Rigorous numerical study of the density of periodic windows for the logistic map

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Numerical study of periodic windows for the logistic map is carried out. Accurate rigorous bounds for periodic windows endpoints are computed using interval arithmetic based tools. An efficient method to find the periodic window with the smallest period lying between two other periodic windows is proposed. The method is used to find periodic windows extremely close to selected points in the parameter space and to find a set of periodic windows to minimize the maximum gap between them. The maximum gap reached is 4×10^{-9}. The phenomenon of existence of regions free from low-period windows is explained.

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It is well known that periodic windows densely fill the parameter space $a \in [0, 4]$ of the logistic map $f_a(x) = ax(1-x)$. However, in numerical simulations one observes relatively wide regions with no low-period windows. To explain this phenomenon, an efficient method to find periodic windows filling the parameter space up to a desired precision is proposed and a rigorous numerical study of periodic windows for the logistic map is carried out.

I. INTRODUCTION

The logistic map $f_a(x) = ax(1 - x)$ is a classical example of a single-parameter nonlinear map with complex dynamics[1]. The parameter range $a \in [0, 4]$ of the logistic map is densely filled by periodic windows[2]. Parameter values belonging to periodic windows are called regular. For regular parameter values a unique periodic attractor exists and its basin of attraction has the full measure. Parameter values for which the logistic map supports an absolutely continuous invariant probability measure are called stochastic. For stochastic parameter values trajectories of the logistic map are chaotic and almost all trajectories are asymptotically distributed according to this measure. The measure of the set of stochastic parameters is positive and together with the set of regular parameters has the full measure in the parameter space[2–4].

For $a < 3$ all trajectories of f_a converge to a fixed point. In this work we consider parameter range $a \in [3, 4]$. It is estimated[5] that the measure of the set of regular and stochastic parameter values is $W^* \approx 0.613960301$ and $S^* \approx 0.386039699$, respectively. A lower bound on the measure of the set of regular parameters $W^* > 0.613942108$ is established in Ref. 6. An improved lower bound $W^* > 0.613960137$ is obtained in Ref. 5. The first non-trivial estimate on the measure of the set of stochastic parameter values $S^* > 0.98 \times 10^{-5000}$ was obtained in Ref. 7.

In this work, a numerical study of the density of periodic windows for the logistic map is performed. The classical method to find periodic windows is based on the construction of bifurcation diagrams. In this approach, equidistant parameter values are selected and for each parameter value a sufficiently long trajectory is computed with the hope to find a periodic attractor. Once a periodic attractor is found the continuation method may be used to find endpoints of the corresponding periodic window[8]. This approach is not feasible for periodic attractors with long periods because long periodic attractors usually correspond to narrow periodic windows. In consequence, locating such periodic windows requires very dense sampling of the parameter space. Moreover, periodic attractors with long periods usually require very long trajectories to observe convergence[9–11].

An alternative approach is based on symbolic dynamics. In this approach symbolic representations of trajectories are used to find unstable periodic orbits, which are then continued to locate bifurcation points where stability of periodic orbits changes and periodic attractors are created. This idea is used in Ref. 12 to find periodic windows for the Rössler system close to the classical case. For the logistic map symbolic dynamics is defined in a natural way by the position of the single maximum[13–15]. Moreover, one may introduce the ordering of periodic symbol sequences which is equivalent to the ordering of periodic windows in the parameter space (see Ref. 16 for a detailed description).

In this work, we use ordering of periodic symbol sequences to locate low-period windows close to a given point in the parameter space. The existence of periodic windows is proved and rigorous bounds for endpoints of periodic windows are calculated by applying the interval Newton operator[17] to a map designed in such a way that its zeros correspond to bifurcations of periodic orbits. The proposed method is applied to find periodic windows extremely close to selected points in the parameter space and to construct sets of periodic windows filling the whole parameter space with the goal to minimize the maximum gap between periodic windows.

Computations are carried out using algorithms written in the `C++` programming language and compiled using the `g++`

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compiler, version 9.3.0. The CAPD library[18] is used for interval arithmetic based computations, and the `GNU MPFR` library[19] is used for multiprecision support. Computation times are reported for a single-core 3.1 GHz processor. Parallel computations are utilized to reduce the wall-clock time to obtain the results presented in this work.

