On rigorous integration of piece-wise linear continuous systems

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***Abstract*— We show how to rigorously integrate piece-wise linear systems in regions containing trajectories tangent to hyperplanes separating the linear regions. The method is applied to compute enclosures of solutions for the Chua's circuit with parameter values where the attractor contains such trajectories.**

I. Introduction

Over the past years there has been a considerable interest in using computers for obtaining rigorous results in the field of continuous dynamical systems. Various numerical techniques have been developed. This includes methods for computing rigorous enclosures of trajectories, finding accurate positions of periodic solutions, finding all short periodic orbits, proving the existence of topological chaos, and proving the existence of chaotic attractors [1]–[4].

In order for computations to be rigorous one has to cope with the problem of rounding errors. Computer implementations of interval arithmetic [5] provide a solution. In interval arithmetic all calculations are performed on intervals in such a way that the true result is always enclosed within the interval found by a computer.

One of the basic tools needed for rigorous study of continuous dynamical systems are algorithms for rigorous integration. Most of methods for rigorous integration work under the assumption that the vector field is smooth. These methods are not directly applicable to piece-wise linear (or more general for piece-wise smooth) systems, which are an important class of nonlinear dynamical systems.

In this work, integration methods for piece-wise linear systems are studied. When intersections of trajectories with hyperplanes separating linear regions (called in the following the C^0 hyperplanes) are transversal it is possible to extend general methods to integration of piece-wise linear systems. This is achieved by using C^0 hyperplanes as transversal sections. When a trajectory intersects a C^0 hyperplane, its intersection with the transversal plane is computed and the resulting set is used as a set of initial conditions for further computations. This approach has been successfully used to find the trapping region for the Poincaré map associated with the Chua's circuit for certain parameter values [6].

On the other hand when some trajectories in the region of interest are tangent to C^0 planes the method described above fails. This situation corresponds to cases when the Poincaré map associated with the C^0 hyperplanes is not continuous.

In this paper, we present a method which can be used to rigorously integrate piece-wise linear systems also in the case when trajectories are tangent to C^0 planes. This technique is based on methods for obtaining estimates for solutions of perturbed continuous dynamical systems.

In Section II the method for integration of piece-wise linear systems when intersections with C^0 planes are transversal is recalled and then the procedure for integrating piece-wise linear systems for the tangent case is presented. In Section III two examples of piece-wise linear systems are considered. The first one is a toy example of a two-dimensional system. The second example is the Chua's circuit with the Roessler-type attractor for which some trajectories embedded within the attractor are tangent to C^0 planes.

In the following, boldface is used to denote intervals, interval vectors and matrices, and the usual math italics is used to denote point quantities. For a given interval $\mathbf{x} = [a, b]$ by $\underline{x}$ and $\overline{x}$ we denote its left and right end points respectively, i.e. $\underline{x} = a$ and $\overline{x} = b$. The diameter of the interval $\mathbf{x}$ is defined as $\operatorname{diam}(\mathbf{x}) = \overline{x} - \underline{x}$.

II. Rigorous integration of piece-wise linear systems

Let the piecewise linear system be defined by

$$\dot{x} = f(x), \tag{1}$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is a piece-wise linear continuous map. By $x(t) = \varphi(t, \hat{x})$ we denote the solution of (1) satisfying the initial condition $x(0) = \hat{x}$.

Let us assume that the state space $\mathbb{R}^n$ is composed of m linear regions $R_1, R_2, \ldots R_m$, separated by hyperplanes $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$. In the following the hyperplanes Σ_j will be referred to as the C^0 hyperplanes. In the region R_k the state equation has the form $\dot{x} = A^{(k)}x + v^{(k)}$, where $A^{(k)} \in \mathbb{R}^{n\times n}$, $v^{(k)} \in \mathbb{R}^n$. If $A^{(k)}$ is invertible then in the linear region R_k solutions can be computed as

$$x(t) = \varphi_k(t, \hat{x}) = \mathrm{e}^{A^{(k)}t}(\hat{x} - p^{(k)}) + p^{(k)}, \tag{2}$$

where $p^{(k)} = -(A^{(k)})^{-1}v^{(k)}$.

