Trapping Region for the Double Scroll Attractor

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***Abstract*— It is shown that a certain set is positively invariant for the return map associated with the Chua's circuit. The set contains the intersection of the numerically observed double-scroll attractor and the planes defining the return map. The proof is based on rigorous numerics.**

I. Introduction

The double scroll is a chaotic attractor observed from a simple third-order electronic circuit with a piece-wise linear characteristics of the nonlinear element [1]. The attractor has been extensively studied in numerical simulations but only a few rigorous results concerning its chaotic dynamics exist. The existence of a Shilnikov-type homoclinic orbit for an unknown parameter value within a certain range is proved in [2] and [3]. In [4], the existence of a nontrivial symbolic dynamics embedded in the double scroll attractor was proved.

The starting point to study the dynamics over the whole attractor is the existence of a trapping region enclosing the attractor. There are several results concerning the existence of a trapping region for the Chua's circuit for different parameter values. First results concerning the spiral attractor for the Chua's circuit were reported in [5]. A trapping region for the spiral Roessler-type attractor was constructed in [6]. The case of the spiral attractor containing trajectories tangent to the planes separating linear regions (the C^0-planes) was considered in [7]. In this work, we deal with the double scroll attractor. A candidate trapping region Ω for the associated Poincaré map is constructed. It is proved that each trajectory based in Ω either returns to it or converges to the origin. The proof uses methods for computing enclosures of trajectories passing arbitrarily close to an unstable equilibrium.

The paper is organized as follows. In Section II, methods for integration of piece-wise linear (PWL) systems are presented. The procedure for integration of PWL systems when intersections of trajectories with the C^0-planes are transversal or tangent is recalled and a method for computing enclosures of trajectories passing close to an unstable equilibrium is presented. In Section III, the existence of a trapping region for the double scroll attractor is proved. The return map is defined, and details of the computer-assisted proof that a certain set is a trapping region for the return map are presented.

During computations, interval arithmetic is used to ensure that the results obtained are rigorous. In the following, boldface is used to denote intervals, interval vectors and matrices, and the usual math italics is used to denote point quantities.

II. Rigorous evaluation of return maps for piece-wise linear systems

Let the piecewise linear system be defined by

$$\dot{x} = f(x), \tag{1}$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is a piece-wise linear continuous map. By $x(t) = \varphi(t, \hat{x})$ we denote the solution of (1) satisfying the initial condition $x(0) = \hat{x}$.

Let us assume that the state space $\mathbb{R}^n$ is composed of m linear regions $R_1, R_2, \ldots R_m$, separated by hyperplanes $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$. In the following, the hyperplanes Σ_j will be referred to as the C^0-hyperplanes. Let $\dot{x} = A_k x + v_k$ be the state equation in the region R_k, where $A_k \in \mathbb{R}^{n\times n}$, $v_k \in \mathbb{R}^n$. If A_k is invertible, the explicit solution has the form $\varphi_k(t, \hat{x}) = \mathrm{e}^{A_k t}(\hat{x} - w^{(k)}) + w^{(k)}$, where $w^{(k)} = -(A_k)^{-1} w^{(k)}$.

Let P be the return map defined by the union S of hyperplanes S_k, i.e. $P(x) = \varphi(\tau(x), x)$, where $\tau(x) > 0$ is the first time at which the trajectory $\varphi(t, x)$ returns to S.

In this section we present a method for finding an enclosure of the set $P(\mathbf{x})$, where $\mathbf{x}$ is an interval vector enclosed in S.

A. Transversal intersections

If for each $x \in \mathbf{x}$ the trajectory $\varphi([0, \tau(x)], x)$ remains in a single region R_k the problem is relatively easy. The first step is to find $t_1 > 0$ such that $\varphi_k((0, t_1], \mathbf{x}) \in R_k$ and $t_2 > t_1$ such that $\varphi_k(t_2, \mathbf{x})$ is on the other side of S. Next, one evaluates the formula for the solution of a linear system in interval arithmetic. More precisely one computes

$$\mathbf{y} = \varphi_k(\mathbf{t}, \mathbf{x}) = \mathrm{e}^{A_k \mathbf{t}}(\mathbf{x} - w^{(k)}) + w^{(k)}, \tag{2}$$

where $\mathbf{t} = [t_1, t_2]$. Finally, the intersection of $\mathbf{y}$ and S is computed. Clearly, $P(\mathbf{x}) \subset \mathbf{y} \cap S$.