For the sake of brevity we use a short notation to denote intervals. For example 3.8^{415}_{284} denotes the interval $[3.8284, 3.8415]$. We will also skip commas when writing symbol sequences and use powers to shorten expressions, for example $s = (0^3 1)$ and $s = (0001)$ both denote the symbol sequence $s = (0, 0, 0, 1)$.

The layout of the paper is as follows. The definition of the logistic map is recalled in Section II. Its basic properties are described and the notations used in the manuscript are defined. Methods to study the existence of periodic windows for the logistic map are presented in Section III. Numerical study of the density of periodic orbits in the parameter space is carried out in Section IV. The set of periodic windows is found with the property that the maximum distance between periodic windows is smaller than 4×10^{-9}. Several parameter values in the interval $a \in [3.5, 4)$ are considered, and periodic windows extremely close to these values are found.

II. LOGISTIC MAP

The *logistic map* is a one-parameter map of the interval $I = [0, 1]$ into itself defined by

$$f_a(x) = ax(1 - x), \tag{1}$$

where $a \in [0, 4]$.

Let us define several notions which will be used in this work and recall some basic properties of the logistic map. All of the material presented in this section is standard and an excellent book on this subject is Ref. 16. Other classical books on analysis of dynamical properties of the logistic map include Refs. 20 and 21. The notions of primary and period-tupling sequences as introduced in Ref. 22 will be used.

The *trajectory* (the *orbit*) of f_a based at x_0 is an infinite sequence of points $(x_k)_{k=0}^{\infty}$ such that $x_{k+1} = f_a(x_k) = f_a^{k+1}(x_0)$ for all $k \geq 0$. We say that x_0 is a fixed point of f_a if $f_a(x_0) = x_0$. We say that $(x_k)_{k=0}^{p-1}$ is a periodic trajectory of f_a with the period p if $f_a^p(x_0) = x_0$ and $x_k = f_a^k(x_0) \neq x_0$ for all k such that $0 < k < p$. Fixed point is a periodic trajectory with the period $p = 1$. We say that a period-p orbit $(x_k)_{k=0}^{p-1}$ is *asymptotically stable* (is a *sink*) if it locally attracts trajectories. Stability of $(x_k)_{k=0}^{p-1}$ can be studied by computing the derivative

$$(f_a^p)'(x_0) = \prod_{k=0}^{p-1} f_a'(x_k) = a^p \prod_{k=0}^{p-1} (1 - 2x_k). \tag{2}$$

If $|(f_a^p)'(x_0)| < 1$ then the orbit is asymptotically stable. If $|(f_a^p)'(x_0)| > 1$ then the orbit is unstable. We say that a is a regular parameter if the map f_a supports a periodic attractor.

A bifurcation of the periodic orbit $(x_k)_{k=0}^{p-1}$ of f_a takes place at values of a at which $|(f_a^p)'(x_0)| = 1$. At a bifurcation point a periodic orbit is born or its stability changes. For the logistic map periodic orbits emerge in a saddle-node or a period-doubling bifurcations when the parameter a is increased. At a saddle node bifurcation the condition $(f_a^p)'(x_0) = 1$ is satisfied and a pair of periodic orbits (one stable and one unstable) is born. At a period-doubling bifurcation the condition $(f_a^p)'(x_0) = -1$ holds, and a period-p orbit becomes unstable and a stable periodic orbit with the period $2p$ is born. Each saddle node bifurcation creating a stable period-p orbit is followed be an infinite sequence of period-doubling bifurcations in which periodic orbits with periods $2^k p$ for $k = 1, 2, \ldots$ emerge. This structure is called a *period-doubling cascade*.

We say that the interval (a_l, a_r) is a *periodic window with the period p* (or *a period-p window*) if for each $a \in (a_l, a_r)$ there exist a period-p orbit $(x_k)_{k=0}^{p-1}$ of f_a satisfying the stability condition $|(f_a^p)'(x_0)| < 1$ and (a_l, a_r) is a maximal interval with this property. Within a periodic window almost all trajectories are attracted to the attractor which is a period-p sink[2].

