we assume that $e > 0$, $f > 0$, $|d| < 1$, and $\Delta > 0$. As explained before these conditions hold for sufficiently small h.

The following two theorems [7] show that for small h the dynamics of the system depends on the sign of a. The first theorem handles the case $af \leq 0$.

Theorem 1: Assume that $af \leq 0$, $|d| < 1$, $\Delta = e(1 - d) + af > 0$. For arbitrary initial conditions the trajectory converges to a period–2 orbit.

The second theorem considers the case $af > 0$. It provides conditions under which a trajectory spends at most two iterations on either side of the sliding surface.

Theorem 2: Let us assume that $0 \leq d < 1$, $af > 0$, $af(1+d)d < e(1-d)$, and $af(1+4d+4d^2+2d^3) < e(1+d)$. Then there exists m such that $r_{m+2k} \neq r_{m+2k+1}$ for each $k \geq 0$.

It can be shown that for sufficiently small $h > 0$ the assumptions of Theorems 1 or 2 are satisfied. It follows that for arbitrary values of parameters of the control system (2) there exists $\bar{h} > 0$ such that if $h \in (0, \bar{h})$ then the trajectory after a sufficient number of iterations spends at most two iterations on each side of the sliding surface.

IV. Periodic orbits

In this section, we characterize periodic orbits of the system (3) in terms of corresponding symbol sequences and derive results on admissibility of periodic orbits with various switching patterns.

First, let us note that systems with fixed value of af/e are equivalent from the dynamical point of view. This can be seen by introducing new coordinates $z_1^{(k)} = y_1^{(k)}/e$, $z_2^{(k)} = y_2^{(k)}/f$. In these coordinates the update equation is $z_1^{(k+1)} = z_1^{(k)} + (af/e)z_2^{(k)} - r_k$, $z_2^{(k+1)} = dz_2^{(k)} - r_k$, where $r_k = \operatorname{sgn}\left(z_1^{(k)}\right)$.

We say that a symbol sequence $r = (r_0, r_1, \ldots, r_{n-1})$ is *admissible* if there exists a point $y^{(0)}$ with the symbol sequence r for which $y^{(0)} = y^{(n)}$. The following lemma shows that in a nondegenerate case the sum of symbols for admissible sequences is zero.

Lemma 3: Let $|d| \neq 1$ and $\Delta = e(1-d) + af \neq 0$. If $r = (r_0, r_1, \ldots, r_{n-1})$ is admissible then $\sum_{j=0}^{n-1} r_j = 0$.

Proof: Since the sequence is admissible there exists $y^{(0)}$ with the symbol sequence r such that $y^{(0)} = y^{(n)}$. From (4) applied to $k = n$ it follows that $a(1 - d^n)y_2^{(0)} = af \sum_{j=0}^{n-2} r_j(1 - d^{n-1-j}) + (1 - d)e \sum_{j=0}^{n-1} r_j$, and $(1 - d^n)y_2^{(0)} = -f \sum_{j=0}^{n-1} d^{n-1-j} r_j$, where the equation (4a) was multiplied by $(1 - d)$. Eliminating $y_2^{(0)}$ from these two equations yields $(af + e(1 - d)) \sum_{j=0}^{n-1} r_j = 0$. Since $\Delta \neq 0$ the assertion follows. ∎

The next result provides conditions under which a symbol sequence is admissible.

Lemma 4: Let $|d| \neq 1$ and $\Delta \neq 0$. The symbol sequence $r = (r_0, r_1, \ldots, r_{n-1})$ is admissible if and only if

$$\sum_{j=0}^{n-1} r_j = 0, \qquad \max_{k\colon r_k=1} \Delta_k(r) < \min_{k\colon r_k=-1} \Delta_k(r), \tag{5}$$

where

$$\Delta_k(r) = \frac{af}{1-d}\left(\frac{1-d^k}{1-d^n}\sum_{j=0}^{n-1} d^{n-1-j} r_j + \sum_{j=0}^{k-2}(1-d^{k-1-j})r_j\right) + e\sum_{j=0}^{k-1} r_j. \tag{6}$$

The initial points of the corresponding periodic orbits are given by $y_1^{(0)} \in [\max_{k\colon r_k=1} \Delta_k(r), \min_{k\colon r_k=-1} \Delta_k(r))$, $y_2^{(0)} = -f(1 - d^n)^{-1} \sum_{j=0}^{n-1} d^{n-1-j} r_j$.