Similar approach can be used when trajectories $\varphi([0, \tau(x)], x)$ visit several linear regions and the intersections with the planes Σ_j are transversal. The problem can be decomposed into passing several linear regions. For each region the minimum t_1 and maximum t_2 travel times are found, formula (2) is used to find an enclosure $\mathbf{y}$ of the intersection area, and the intersection of $\mathbf{y} \cap \Sigma$ is computed.

B. Tangent intersections

Let us now assume that some trajectories based at $\mathbf{x} \subset R_k$ are tangent to the plane Σ_j separating the linear regions R_k and R_l. The method presented above cannot be used since some trajectories based in $\mathbf{x}$ intersect Σ_j, while others do not.

To pass the tangency area, we consider the PWL system (1) as a perturbation of the linear system:

$$\dot{x} = g(x) = A_k x + v_k, \tag{3}$$

The following theorem provides bounds for the solutions of (1) based on the solutions of the linear system (3).

Theorem 1: Let $x(t)$ and $y(t)$ be solutions of $\dot{x} = g(x)$ and $\dot{y} = f(y)$, respectively. Let us assume that $x(0) = y(0)$, and $x(t), y(t) \in D \subset \mathbb{R}^n$ for $t \in [0, h]$, where the set D is bounded, closed, and convex. Then for $t \in [0, h]$

$$|y_i(t) - x_i(t)| \leq \Delta_i = \left(\int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}s \right)_i, \tag{4}$$

where $B_{ij} = |(A_k)_{ij}|$ for $i \neq j$, $B_{ii} = (A_k)_{ii}$, and $c_i \geq |g_i(x(t)) - f_i(x(t))|$, for $t \in [0, h]$.

The above theorem is a conclusion from the results on integration of differential inequalities developed in [8], [9].

The procedure for passing the tangency area starts by finding $s_1 > 0$ such that $\varphi_k([0, s_1], \mathbf{x}) \subset R_k$. The set $\mathbf{u} = \varphi_k(s_1, \mathbf{x})$ serves as an initial condition for integration along the tangency. For $t \geq s_1$ the PWL system is treated as a perturbed linear system. We select s_2, compute an enclosure $\mathbf{v}$ of the solution $\varphi_k([0, s_2], \mathbf{u})$ of the linear system (3). Next, the set $\mathbf{v}$ is increased to form the interval vector $\mathbf{w}$, which serves as a guess of the set containing the solution $\varphi([0, s_2], \mathbf{u})$ of the PWL system. Finally, one computes Δ and $c = \sup_{x \in \mathbf{w}} |g(x) - f(x)|$. The difference between g and f is zero over the region R_k and for the region R_l can be computed as $g(x) - f(x) = (A_k - A_l)x + v_k - v_l$. If $\mathbf{v} + [-1, 1]\Delta \subset \mathbf{w}$, it follows from the Theorem 1 that the solution of the PWL system is enclosed in $\mathbf{v} + [-1, 1]\Delta$ and that $\varphi(s_2, \mathbf{u}) \subset \mathbf{z} = \varphi_k(s_2, \mathbf{u}) + [-1, 1]\Delta$. If $\mathbf{z} \subset R_k$ and the vector field f over the set $\mathbf{z}$ points away from the plane Σ_j we continue integration from the initial set $\mathbf{z}$ using the standard algorithm. For the details see [7].

C. *Integration of a linear vector field near the origin*

The methods described so far are applicable when the return times $\tau(x)$ for $x \in \mathbf{x}$ are bounded. If the attractor is singular (which is the case for the Chua's circuit double scroll attractor) and it contains an unstable equilibrium, the algorithm presented above fails for some points belonging to the attractor. This is due to the fact that some trajectories pass arbitrarily close to the equilibrium and for such trajectories the return time may be arbitrarily large. Therefore, we have to develop another method to cope with such trajectories.

In this section, we present a method to integrate a linear 3D vector field near the origin. We assume that the matrix defining the vector field has one positive real eigenvalue, and two complex eigenvalues with negative real parts. For the Chua's circuit, the origin, which is the only equilibrium belonging to the attractor, has this type of stability. The method is similar to the method presented in [10], where the case of one positive and two negative real eigenvalues was considered. In [10], the method was applied to the integration of the Lorenz system near the origin.