For $a \in [0, 1]$ the map f_a has a single fixed point $x = 0$, which is a global attractor (all trajectories converge to this fixed point). At $a = 1$ the fixed point $x = 0$ loses stability ($f_a'(0) = a$) and the second fixed point $x^* = 1 - 1/a$ is born. For $a \in (1, 3)$ the condition $|f_a'(x^*)| = |a(1-2x^*)| = |-a+2| < 1$ holds, and hence x^* is asymptotically stable. For $a \in \{1, 3\}$ the derivative $f_a'(x^*)$ is either 1 or -1. It follows that the interval $(1, 3)$ is a period-1 window. One may also show that for $a \in (1, 3)$ the fixed point x^* attracts all trajectories apart from the ones starting at $x = 0$ or $x = 1$. In this work, we consider parameter values $a \in [3, 4]$.

Each period-2 point satisfies the equation $f_a(f_a(x)) - x = 0$, which can be rewritten as $q(x) = -a^3\,x^4 + 2\,a^3\,x^3 - a^3\,x^2 - a^2\,x^2 + a^2\,x - x = 0$. Dividing $q(x)$ by the polynomial $x(ax - a + 1)$ whose zeros are fixed points of f_a leads to the polynomial $-a^2\,x^2 + a^2\,x + a\,x - a - 1$ whose zeros are period–2 points $x_{0,1} = \left(a + 1 \pm \sqrt{(a-3)(a+1)}\right)/(2a)$. These solutions exist and are different for $a > 3$. It follows that for $a > 3$ there exist period-2 orbit (x_0, x_1). The derivative of f_a^2 at period-2 points is $(f_a^2)'(x_{0,1}) = -a^2 + 2a + 4$. From the stability condition $-1 < (f_a^p)'(x_0) < 1$ we obtain $a \in (a_l, a_r) = (3, 1 + \sqrt{6}) \approx (3, 3.44949)$. This is the only period-2 window—the widest periodic window in the considered parameter range $a \in [3, 4]$.

In this work, we use symbolic representations of trajectories and ordering of symbol sequences presented in Ref. 16 (compare also the notion of U-sequences introduced in Ref. 13). With a trajectory $(x_k)_{k=0}^{n-1}$ we associate a symbol sequence $(s_k)_{k=0}^{n-1}$ in such a way that $s_k = 0$ if $x_k < 0.5$ and $s_k = 1$ otherwise. With a periodic orbit $(x_k)_{k=0}^{p-1}$ we associate a periodic symbol sequence $(s_k)_{k=0}^{p-1}$. In the set of infinitely long symbol sequences we introduce the ordering '$\prec$' defined in the following way: $s \prec t$ if $s_k < t_k$ and $\sum_{j=0}^{k-1} s_k$ is even or $s_k > t_k$ and $\sum_{j=0}^{k-1} s_k$ is odd, where k is the smallest index such that $s_k \neq t_k$. For example we have $(0011\ldots) \prec (0010\ldots) \prec (010\ldots) \prec (111\ldots) \prec (101\ldots) \prec (100\ldots)$.

Periodic symbol sequences (also with different periods) are compared using the ordering '$\prec$' after expansion to infinitely long sequences. We say that the periodic sequence $(s_k)_{k=0}^{p-1}$ is *minimal* if it is not larger that any of its cyclic permutations according to the ordering '$\prec$'. For example (01101) is minimal since (01101) $\prec$ (01011) $\prec$ (11010) $\prec$ (10101) $\prec$ (10110). We say that $(s_k)_{k=0}^{p-1}$ is *odd-parity* (*even-parity*) if it contains an odd (even) number of nonzero symbols. Each minimal odd-parity symbol sequence corresponds to a single periodic window. An important property of the ordering '$\prec$' is the relation between ordering of two minimal odd-parity symbol sequences and the position of corresponding periodic windows in the parameter space. Let $\hat{s}$ and $\tilde{s}$ be two minimal odd-parity symbol sequences corresponding to periodic windows $\hat{\mathbf{a}} = (\hat{a}_l, \hat{a}_r)$ and $\tilde{\mathbf{a}} = (\tilde{a}_l, \tilde{a}_r)$. If $\hat{s} \prec \tilde{s}$ then $\hat{\mathbf{a}} > \tilde{\mathbf{a}}$. For example for $\hat{s} = (01101)$ and $\tilde{s} = (001)$ we have $\hat{s} \prec \tilde{s}$ and the corresponding periodic windows are $\hat{\mathbf{a}} \approx 3.8^{415}_{284}$ and $\tilde{\mathbf{a}} \approx 3.7^{411}_{382}$.