Proof: First let us assume that r is admissible. Let $y^{(0)}$ be an initial point with the symbol sequence r such that $y^{(0)} = y^{(n)}$. From Lemma 3 it follows that $\sum_{j=0}^{n-1} r_j = 0$. From the proof of Lemma 3 it follows that $y_2^{(0)} = -f(1 - d^n)^{-1} \sum_{j=0}^{n-1} d^{n-1-j} r_j$. Substituting $y_2^{(0)}$ into (4a) yields $y_1^{(k)} = y_1^{(0)} - \Delta_k(r)$. Since r is the symbol sequence for $y_1^{(0)}$ it follows that $\operatorname{sgn}\left(y_1^{(0)} - \Delta_k(r)\right) = r_k$ for $0 \leq k < n$, i.e. $y_1^{(0)} - \Delta_k(r) \geq 0$ if $r_k = 1$ and $y_1^{(0)} - \Delta_k(r) < 0$ if $r_k = -1$. It follows that $\max_{k\colon r_k=1} \Delta_k(r) \leq y_1^{(0)} < \min_{k\colon r_k=-1} \Delta_k(r)$ and hence (5) is satisfied.

Now, let us assume that the conditions (5) holds. Let us define $y_2^{(0)} = -f(1-d^n)^{-1} \sum_{j=0}^{n-1} d^{n-1-j} r_j$ and let us select $y_1^{(0)} \in [\max_{k\colon r_k=1} \Delta_k(r), \min_{k\colon r_k=-1} \Delta_k(r))$. It follows that $\operatorname{sgn}\left(y_1^{(0)} - \Delta_k(r)\right) = r_k$ for $0 \leq k < n$, and hence r is the symbol sequence for the initial point $y^{(0)}$. From the definition of $y_2^{(0)}$ it follows that $y_2^{(n)} = y_2^{(0)}$ and using (5) similarly as in the proof of Lemma 3 one can show that $y_1^{(n)} = y_1^{(0)}$. Thus

$y^{(0)}$ is periodic with the symbol sequence r, and hence r is admissible. ∎

Lemma 5: Let $|d| \neq 1$ and $\Delta \neq 0$. Let us consider two opposite symbol sequences, i.e. $s = (s_0, s_1, \ldots, s_{n-1})$, $r = (r_0, r_1, \ldots, r_{n-1})$, with $r_k = -s_k$. The admissibility conditions (5) are the same for s and r.

Proof: The assertion follows from $\Delta_k(r) = -\Delta_k(s)$. ∎

Lemma 6: Let $|d| \neq 1$ and $\Delta \neq 0$. If $y^{(n)} = y^{(0)}$ and the symbol sequence r corresponding to $y^{(0)}$ has the principal period $p < n$ then $y^{(p)} = y^{(0)}$.

Proof: Let us denote $m = n/p$. From (4b) it follows that $y_2^{((k+1)p)} = \beta_1 y_2^{(kp)} + \beta_2$, where $\beta_1 = d^p \neq 1$ and $\beta_2 = -f\sum_{j=0}^{p-1} d^{p-1-j} r_j$. Hence $y_2^{(n)} = y_2^{(mp)} = \beta_1^m y_2^{(0)} + \beta_2 \sum_{j=o}^{m-1} \beta_1^j = \beta_1^m y_2^{(0)} + \beta_2(1-\beta_1^m)/((1-\beta_1))$. Since $y_2^{(n)} = y_2^{(0)}$ we obtain $y_2^{(0)} = \beta_2/(1-\beta_1)$. It follows that $y_2^{(p)} = \beta_1 y_2^{(0)} + \beta_2 = \beta_2/(1-\beta_1) = y_2^{(0)}$.