Let us consider a linear dynamical system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Dx. \tag{5}$$

with $\lambda > 0$, $\alpha < 0$, and $\beta > 0$. The origin is an unstable equilibrium with the two-dimensional stable manifold $W_s = \{x = (x_1, x_2, x_3) \colon x_1 = 0\}$, and one-dimensional unstable manifold $W_u = \{x \colon x_2 = x_3 = 0\}$. The explicit solution has the form: $x_1(t) = \mathrm{e}^{\lambda t} x_1(0)$, $x_2(t) = \mathrm{e}^{\alpha t}(\cos(\beta t) x_2(0) + \sin(\beta t) x_3(0))$, $x_3(t) = \mathrm{e}^{\alpha t}(-\sin(\beta t) x_2(0) + \cos(\beta t) x_3(0))$.

Consider a cylinder $C(\bar{h}, \bar{r}) = \{x \colon |x_1| \leq \bar{h}, x_2^2 + x_3^2 \leq \bar{r}^2\}$. The half of the cylinder with $x_1 \geq 0$ for the case $\bar{h} = 1$ and $\bar{r} = 1$ is shown in Fig. 1.

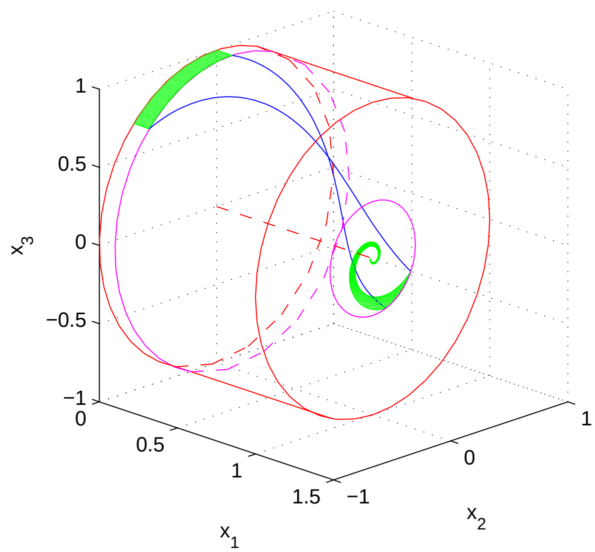

Fig. 1. Integration of the linear system (5) near the origin, $\lambda = 0.728$, $\alpha = -0.319$, $\beta = 0.892$, the cylinder $C(1, 1)$, two trajectories with initial points $(0.1, \cos(\pi/2), \sin(\pi/2))$ and $(0.1, \cos(3\pi/4), \sin(3\pi/4))$, the exit set for the stripe of initial conditions enclosed in the cylinder side is the spiral-shaped region enclosed in the cylinder base

Let us select an initial point $x(0) \in C(\bar{h}, \bar{r})$. For $x_1(0) = 0$ the trajectory converges to the origin along the stable manifold W_s. For $x_1(0) \neq 0$, since $\alpha < 0$ and $\lambda > 0$, the trajectory exits the cylinder through one of its bases after the time τ satisfying $\bar{h} = \mathrm{e}^{\lambda \tau} |x_1(0)|$, i.e.

$$\tau = \lambda^{-1} \log(\bar{h}/|x_1(0)|).$$

The position of the exit point is $x_1(\tau) = \mathrm{sgn}(x_1(0))\bar{h}$, $x_2(\tau) = \mathrm{e}^{\alpha\tau}(\cos(\beta\tau) x_2(0) + \sin(\beta\tau) x_3(0))$, $x_3(\tau) = \mathrm{e}^{\alpha\tau}(-\sin(\beta\tau) x_2(0) + \cos(\beta\tau) x_3(0))$.

Two example trajectories for $\lambda = 0.728$, $\alpha = -0.319$, $\beta = 0.892$ are shown in Fig. 1. The initial points are $(0.1, \cos(\pi/2), \sin(\pi/2))$ and $(0.1, \cos(3\pi/4), \sin(3\pi/4))$. One can see that both trajectories exit the cylinder $C(1, 1)$ through the circle centered at the x_1 axis. More generally, one can show that $x_2^2(\tau) + x_3^2(\tau) = \mathrm{e}^{2\alpha\tau}(x_2^2(0) + x_3^2(0))$.