Let $(s_k)_{k=0}^{p-1}$ be a minimal odd-parity symbol sequence with the period p. Its even-parity partner is the sequence obtained from s by flipping its second to last symbol. We say that s is a *saddle-node* sequence, if the period of its odd-parity partner is p. Otherwise, the sequence s is called a *period-doubling* sequence. Saddle-node and periodic-doubling sequences correspond to saddle-node and periodic-doubling windows[16]. Each saddle-node window is followed by an infinite sequence of period-doubling windows creating a period-doubling cascade. In this work, the sequence $s = (01)$ will also be called the saddle-node sequence since it starts the first period-doubling cascade in the interval $a \in [3, 4]$.

For example, let us consider two minimal odd-parity period-4 sequences (0001), and (0111). Other symbol sequences of the length 4 are either even-parity, or not minimal, or their minimal period is smaller than 4. The even-parity partner of (0001) is (0011). It has period 4, which means that (0001) is a saddle-node sequence. It corresponds to a period-4 saddle-node window $\mathbf{a} \approx 3.96^{077}_{010}$. The period of the even parity partner (0101) of (0111) is 2, which means that (0111) is a period-doubling sequence. It corresponds to a period-4 window $\mathbf{a} \approx 3.^{5541}_{4495}$ starting at the endpoint of the period-2 window $\mathbf{a} \approx 3.^{4495}_{0000}$ with the symbol sequence (01).

Let s be a minimal odd-parity sequence, and s' its even-parity partner. The sequence which is a concatenation (in arbitrary order) of $k_1 > 0$ copies of s and $k_2 = k - k_1 > 0$ copies of s', where k_1 is odd is called a *period-k-tupling* sequence[10,23,24] generated from s. The assumption that k_1 is odd ensures that the resulting sequence is odd-parity. A minimal odd-parity sequence which is not period-tupling is called *primary*. Period-doubling sequences are the period-2-tupling sequences with $k_1 = 1$ and $k_2 = 1$. We will use the following notation to denote period-tupling sequences. Let t be an odd-parity sequence of the length m. The period-tupling sequence generated from s by the sequence t is denoted by s^t and is defined as $s^t = (\kappa(s, s', t_0), \kappa(s, s', t_1), \ldots, \kappa(s, s', t_{m-1}))$ where $\kappa(s, s', 1) = s$ and $\kappa(s, s', 0) = s'$. As an example let us consider $s = (001)$ and its odd-parity partner $s' = (011)$. $s^{(10)} = (ss') = (001\,011)$ is the period-doubling sequence generated from s, $s^{(100)} = (ss's') = (001\,011\,011)$ is the only period-tripling sequence generated from s, while $s^{(1000)} = (ss's's') = (001\,011\,011\,011)$ and $s^{(1011)} = (ss'ss) = (001\,011\,001\,001)$ are the two period-quadrupling sequences generated from s. Period-tupling sequences are important because the corresponding windows have usually larger widths than periodic windows corresponding to primary sequences with the same period[5].

Let us assume the s is a saddle-node sequence. The sequence s corresponds to a saddle-node periodic window followed by an infinite sequence of period-doubling windows. The corresponding periodic sequences will be denoted by $\delta(s, k)$ $(k = 0, 1, 2, \ldots)$ in a recursive way. More precisely, we define $\delta(s, 0) = s$ and $\delta(s, k+1) = \delta(s, k)^{(10)}$. For example for $s = (01)$ we have $\delta(s, 1) = (01)^{10} = (01\,11)$, $\delta(s, 2) = (0111)^{10} = (0111\,0101)$, and $\delta(s, 3) = (01110101)^{10} = (01110101\,01110111)$. For $s = (001)$ we have $\delta(s, 1) = (001)^{10} = (001\,011)$, $\delta(s, 2) = (001011)^{10} = (001011\,001001)$, and $\delta(s, 3) = (001011001001)^{10} = (001011001001\,001011001011)$.