Now, we show that $y_1^{(p)} = y_1^{(0)}$. Since $y_2^{(p)} = y_2^{(0)}$, and the principal period of r is p it follows from (4a) that

$$y_1^{(kp)} = y_1^{(0)} + k\left(a y_2^{(0)} \sum_{j=0}^{p-1} d^j - af\sum_{j=0}^{p-2} s_j \sum_{i=0}^{p-2-j} d^i - e\sum_{j=0}^{p-1} s_j\right),$$

Since $y_1^{(mp)} = y_1^{(0)}$, the expression in the parentheses is zero, and $y_1^{(p)} = y_1^{(0)}$. ∎

Below, we find admissibility conditions for some specific symbol sequences. To make the notation shorter we will write for example $(-++--+)$ instead of $(-1,+1,+1,-1,-1,+1)$ and $(+-(-+)^3)$ instead of $(+--+-+-+)$.

A. Periodic orbits with three equal consecutive symbols

We will show that sequences containing three or more equal consecutive symbols are not admissible for small h.

Lemma 7: A periodic symbol sequence r such that $r_k = r_{k+1} = r_{k+2}$ for some k is not admissible for sufficiently small $h > 0$.

Proof: From Theorem 1 it is sufficient to consider the case $af > 0$. Let us consider a period–n symbol sequence containing at least 3 equal consecutive symbols. The sequence can be transformed by shifting and/or negating all symbols into the form: $r = (+1, -1, -1, -1, r_4, \ldots, r_{n-1})$. These transformation do not change the admissibility conditions. From Lemma 4 it follows that if the sequence r is admissible then $\Delta_0 < \Delta_3$. We will show that $\Delta_3 < \Delta_0 = 0$ for small h.

$$\begin{aligned}\Delta_3 =& \frac{af}{1-d}\left(\frac{1-d^3}{1-d^n}\sum_{j=0}^{n-1} r_j d^{n-1-j} + 1 - d^2 - 1 + d\right) - e\\ =& af\left(\frac{1+d+d^2}{1+d+\cdots+d^{n-1}} \cdot \frac{\sum_{j=0}^{n-1} r_j d^{n-1-j}}{1-d} + d\right) - e.\end{aligned}$$

Since $\sum_{j=0}^{n-1} r_j = 0$ it follows that $\left(\sum_{j=0}^{n-1} r_j d^{n-1-j}\right)/(1-d)$ is a polynomial in d. Hence, for small h the expression in the parentheses is of order of a constant. It follows that the first component of the above sum is of order h^3, while the second one is of order h. Therefore $\Delta_3 < 0$ for small h, and in consequence r is not admissible for small h. ∎

The condition $\Delta_3 < 0$ can be used to find the interval $(0, \bar{h})$ of discretization steps for which a given sequence with 3 or more equal consecutive symbols is not admissible.

B. Periodic orbits of type $(+-(-+)^m)$

Now, we prove that when $af > 0$ and h is sufficiently small there exist an infinite number of admissible sequences. Let us consider sequences of length $n = 2+2m \geq 6$ with the pattern $r = (r_0, r_1, \ldots, r_{n-1}) = (+-(-+)^m)$.

Lemma 8: Let $|d| < 1$, $m \geq 2$, $n = 2m+2$. The symbol sequence $r = (+-(-+)^m)$ is admissible if and only if $d \neq 0$ and $0 < af < eg_n(d)$, where

$$g_n(d) = \frac{(1+d)(1-d^n)}{1+2d(1+d)-2d^{n-4}(1+d)-d^n}, \text{ for } d \in (0,1),$$

$$g_n(d) = \frac{(1+d)(1-d^n)}{1-2d^{n-4}+2d^{n-2}-d^n}, \text{ for } d \in (-1,0).$$

Proof: For the sequence r we have $\Delta_0 = 0$, $\Delta_1 = af(1-2d^{n-2}+d^n)/((1+d)(1-d^n)) + e$, $\Delta_{2+2p} = 2af(d^{2p}-d^{n-2})/(1-d^n)$, $\Delta_{2+2p+1} = af(1+2(1+d)(d^{2p+1}-d^{n-2})-d^n)/((1+d)(1-d^n)) - e$ for $0 \leq p < m$.