When $x_1(0) > 0$ is decreased to zero and $x_2(0)$, $x_3(0)$ are kept constant, the exit time increases exponentially, and the exit point moves closer to the x_1 axis along a spiral. An example is shown in Fig. 1. The set of initial conditions $x_1(0) \in (0, 0.1]$, $(x_2(0), x_3(0)) = (\cos\gamma, \sin\gamma)$, $\gamma \in [\pi/2, 3\pi/4]$ is a stripe belonging to the cylinder side. The corresponding exit set is a spiral-shaped region enclosed in the the cylinder base.

From the discussion presented above, it follows that the set of initial points $\{x_1(0) \in (0, h], x_2^2(0) + x_3^2(0) \leq r^2\}$ with $h \leq \bar{h}$ and $r \leq \bar{r}$ exits the cylinder $C(\bar{h}, \bar{r})$ through the disc $\{x_1 = \bar{h}, x_2^2 + x_3^2 \leq \mathrm{e}^{2\alpha\tau} r^2\}$, where $\tau = \lambda^{-1} \log(\bar{h}/h)$ is the minimum flow time. Flow times belong to the interval $[\tau, \infty)$.

The numerical procedure for the integration close to the origin is following. First, we select $\bar{h}$ and $\bar{r}$. A trajectory is integrated using standard methods until the solution set X is enclosed in the cylinder $C(\bar{h}, \bar{r})$. Next, we find $-h_2 < 0 < h_1$ and $r > 0$ such that $X \subset \{(x_1, x_2, x_3) \colon x_1 \in [-h_2, h_1], x_2^2 + x_3^2 \leq r^2\}$ and compute the exit sets $E_1 = \{x_1 = \bar{h}, x_2^2 + x_3^2 \leq \mathrm{e}^{2\alpha\tau_1} r^2\}$, and $E_2 = \{x_1 = -\bar{h}, x_2^2 + x_3^2 \leq \mathrm{e}^{2\alpha\tau_2} r^2\}$, where $\tau_1 = \lambda^{-1} \log(\bar{h}/h_1)$ and $\tau_2 = \lambda^{-1} \log(\bar{h}/h_2)$. The flow times for the exit sets E_1 and E_2 are $[\tau_1, \infty)$ and $[\tau_2, \infty)$, respectively.

For a general linear vector field with one positive real eigenvalue, and a pair of complex eigenvalues with negative real parts one has to apply an appropriate transformation converting the vector field to the diagonal form (5) prior to applying the computation procedure described above.

III. Analysis of the Chua's circuit

The Chua's circuit [1] is described by the following equation

$$
\begin{aligned}
C_1 \dot{x}_1 &= (x_2 - x_1)/R - g(x_1), \\
C_2 \dot{x}_2 &= (x_1 - x_2)/R + x_3, \\
L \dot{x}_3 &= -x_2 - R_0 x_3,
\end{aligned}
\tag{6}
$$

where $g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)$ is a three-segment piecewise linear characteristics.

There are three linear regions $R_1 = \{x \in \mathbb{R}^3 \colon x_1 < -1\}$, $R_2 = \{x \colon |x_1| < 1\}$ and $R_3 = \{x \colon x_1 > 1\}$ separated by the C^0-planes $\Sigma_1 = \{x \colon x_1 = -1\}$ and $\Sigma_2 = \{x \colon x_1 = 1\}$.

The circuit is studied with the following parameter values (after parameter rescaling): $C_1 = 1$, $C_2 = 9.3515$ $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$, for which the double scroll attractor is observed in computer simulations (see Fig. 2). It can be seen that some trajectories turn close to the planes Σ_k, which means that intersections with the C^0-planes are not always transversal. Moreover, some trajectories pass very close to the origin, which is an equilibrium of the system. In the region R_2 the vector field has the form $\dot{x} = A_2 x$. The eigenvalues of A_2 are $\lambda \approx 0.728$, $\alpha \pm i\beta \approx -0.319 \pm i0.892$, and hence the origin has the type of stability studied in Section II-C.

A. The return map

Let P be the return map defined by the planes $S_1 = \{x \colon x_1 = -1.5\}$, $S_2 = \{x \colon x_1 = 1.5\}$. The attractor intersects these plane transversally (compare Fig.2).