III. METHODS TO FIND PERIODIC WINDOWS IN A GIVEN REGION OF THE PARAMETER SPACE

In this section, we first recall the method introduced in Ref. 5 to find endpoints of the periodic window corresponding to a given minimal symbol sequence. Then, we present a procedure to find the minimal symbol sequence with the smallest period lying between two minimal symbol sequences according to the ordering '$\prec$'. Based on these procedures we propose methods to construct periodic windows filling the whole parameter space to reduce the maximum gap between periodic windows to a given value $g_{\max}$, and to find a sequence of periodic windows arbitrarily close to a given point in the parameter space.

A. Finding the periodic orbit of f_a with a given symbol sequence

Let $s = (s_k)_{k=0}^{p-1}$ be an odd-parity minimal symbol sequence. Let a be a parameter value such that the periodic orbit $(x_k)_{k=0}^{p-1}$ of f_a with the symbol sequence s exists and is unstable. Here, we present a method to find an accurate approximation of the position of the periodic orbit $(x_k)_{k=0}^{p-1}$.

First, we construct a sequence of intervals $\mathbf{x}_{-k}$ for $k = 0, 1, 2, \ldots$ with decreasing widths each containing a point belonging to the orbit $(x_k)_{k=0}^{p-1}$. More precisely, we require that the condition $x_{(-k) \bmod p} \in \mathbf{x}_{-k}$ is satisfied for each $k \geq 0$.

The initial interval interval is selected as $\mathbf{x}_0 = [0, f_a(0.5)]$. In the kth step we consider the interval $\mathbf{x}_{-k}$. Its preimage under f_a is the union of two intervals. As $\mathbf{x}_{-(k+1)}$ we select the interval for which the symbol is $s_{(-k-1) \bmod p}$. Widths of intervals $\mathbf{x}_{-k}$ decrease because the periodic orbit $(x_k)_{k=0}^{p-1}$ is unstable. This process is continued until the width of $\mathbf{x}_{-mp}$ for some $m > 0$ is smaller then a predefined small value ε (in computations $\varepsilon = 10^{-16}$ is used). The middle of the interval $\mathbf{x}_{-mp}$ is selected as an approximation of x_0. To obtain the whole orbit we compute $p - 1$ preimages of this point selecting in each step from the two preimages of the current iterate the one with the correct symbol sequence.

B. Finding rigorous bounds of periodic windows endpoints

Let $s = (s_k)_{k=0}^{p-1}$ be an odd-parity minimal symbol sequence. Let us briefly recall the method to find endpoints of the periodic windows corresponding to s (for a detailed description see Ref. 5). Endpoints of periodic windows are studied using the map $H_{\lambda_0}\colon \mathbb{R}^{p+1} \mapsto \mathbb{R}^{p+1}$ defined by

$$H_\lambda \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \\ a \end{pmatrix} = \begin{pmatrix} x_1 - f_a(x_0) \\ x_2 - f_a(x_1) \\ \vdots \\ x_0 - f_a(x_{p-1}) \\ (f_a^p)'(x_0) - \lambda \end{pmatrix} \tag{3}$$

where $\lambda = \pm 1$. Zeros of $H_{\lambda=-1}$ and $H_{\lambda=+1}$ correspond to saddle-node and period-doubling bifurcations of the map f_a. To prove the existence of bifurcations and to obtain rigorous bounds for endpoints of periodic windows one applies the interval Newton operator[17] to the map H_λ. Techniques for efficient evaluations of the standard (real valued) Newton method and the interval Newton method using forward and backward shooting are presented in Refs. 5 and 25.

To find the right endpoint we select a lying to the right of the periodic window of interest. One may always select $a = 4$, however if a better upper bound is known, then it should be used to improve the convergence of the Newton method. Next, using the method presented in Sec.III A, we find an approximate position of the periodic orbit $(x_k)_{k=0}^{p-1}$ of f_a with the symbol sequence s. Then using the backward shooting version of the Newton method applied to the map $H_{\lambda=-1}$ an approximate position of the periodic window's right endpoint a_r and the corresponding periodic orbit are found. Next, we prove the existence of a zero of $H_{\lambda=-1}$ by applying the interval Newton operator to the map $H_{\lambda=-1}$. In this way we obtain rigorous bounds for the position of the right endpoint of the periodic window.