According to Lemma 4 the symbol sequence $(+-(-+)^m)$ is admissible if and only if (a) $\Delta_0 < \Delta_1$, (b) $\Delta_0 < \Delta_{2+2p}$, (c) $\Delta_{2+2p+1} < \Delta_1$, (d) $\Delta_{2+2p+1} < \Delta_{2+2k}$ for $k, p = 0, 1, \ldots m-1$.

The condition (a) is equivalent to $af(1-2d^{n-2}+d^n)/((1+d)(1-d^n)) + e > 0$. It holds provided that $|d| < 1$, $e > 0$, $af > 0$. This can be seen by noting that for $|d| < 1$ we have $1+d > 0$, $1-d^n > 0$, $1-2d^{n-2}+d^n = (1-d^{n-4}) + (d^{n-4}-2d^{n-2}+d^n) = (1-d^{n-4}) + d^{n-4}(1-d^2)^2 > 0$.

The condition (b) can be written as $2af(d^{2p}-d^{2m})/(1-d^n) > 0$. When $d = 0$ this condition does not hold for $p = m-1$. For $d \neq 0$, since $|d| < 1$, we have $1-d^n > 0$ and $d^{2p}-d^{2m} = d^{2p}(1-d^{2(m-p)}) > 0$ for $0 \leq p < m$. It follows that the condition (b) is equivalent to $af > 0$.

The condition (c) can be written as $afd(d^{2p}-d^{2m})/(1-d^n) < e$. Since $1-d^n > 0$ and $d^{2p}-d^{2m} = d^{2p}(1-d^{2(m-p)}) \geq 0$ for $0 \leq p < m$ the condition (c) holds for $d \in (-1,0)$, $e > 0$, and $af > 0$. For positive d it is sufficient to consider the case $p = 0$. Indeed for $d \in (0,1)$ and $p > 0$ we have $d^{2p}-d^{2m} \leq d^0-d^{2m}$. The condition (c) for $p = 0$ has the form $afd(1-d^{n-2})/(1-d^n) < e$. Later, we will show that it is satisfied when the condition (d) is fulfilled.

The condition (d) can be written as $af(1+2d^{2p+1}(1+d)-2d^{2k}(1+d)-d^n)/((1+d)(1-d^n)) < e$. Since for $|d| < 1$ the denominator $(1+d)(1-d^n)$ is positive and $-2d^{2k}(1+d) \leq -2d^{2(m-1)}(1+d)$ for $0 \leq k \leq m-1$ it is sufficient to consider the case $k = m-1$. For positive d the condition (d) is the most restrictive for $p = 0$. Indeed, when $d \in (0,1)$ we have $2d^{2p+1}(1+d) < 2d(1+d)$ for $0 \leq p < m$. For $p = 0$ we obtain $af(1+2d(1+d)-2d^{n-4}(1+d)-d^n)/((1+d)(1-d^n)) < e$. Since $1 > d^n$ and $2d(1+d) > 2d^{n-4}(1+d)$ the numerator $(1+2d(1+d)-2d^{n-4}(1+d)-d^n)$ is positive and we obtain the first part of the definition of g_n. For $d < 0$ the condition (d) is the most restrictive for $p = m-1$. Indeed, when $d \in (-1,0)$ we have $2d^{2p+1}(1+d) < 2d^{2(m-1)+1}(1+d)$ for $0 \leq p < m$. This leads to the inequality $af(1-2d^{n-4}+2d^{n-2}-d^n)/((1+