A trajectory of P is shown in Fig. 3. It consists of four parts. The two top ones $P_{3,4}$ (depicted with red) correspond to intersections with the plane S_1, while the two bottom ones $P_{1,2}$ (depicted with blue) correspond to intersections with the plane S_2. P_1 is mapped into P_2. P_2 is mapped into the dark blue part of P_1 and the light red part of P_3. Similarly, P_3 is mapped into P_4, and P_4 is mapped to the light-blue part of P_1 and the dark-red part of P_3.

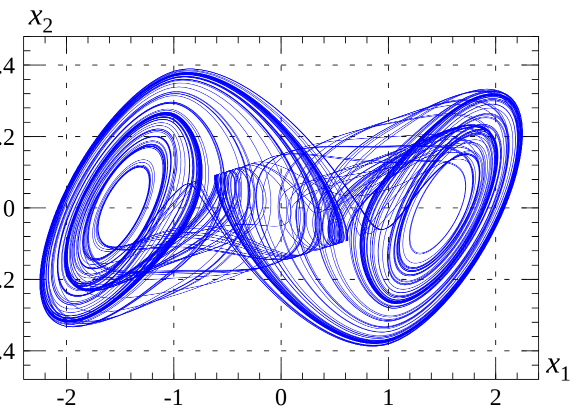

Fig. 2. Computer generated double scroll attractor for the Chua's circuit

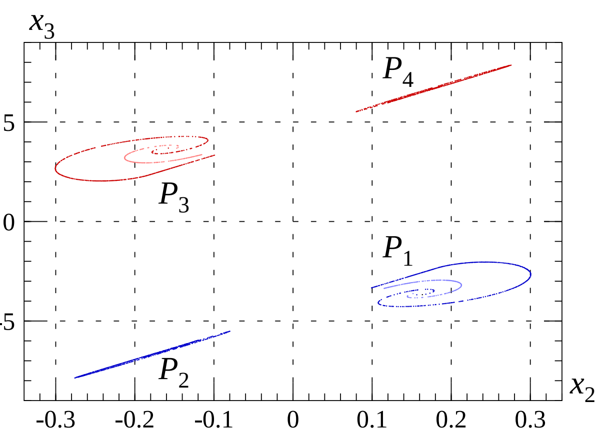

Fig. 3. A trajectory of the return map P defined by the planes $S_1 = \{x \colon x_1 = -1.5\}$, $S_2 = \{x \colon x_1 = 1.5\}$

B. A trapping region

In this section, we prove that there exist a trapping region for the return map P defined above. We say that a set A is a trapping region for the map f if it is positively invariant under the action of this map, i.e. $f(x) \in A$ for every $x \in A$.

We start with the construction of a candidate set Ω satisfying the following two properties: (a) Ω encloses an observed attractor, so that the dynamics is captured by the set selected, (b) the image of the border $\partial\Omega$ of Ω is enclosed in Ω. The candidate set $\Omega = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ is shown in Fig. 4. It has been constructed by drawing four polygons enclosing the numerically observed trajectory and then adjusting their corners so that $P(x_i) \in \Omega$, where $x_i \in \partial\Omega$ are test points. For now, the evaluation of P is not performed rigorously.