The problem discussed in this paper is how to rigorously calculate an enclosure of the set $\varphi(\mathbf{t}, \mathbf{x})$ for a given interval $\mathbf{t}$

with $\underline{t} > 0$ and an interval vector $\mathbf{x} \in \mathbb{R}^n$. Without loss of generality we can assume that $\mathbf{x}$ is located within a single linear region R_k.

If all trajectories based at $\mathbf{x}$ remain in R_k for $s \in [0, \bar{t}]$ the problem is simple. The enclosure can be found by evaluating the formula (2) in interval arithmetic. More precisely, one computes

$$\mathbf{y} = \varphi_k(\mathbf{t}, \mathbf{x}) = \mathrm{e}^{A^{(k)}\mathbf{t}}(\mathbf{x} - p^{(k)}) + p^{(k)}. \tag{3}$$

For the evaluation of the above formula one can use the mean value form to obtain a narrower enclosure of the set of solutions. For the details see [7].

A. Transversal intersections

Another relatively easy case is when all trajectories based at $\mathbf{x}$ enter another linear region R_l through the plane Σ, and intersections of trajectories with Σ are transversal. In this case the first step is to find $s_1 > 0$ such that $\varphi_k([0, s_1], \mathbf{x}) \in R_k$. Then we find $s_2 > s_1$ such that $\varphi_k(s_2, \mathbf{x}) \subset R_l$. Next, one evaluates $\mathbf{y} = \varphi_k(\mathbf{s}, \mathbf{x})$, where $\mathbf{s} = [s_1, s_2]$ and finally, the intersection of $\mathbf{y}$ and Σ is computed. The intersection serves as a set of initial conditions for further computations. It remains to find the solution of the problem $\varphi(\mathbf{t} - \mathbf{s}, \mathbf{y} \cap \Sigma)$. This algorithm is summarized below.

Algorithm 1. Computation of $\varphi(\mathbf{t}, \mathbf{x})$, transversal case:

1) Find s_1 such that $\varphi_k(s_1, \mathbf{x}) \subset R_k$,
2) If $s_1 > \bar{t}$ return $\mathbf{y} = \varphi_k(\mathbf{t}, \mathbf{x})$,
3) Find $s_2 > s_1$ such that $\varphi_k(\mathbf{t}, \mathbf{x}) \subset R_l$,
4) Define $\mathbf{s} = [s_1, s_2]$ and compute $\mathbf{y} = \varphi_k(\mathbf{s}, \mathbf{x})$,
5) Go to step 1 with $\mathbf{x} = \mathbf{y} \cap \Sigma$, $\mathbf{t} = \mathbf{t} - \mathbf{s}$.

The method presented above works when trajectories of interest transversally intersect the C^0 hyperplanes. It has been successfully applied to the analysis of the Chua's circuit for parameter values, for which the attractor does not contain trajectories tangent to planes Σ_i (see [6]).

However, this method fails if some trajectories based at $\mathbf{x}$ are tangent to a plane separating linear regions. This case is handled in the following section.

B. Integration of perturbed dynamical systems

We start by formulating a theoretical result which allows one to compute enclosures of solutions of perturbed systems.

Let us consider an ordinary differential equation

$$\dot{x} = f(x), \tag{4}$$

where $x \in \mathbb{R}^n$ and $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$. Let us assume that we know how to rigorously integrate

$$\dot{x} = g(x), \tag{5}$$

which is a perturbation of (4). The following theorem allows one to study solutions of (4) based on solutions of (5).