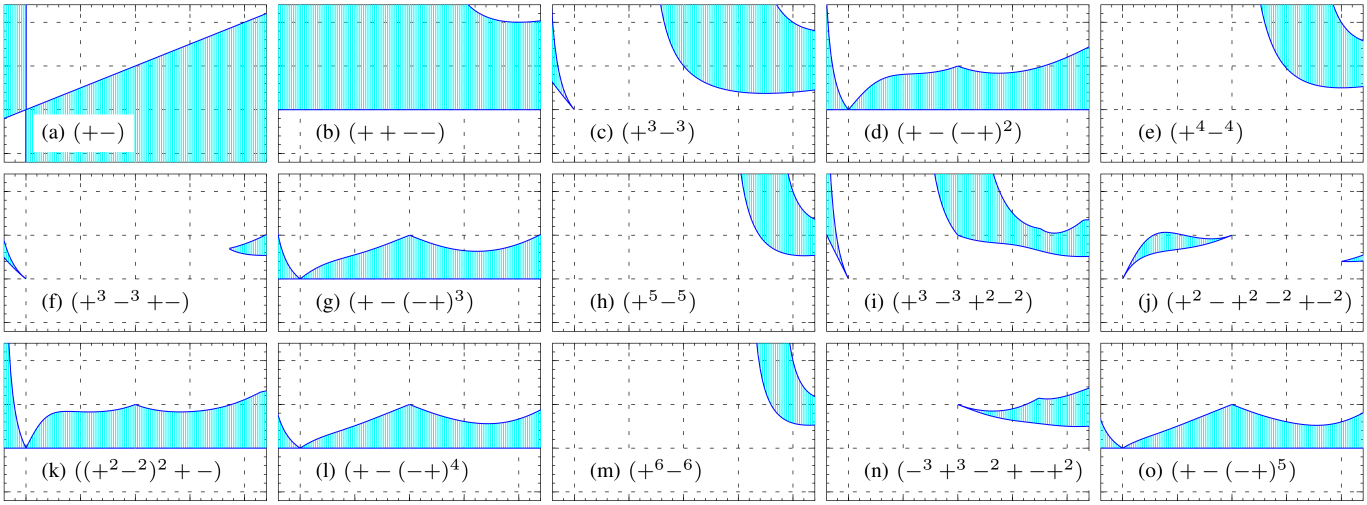

Fig. 1. Admissibility of symbol sequences with period $n \leq 12$ in the $(d, af/e)$ plane, $d \in [-1.2, 1.2]$, $af/e \in [-1.2, 2.4]$

$d)(1-d^n)) < e$. The expression $(1-2d^{n-4}+2d^{n-2}-d^n) = (1-d^{n-6})+(d^{n-6}-2d^{n-4}+d^{n-2})+(d^{n-2}-d^n) = (1-d^{n-6})+d^{n-6}(1-d^2)^2+d^{n-2}(1-d^2)$ is positive since it is a sum of three positive components and hence we obtain the second part of the definition of g_n.

It remains to show that for $d \in (0,1)$ the condition (c) is weaker than the condition (d). It is sufficient to show that $af(1+2d(1+d)-2d^{n-4}(1+d)-d^n)/((1+d)(1-d^n)) < e$ implies that $afd(1-d^{n-2})/(1-d^n) < e$. We will show that for $d \in (0,1)$ the left hand side of the first inequality is larger that the left hand side of the second inequality. In the following we multiply both expressions by $(1+d)(1-d^n)/af$. The difference between the two expressions is $(1+2d(1+d)-2d^{n-4}(1+d)-d^n)-(d(1-d^{n-2})(1+d)) = 1+d+d^2-2d^{n-4}-2d^{n-3}+d^{n-1} = (1-d^{n-5})+d(1-d^{n-5})+d^2(1-d^{n-6})+d^{n-5}(1-2d^2+d^4)$. It is positive as it is a sum of four positive components. This completes the proof. ∎

When n goes to infinity the functions g_n decrease monotonically (compare Fig. 2) to the function

$$g(x) = \begin{cases} (1+d)/(1+2d+2d^2), & \text{for } d \in (0,1), \\ 1+d, & \text{for } d \in (-1,0). \end{cases} \tag{7}$$

It follows that for $d \in (0,1) \cup (0,1)$, $0 < af < eg(d)$ there exist periodic orbits of type $(+-(-+)^m)$ with arbitrary $m \geq 2$.