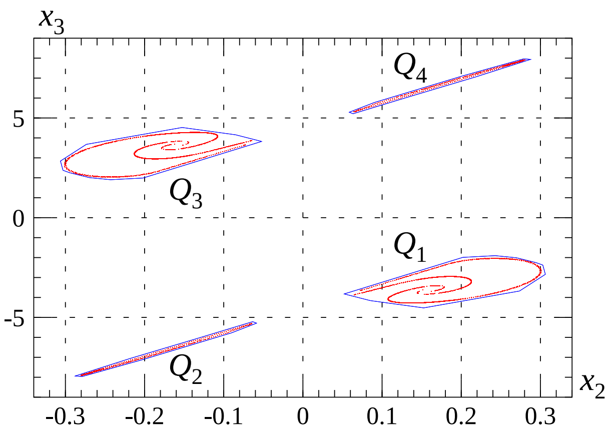

Fig. 4. The set Ω and the image of its border computed non-rigorously

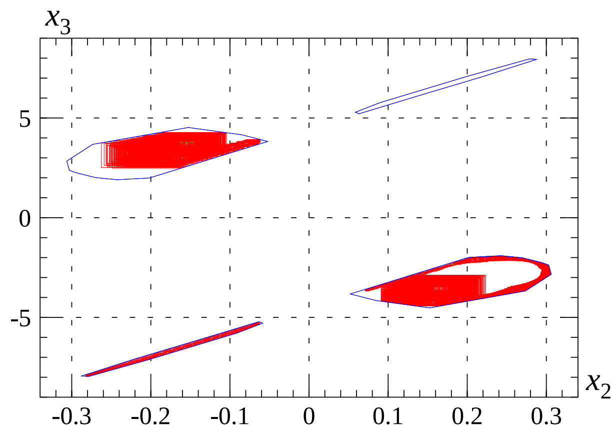

Fig. 5. An enclosure of the image of $\partial(Q_1 \cup Q_2)$ computed rigorously, $P(\partial(Q_1 \cup Q_2)) \subset Q_1 \cup Q_2 \cup Q_3$

P is not defined on the whole Ω. Some trajectories based in Ω converge to the origin and never come back to S. Therefore, we study the intersection of Ω with the domain $\mathrm{Dom}(P)$ of P. Let us denote $\bar{\Omega} = \Omega \cap \mathrm{Dom}(P)$, and $\bar{Q}_k = Q_k \cap \mathrm{Dom}(P)$ for $k = 1, 2, 3, 4$. In order to prove that $\bar{\Omega}$ is a trapping region one has to show that $P(x) \in \Omega$ for each $x \in \bar{\Omega}$. From the symmetry of the problem it follows that it is sufficient to prove that $P(\bar{Q}_1 \cup \bar{Q}_2) \subset \Omega$. From the uniqueness of solutions of the dynamical system considered it follows that it is sufficient to verify that (a) the image of the set $(\partial Q_1 \cup \partial Q_2) \cap \mathrm{Dom}(P)$ is enclosed in Ω and (b) the map P is well defined on $\bar{Q}_1 \cup \bar{Q}_2$.

In theory, one could use the method presented in the previous section to prove the conditions (a) and (b). In practice however, this approach is too slow due to long integration times and strong overestimation in computed enclosures for trajectories passing close to the equilibrium at the origin. Therefore, such trajectories are treated in a different way.

Let E be the matrix transforming the vector field A_2 into the diagonal form (5), i.e. $D = E^{-1} A_2 E$. We have shown that each trajectory starting at the base of the cylinder $C(0.5, 0.02)$ (in transformed coordinates) hits the set $Q_1 \cup Q_3$. Specifically, it was shown that $P(Ey) \in (1, [-0.278, -0.028], [2.50, 4.87]) \subset Q_3$ for all $y = (0.5, r\cos\phi, r\sin\phi)$, with $r \in (0, 0.02]$, $\phi \in [0, 2\pi]$.

To prove the condition (a), the set $\partial Q_1 \cup \partial Q_2$ was covered by 4312 boxes $\mathbf{x}_i$. For 4218 boxes enclosures of $P(\mathbf{x}_i)$ were found and it was verified that $P(\mathbf{x}_i) \subset Q_1 \cup Q_2 \cup Q_3$. For the remaining 94 boxes, it was shown that the corresponding trajectories enter the cylinder $C(0.5, 0.02)$. Enclosures of the images $P(\mathbf{x}_i)$ are shown in Fig. 5.

To prove that P is well defined on $\bar{Q}_1 \cup \bar{Q}_2$, the set $Q_1 \cup Q_2$ was covered by 616 boxes $\mathbf{x}_i$. For 606 boxes enclosures of $P(\mathbf{x}_i)$ have been found. For the remaining 8 boxes, it was shown that the corresponding trajectories enter the cylinder $C(0.5, 0.2)$. It follows that each trajectory based in $Q_1 \cup Q_2$ either returns to S or converges to the origin.

Summarizing, we have proved that each trajectory based at Ω either returns to Ω or converges to the origin. It follows that $\bar{\Omega}$ if a trapping region for P, and the set $\{\varphi(t, x) \colon x \in \Omega, t \in [0, \tau(x))\}$ is a trapping region for the double scroll attractor.

IV. Conclusion

Rigorous integration methods for piece-wise linear systems have been studied. An algorithm handling also the case of trajectories passing arbitrarily close to an equilibrium has been described. The method can be applied directly if the equilibrium has one positive real eigenvalue, and a pair of complex eigenvalues with negative real parts. Other stability types can be handled in a similar way.

The existence of a trapping region for the return map associated with the Chua's circuit double scroll attractor has been proved.

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