Theorem 1: Let $x(t)$ and $y(t)$ be solutions of (4) and (5), respectively. Let us assume that $x(0) = y(0)$, and $x(t), y(t) \in D \subset \mathbb{R}^n$ for $t \in [0, h]$. We also assume that D is a bounded, closed, convex set, and the map g is C^1. Then for $t \in [0, h]$

$$|y_i(t) - x_i(t)| \le \Delta_i, \tag{6}$$

where

$$\Delta = \int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}s, \tag{7}$$

$b_{ij} \ge \sup_{x \in D} \left| \frac{\partial g_i}{\partial x_j}(x) \right|$ for $i \ne j$, $b_{ii} \ge \sup_{x \in D} \frac{\partial g_i}{\partial x_i}(x)$, and $c_i \ge |g_i(x(t)) - f_i(x(t))|$, for $t \in [0, h]$.

The above theorem is a simple conclusion from the results on integration of differential inequalities developed in [8], [9].

C. Tangent intersections

We will use Theorem 1 for rigorous integration of piece-wise linear systems in regions where trajectories are tangent to C^0 planes.

Let us assume that $\hat{x} \in R_k$, the trajectory $\varphi([0, \tau), \hat{x}) \subset R_k$, $\varphi(\tau, \hat{x}) \subset \Sigma$, where Σ is the hyperplane separating the linear regions R_k and R_l. Further, assume that the trajectory $\varphi(t, \hat{x})$ is tangent to Σ at the intersection point $\varphi(\tau, \hat{x})$. The goal is to compute an enclosure of the set $\varphi(\mathbf{t}, \mathbf{x}) = \{\varphi(t, x)\colon x \in \mathbf{x}, t \in \mathbf{t}\}$, for a given interval vector $\mathbf{x}$ containing $\hat{x}$ and $\mathbf{t}$ such that $\underline{t} > \tau$.

To solve the problem we consider the piece-wise linear system (1) as a perturbation of the linear system:

$$\dot{x} = g(x) = A^{(k)} x + v^{(k)}, \tag{8}$$

In this case the matrix B as defined in Theorem 1 is constant, and can be computed as $b_{ij} = |a_{ij}^{(k)}|$ for $i \ne j$ and $b_{ii} = a_{ii}^{(k)}$. One can easily show that

$$\Delta = \int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}s = \int_0^t \mathrm{e}^{Bs} c \mathrm{d}s = t \sum_{i=0}^{\infty} \frac{(Bt)^i}{(i+1)!} \cdot c. \tag{9}$$

When B is invertible the above formula reduces to

$$\Delta = \int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}x = B^{-1} \left(\mathrm{e}^{Bt} - I \right) c. \tag{10}$$

The difference between g and f is zero over the region R_k and for the region R_l can be computed as:

$$g(x) - f(x) = (A^{(k)} - A^{(l)})x + v^{(k)} - v^{(l)}. \tag{11}$$

From the continuity of the vector field f it follows that when the trajectory remains close to the plane Σ the above difference is small.

The procedure starts by finding $s_1 > 0$ such that $\varphi_k([0, s_1], \mathbf{x}) \subset R_k$. The set $\mathbf{u} = \varphi_k(s_1, \mathbf{x})$ serves as an initial condition for integration along the tangency. To reduce overestimation s_1 should be as large as possible. In the second part of the procedure the piece-wise linear system is treated as a perturbed linear system. We select s_2, compute enclosure $\mathbf{v}$ of the solution $\varphi_k([0, s_2], \mathbf{u})$ of the linear system (8). Next, the set $\mathbf{v}$ is increased to form the interval vector $\mathbf{w}$, which serves as a guess of the set containing the solution $\varphi([0, s_2], \mathbf{u})$ of the nonlinear system.

Finally, one computes the vector $c = \sup_{x\in\mathbf{w}} |g(x) - f(x)|$ using (11) and the vector Δ using formula (9) or (10). If $\mathbf{v} + [-1,1]\Delta \subset \mathbf{w}$ we know from the Theorem 1 that the solution of the piecewise linear system is enclosed in $\mathbf{v} + [-1,1]\Delta$. It follows from the Theorem 1 that $\varphi(s_2,\mathbf{u}) \subset \mathbf{z} = \varphi_k(s_2,\mathbf{u}) + [-1,1]\Delta$. If $\mathbf{z} \subset R_k$ and the vector field f over the set $\mathbf{z}$ points away from the plane Σ we can break the computation along the tangency and start integration using the Algorithm 1.