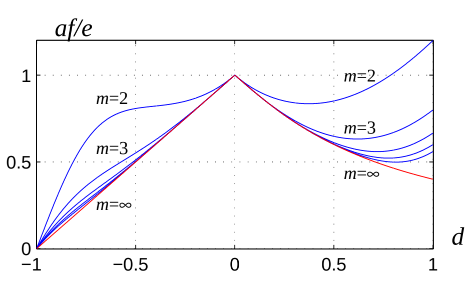

Fig. 2. Admissibility of periodic orbits of type $(+-(-+)^m)$

Combining this result with the observation that for sufficiently small discretization steps $f > 0$, $e > 0$, $d \in (0,1)$ and $\mathrm{sgn}(a) = \mathrm{sgn}(a_2c_1 - c_1^2 - a_1)$, we conclude that if $a_2c_1 - c_1^2 - a_1 > 0$ then periodic orbits with the symbol sequence $(+-(-+)^m)$ exist for arbitrarily small time steps. Admissibility regions for these symbol sequences for $m = 2,3,4,5$ are shown in Fig. 1(d,g,l,o).

C. Admissibility of short periodic symbol sequences

All period–n solution can be found by considering symbol sequences of length n and verifying the admissibility conditions (5). The number of symbol sequences to be considered can be reduced by using Lemmas 5 and 6. This analysis is carried out below for small n.

1) Period–2 orbits: The only period–2 symbol sequence which has to be considered is $(+-)$. In this case $\Delta_0 = 0$, $\Delta_1 = e - af/(1+d)$. It follows from Lemma 4 that this sequence is admissible if and only if $e > af/(1+d)$ (compare Fig. 1(a)).

2) Period–4 orbits: It is sufficient to check the sequence $(++--)$. This is a sequence of type $(+^m-^m)$ with $m=2$. For these type of sequences from symmetry of the symbolic sequence it follows that $\Delta_{m+k}(r) = \Delta_m(r) - \Delta_k(r)$ for $0 \leq k < m$. Therefore, the admissibility condition (5) can be rewritten as $\max_{0\leq k<m} \Delta_k(r) < \min_{m \leq k < 2m} \Delta_k(r) = \Delta_m(r) + \min_{0\leq k<m}(-\Delta_k(r)) = \Delta_m(r) - \max_{0\leq k<m} \Delta_k(r)$, and finally $2\max_{0\leq k<m} \Delta_k(r) < \Delta_m(r)$. For $m=2$ we have $\Delta_0 = 0$, $\Delta_1 = e - af(1+d)/(1+d^2)$, $\Delta_2 = 2e - 2afd/(1+d^2)$. The sequence is admissible if $2\max(\Delta_0(r), \Delta_1(r)) < \Delta_2(r)$, which is equivalent to $af > 0$, and $afd < e(1+d^2)$ (compare Fig. 1(b)).

3) Period–6 orbits: It is sufficient to check the sequences $(+^3-^3)$ and $(+-(-+)^2)$. The first sequence is of type $(+^m-^m)$ for $m=3$ with $\Delta_0 = 0$, $\Delta_1 = e - af(1+d+d^2)/(1+d^3)$, $\Delta_2 = 2e - 2afd(d+1)/(1+d^3)$, $\Delta_3 = 3e - af(3d^2+d-1)/(1+d^3)$. From the argument presented when considering the sequence $(+^2-^2)$ we know that this sequence is admissible if and only if $2\max(\Delta_0(r), \Delta_1(r), \Delta_2(r)) < \Delta_3(r)$, which is equivalent to $3e > af(3d^2+d-1)/(1+d^3)$, $e > af(d^2-d-3)/(1+d^3)$, and $e < af(d^2+3d+1)/(1+d^3)$.

Note that for $|d| < 1$ and small af/e this symbol sequence is not admissible (see Fig. 1(c)).