The above procedure can be summarized in the following way (we assume that the transversal case has already been excluded).

Algorithm 2. Computation of $\varphi(\mathbf{t},\mathbf{x})$, tangent case:

1) Find maximum s_1 such that $\varphi_k(s_1,\mathbf{x}) \subset R_k$,
2) Compute $\mathbf{u} = \varphi_k(s_1,\mathbf{x})$,
3) Select $s_2 > 0$ and compute $\mathbf{v} = \varphi_k([0,s_2],\mathbf{u})$,
4) Select $\mathbf{w} \supset \mathbf{v}$,
5) Compute $c = \sup_{x\in\mathbf{w}} |g(x) - f(x)|$ using (11),
6) Compute Δ using (9) or (10),
7) Compute $\mathbf{z} = \varphi_k(s_2,\mathbf{u}) + [-1,1]\Delta$,
8) If $\mathbf{v} + [-1,1]\Delta \subset \mathbf{w}$, $\mathbf{z} \subset R_k$ and the vector field f over the set $\mathbf{z}$ points away from the plane Σ call the Algorithm 1 with $\mathbf{x} = \mathbf{z}$ and $\mathbf{t} = \mathbf{t} - s_1 - s_2$,
9) Go back to step 4 and select larger $\mathbf{w}$ or go back to step 3 and select larger s_2.

III. Examples

A. An illustrative example

As a first example let us consider a simple piecewise-linear planar system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = g\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}a_{11}x_1 + a_{12}x_2 + (|x_1 - 1| - 1)e\\ a_{21}x_1 + a_{22}x_2\end{pmatrix}. \tag{12}$$

For this system the line $\Sigma_1 = \{x\colon x_1 = 1\}$ separates the two linear regions $U_1 = \{x\colon x_1 < 1\}$ and $U_2 = \{x\colon x_1 > 1\}$. The only point at which trajectories are tangent to Σ is $(x_1,x_2) = (1, e - a_{11}/a_{12})$. This can be found be solving the equations: $x_1 = 1$, $\dot{x}_1 = 0$.

To find solutions of the nonlinear system (12) we treat it as a perturbation of the linear system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = f\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}a_{11}x_1 + a_{12}x_2 + (x_1 - 2)e\\ a_{21}x_1 + a_{22}x_2\end{pmatrix}, \tag{13}$$

for which the vector field is equal to the vector field of the nonlinear system when $x_1 > 1$.

Hence, we can get bounds for the solution $y(t)$ of (12) from the solution $x(t)$ of (13) using bound (6) with the following constants:

$$B = \begin{pmatrix}a_{11} + e & |a_{12}|\\ |a_{21}| & a_{22}\end{pmatrix}, c = \begin{pmatrix}\sup_{x\in\mathbf{w}} |(|x_1 - 1| - x_1 + 1)e|\\ 0\end{pmatrix}.$$

Let us select the following parameter values: $a_{11} = 2$, $a_{12} = 1$, $a_{21} = 1$, $a_{22} = 1$, and $e = 2$. In this case the point of tangency is $(x_1,y_1) = (1,0)$. The integration procedure is tested using the set of initial conditions $\mathbf{x} = ([1.004, 1.0045], [-0.099, -0.091]) \subset U_2$ and the integration time $t = 0.2$. The interval vector $\mathbf{x}$ and three trajectories starting from the center and two corners of $\mathbf{x}$ are plotted in Fig. 1. One can see that the trajectory based at the center of $\mathbf{x}$ passes close to the point of tangency, the trajectory based at the upper right corner of $\mathbf{x}$ does not leave the linear region $x > 1$, while the trajectory based at the second corner intersects the line $x = 1$ two times. No matter how we divide $\mathbf{x}$ into smaller rectangles there will always be at least one interval vector containing all three types of trajectories (tangent to Σ, with no intersections with Σ and with two intersections). The tangent case is therefore unavoidable if we want to integrate the system for the whole $\mathbf{x}$.

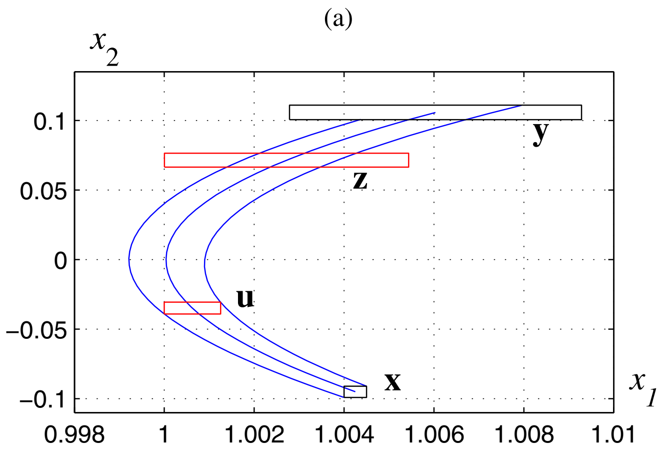

Fig. 1. Two-dimensional piece-wise linear system, rigorous integration along the tangency

Results of applying the Algorithm 2 for computation of enclosure $\varphi(0.2,\mathbf{x})$ are plotted in Fig. 1. Intermediate enclosures $\mathbf{u}$, $\mathbf{z}$, and the final result $\mathbf{y}$ are shown. The set $\mathbf{u} = \varphi_2(s_1,\mathbf{x})$ encloses the solution set at time s_1 when all trajectories are just before intersection with Σ. The rectangle $\mathbf{z} = \varphi_2(s_2,\mathbf{u}) + \Delta$ contains the solution set $\varphi(s_2,\mathbf{u})$ when all trajectories has already passed the tangency area.

In this case $s_1 = 0.06419889$, and one can see that $\mathbf{u} = \varphi_k(s_1,\mathbf{x})$ is a very narrow enclosure of the set of true trajectories. The set $\mathbf{u}$ is relatively large and in consequence the time $s_2 = 0.1044$ needed to pass the tangency for all initial conditions is also large. This results in a considerable overestimation, the width of the set $\mathbf{z}$ is much larger than the width of the true solution set. After the set $\mathbf{z}$ is found the final result $\mathbf{y} = \varphi_2(0.2 - s_1 - s_2, \mathbf{z})$ is computed using formulas for solutions of linear systems.

The diameters of the initial set and the result are $\mathrm{diam}(\mathbf{x}) = (0.0005, 0.008)$, $\mathrm{diam}(\mathbf{y}) = (0.0065, 0.0104)$.

When the diameter of the initial set is reduced to $\mathrm{diam}(\mathbf{x}) = (10^{-5}, 10^{-5})$ the time s_2 needed to pass along the tangency is significantly smaller $s_2 = 0.0215$ and in consequence the overestimation is also reduced. In this case the diameter of the result is $\mathrm{diam}(\mathbf{y}) = (6.63\cdot 10^{-5}, 1.92\cdot 10^{-5})$.

B. The Chua's circuit

As a second example, let us consider a third order piecewise linear electronic circuit [10] described by the following state equation

$$\begin{aligned} C_1\dot{x}_1 &= (x_2 - x_1)/R - g(x_1), \\ C_2\dot{x}_2 &= (x_1 - x_2)/R + x_3, \\ L\dot{x}_3 &= -x_2 - R_0 x_3, \end{aligned} \tag{14}$$

where $g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)$ is a three segment piecewise linear characteristics.

The circuit is studied with the following parameter values (after appropriate parameter rescaling): $C_1 = 1$, $C_2 = 8.3$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$.