The sequence $(+-(-+)^2)$ is of type $(+-(-+)^m)$ with $m = 2$. According to Lemma 8 it is admissible for $d \in (0,1)$ if $0 < af < e(1+d+d^2)(1+d^3)/(1+2d+d^2+d^4)$, and for $d \in (-1,0)$ if $0 < af < e(1+d+d^2)(1+d^3)/(1-d^2+d^4)$ (compare Fig. 1(d)).

4) Longer orbits: Analysis of the existence of periodic orbits for larger periods becomes more difficult. When d is fixed, using Lemma 4 one can numerically compute the admissible range for af/e. These calculation were performed for all symbol sequences of period $n \le 12$ for $d \in [-2,2]$, $|d| \ne 1$, $d = k/50$ with integer k. Rational arithmetic was used to avoid rounding errors. Admissibility regions are shown in Fig. 1. Sequences, which are not admissible for $|d| < 1$ are skipped. For $a < 0$ the only admissible sequence is $r = (-+)$. This is in full agreement with the Theorem 1. For $a > 0$ other sequences are also admissible. Observe that only a small fraction of symbol sequences is admissible for $|d| < 1$. As a rule, patterns containing both long and short subsequences of equal symbols are forbidden for $|d| < 1$. As predicted by Lemma 7 patterns with more than three equal consecutive symbols are not admissible for $|d| < 1$ and small $|af/e|$.

V. Simulation results

As an example, let us consider the SMC system with parameters $a_1 = -2$, $a_2 = 2$, $c_1 = 1$, $\alpha = 1$, $c_1 a_2 - c_1^2 - a_1 = 3 > 0$. For $h = 0.1$ we have $\gamma_1 \approx 0.00469$, $\gamma_2 \approx 0.09094$, $v \approx 0.2701$, $a \approx 0.01407$, $e \approx 0.09563$, $f \approx 0.09094$, $d \approx 0.9184$.

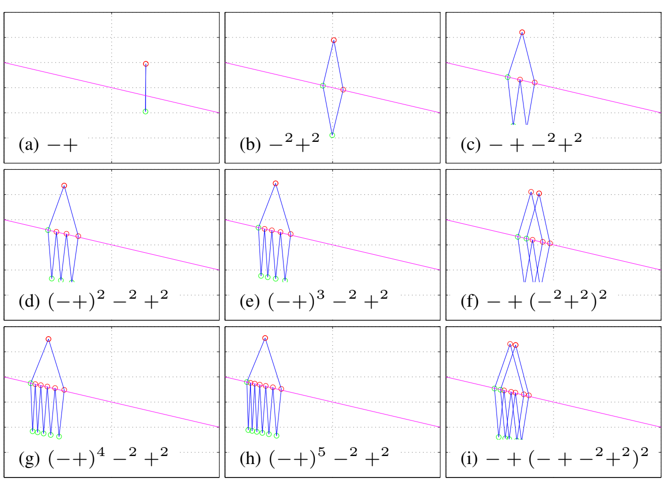

Fig. 3. Examples of periodic orbits, $x_1 \in [-0.05, 0.05]$, $x_2 \in [-0.15, 0.15]$, $a_1 = -2$, $a_2 = 2$, $c_1 = 1$, $h = 0.1$, $\alpha = 1$

Periodic orbits with the period $n \le 20$ have been found using Lemma 4. In Table I all 17 admissible patterns of symbols with period $n \le 20$ are shown. Patterns corresponding to the same cycle are not reported. Examples of periodic orbits corresponding to the first nine admissible patterns are shown in Fig. 3. Let us note that the condition $0 < af < eg(d)$ is satisfied. It follows that symbol sequences of type $(+-(-+)^m)$ with arbitrary m are admissible. There are 8 such patterns in Table I (rows c, d, e, g, h, k, m, p).

It is interesting to note that all admissible sequences found can be obtained by concatenating two short patterns: $P = (-+)$ and $Q = (--++)$ (compare Table I). For example sequences of type $(+-(-+)^m)$ after shifting can be expressed as $P^{m-1}Q$. In Table I one can also see examples of other infinite families of admissible sequences, for example of type PQ^k (rows c, f, j, o), or $P(PQ)^k$ (rows d, i, r).