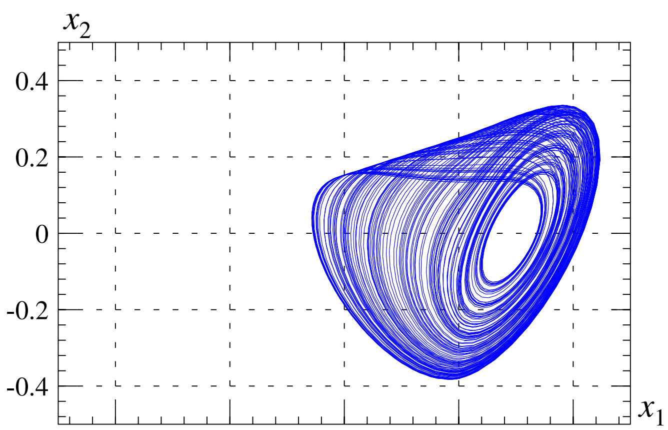

Fig. 2. Computer generated Roessler-type attractor for the Chua's circuit

For the Chua's circuit there are three linear regions $R_1 = \{x \in \mathbb{R}^3 \colon x_1 < -1\}$, $R_2 = \{x \colon |x_1| < 1\}$ and $R_3 = \{x \colon x_1 > 1\}$ separated by planes $\Sigma_1 = \{x \colon x_1 = -1\}$ and $\Sigma_2 = \{x \colon x_1 = 1\}$.

For the parameter values considered the Roessler-type attractor is observed in computer simulations (compare Fig. 2). It can be seen that some trajectories turn close to the plane $x_1 = 1$, which means that intersections with this plane are not always transversal.

Let us consider the problem of finding an enclosure of $\varphi(t, \mathbf{x})$ for the integration time $t = 2$ and the set of initial conditions $\mathbf{x} = ([1.2412, 1.2432], [-0.2141, -0.2121], [-4.7623, -4.7603])$. This initial set has non-empty intersection with the numerically observed attractor and some trajectories based in $\mathbf{x}$ are tangent to the plane Σ_2.

The result and an example trajectory based in $\mathbf{x}$ are shown in Fig. 3. One can see that the trajectory passes along the tangency. The diameters of the initial set and the result are $\mathrm{diam}(\mathbf{x}) = (0.002, 0.002, 0.002)$, $\mathrm{diam}(\mathbf{y}) = (0.0098, 0.0042, 0.041)$. In this case the time the system was integrated as a perturbed linear system was $s_2 = 0.1936$.

In this example the size of initial set is relatively large and the integration time is relatively long thus showing usefulness of the proposed method.

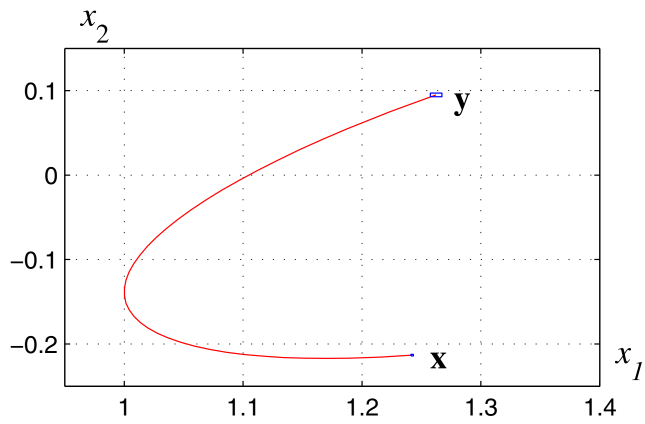

Fig. 3. Chua's circuit, rigorous integration along the tangency

IV. Conclusion

We have studied rigorous integration methods for piecewise linear systems. An algorithm handling also the case of trajectories tangent to hyperplanes separating linear regions has been described. Several examples have been considered to show the effectiveness of this technique.

The methods presented in this work can be used without major modifications for rigorous integration of piece-wise smooth systems, i.e. systems with a continuous vector field, where the state space can be divided into regions where the vector field is C^1. The only difference when compared to piece-wise linear systems is that one cannot use exact formulas for integration in smooth regions and standard techniques for rigorous integration of nonlinear systems has to be used.

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