TABLE I
Admissible symbol sequences with period $n \le 20$ for $a_1 = -2$, $a_2 = 2$, $c_1 = 1$, $h = 0.1$, $\alpha = 1$, $P = (-+)$, $Q = (--++)$

	n	r	r	y_1	y_2
a	2	$-+$	P	0.0^{9496}_{0000}	0.04740
b	4	$-^2+^2$	Q	0.09^{568}_{430}	0.09463
c	6	$-+-^2+^2$	PQ	0.095^{92}_{08}	0.11022
d	8	$(-+)^2-^2+^2$	P^2Q	0.09^{604}_{546}	0.11791
e	10	$(-+)^3-^2+^2$	P^3Q	0.09^{611}_{569}	0.12243
f	10	$-+(-^2+^2)^2$	PQ^2	0.095^{80}_{38}	0.10237
g	12	$(-+)^4-^2+^2$	P^4Q	0.09^{615}_{584}	0.12538
h	14	$(-+)^5-^2+^2$	P^5Q	0.09^{619}_{594}	0.12742
i	14	$-+(-+-^2+^2)^2$	$P(PQ)^2$	0.095^{97}_{73}	0.11350
j	14	$-+(-^2+^2)^3$	PQ^3	0.095^{75}_{51}	0.09917
k	16	$(-+)^6-^2+^2$	P^6Q	0.096^{21}_{02}	0.12890
l	16	$(-+-^2+^2)^2-^2+^2$	$(PQ)^2Q$	0.095^{83}_{64}	0.10418
m	18	$(-+)^7-^2+^2$	P^7Q	0.096^{23}_{07}	0.13002
n	18	$-+((-+)^2-^2+^2)^2$	$P(P^2Q)^2$	0.09^{607}_{591}	0.11959
o	18	$-+(-^2+^2)^4$	PQ^4	0.095^{73}_{57}	0.09750
p	20	$(-+)^8-^2+^2$	P^8Q	0.096^{24}_{11}	0.13087
r	20	$-+(-+-^2+^2)^3$	$P(PQ)^3$	0.095^{95}_{82}	0.11190

VI. Conclusion

Admissibility conditions for the existence of periodic orbits with a given symbol sequence have been formulated and regions of the existence of short periodic orbits have been found. It was shown that even for arbitrarily small discretization steps complex switching patterns including arbitrarily long periodic solutions are possible.

References

[1] V. Utkin, *Sliding Modes in Control and Optimization*, ser. Communications and Control Engineering Series. Berlin: Springer Verlag, 1992.

[2] X. Yu and G. Chen, "Discretization behaviors of equivalent control based sliding mode control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1641–1646, 2003.

[3] X. Xia and A. S. I. Zinober, "Periodic orbits from delta-modulation of stable linear systems," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1376–1380, 2004.

[4] ——, "Delta-modulated feedback in discretization of sliding mode control," *Automatica*, vol. 42, no. 5, pp. 771–776, 2006.

[5] M. di Bernardo and F. Vasca, "Discrete-time maps for the analysis of bifurcations and chaos in DC/DC converters," *IEEE Trans. Circ. Syst. I*, vol. 47, no. 2, pp. 130–143, 2000.

[6] Z. Galias and X. Yu, "Euler's discretization of single input sliding mode control systems," *IEEE Trans. Automatic Control*, vol. 52, no. 9, pp. 1726–1730, 2007.

[7] ——, "Analysis of zero-order holder discretization of two-dimensional sliding-mode control systems," *IEEE Trans. Circ. Syst. II*, vol. 55, no. 12, pp. 1269–1273, 2008.

[8] C. Milosavljevic, "General conditions for the existence of a quasi-sliding mode on the switching hyperplane in discrete variable structure systems," *Automation and Remote Control*, vol. 3, pp. 36–44, 1985.

[9] Z. Galias and X. Yu, "Complex discretization behaviors of a simple sliding-mode control system," *IEEE Trans. Circ. Syst. II*, vol. 53, no. 8, pp. 652–656, August 2006.