The method to find the left endpoint a_l depends on the type of the symbol sequence s. If s is a saddle-node symbol sequence, then we use the forward shooting version of the Newton method applied to the map $H_{\lambda=1}$ to find an approximate position of the periodic window's left endpoint and the corresponding periodic orbit. The initial condition for the Newton method is an approximate position of the periodic window's right endpoint a_r and the corresponding periodic orbit. The existence of a zero of the map $H_{\lambda=1}$ is proved and rigorous bounds for the position of the left endpoint are found using the interval Newton operator applied to the map $H_{\lambda=1}$. When s is a period-doubling sequence then its left endpoint is the same as the right endpoint of its parent window and can be found using the method to find the right endpoint described before.

C. Bisection in the set of periodic symbols sequences

In this section, we propose a procedure to find the minimal odd-parity symbol sequence with the smallest period lying between two minimal odd-parity symbol sequences.

Let us first describe a procedure to increment a finite length sequence $s = (s_0, s_1, \ldots, s_{k-1})$. If s is even-parity then incrementing s is equivalent to flipping the last symbol. For example incrementing (110000) yields (110001) and incrementing (110011) yields (110010). To increment an odd-parity sequence we need to find the last nonzero symbol and flip the previous one. For example incrementing (101100) yields (100100). If the last nonzero symbol is the first element of the sequence s (for example $s = (100000)$) then this is already the largest sequence of a given length and it cannot be increased.

Let us now consider two minimal odd-parity symbol sequences $\underline{s}$ and $\overline{s}$ with periods $\underline{n}$ and $\overline{n}$ such that $\underline{s} \prec \overline{s}$. To find s such that $\underline{s} \prec s \prec \overline{s}$ we first find the largest m such that $\underline{s}_j = \overline{s}_j$ for all $j < m$. The sequence $\underline{s}_0\underline{s}_1 \cdots \underline{s}_{m-1} = \overline{s}_0\overline{s}_1 \cdots \overline{s}_{m-1}$ is the common beginning of sequences $\underline{s}$ and $\overline{s}$.

Next, for $l = 1, 2, \ldots$ we search for a minimal odd-parity sequence s of the length $m + l$ with the property $\underline{s} \prec s \prec \overline{s}$. For a given $l > 0$ we start with the sequence s such that $s_j = \underline{s}_{j \bmod \underline{n}}$ for all $0 \leq j < m+l$. In each step, we verify whether s is minimal, odd-parity and $\underline{s} \prec s \prec \overline{s}$. If all these conditions hold then the search process is stopped—the sequence has been found. Otherwise we increase the sequence s and repeat the computations. Computations for a given l are also stopped when we reach a sequence s such that $\overline{s} \prec s$.

This procedure works properly when sequences $\underline{s}$ and $\overline{s}$ are of a similar length. For example, for $\underline{s} = (01^3010101^301^30101)$ and $\overline{s} = (01^3010101^301^3)$ of the length 20 and 16 finding the sequence $s = (01^3010101^301^301^301^3)$ of the length 24 such that $\underline{s} \prec s \prec \overline{s}$ requires considering 38 test sequences. Sometimes however it is necessary to consider sequences of completely different length for which the proposed approach becomes infeasible. For example, for $\underline{s} = (001)$ and $\overline{s} = ((011)^901)$ we need to consider 272696402 symbols sequences to find the sequence $s = ((011)^{10}1)$ satisfying all the conditions. The computations last approximately one minute.

One possible improvement is to skip non-smallest sequences once such a sequence s is detected. The idea is to find the smallest position of the sequence s such that the sequence starting at this position is smaller than s and replace this sequence by the beginning of the sequence s. Introducing this modification reduces the number of test sequences in the considered example to 479. Another possibility is to record the last acceptable sequence for a given l and use it as a starting point for the next step with $l + 1$. Using this modification reduces the number of test sequences to 124. It is even better to use both improvements simultaneously. In this case the number of test sequences is reduced to 118.

Note that without these two improvements we would not be able to obtain results presented in this work where the length of some sequences exceeds 20000.

D. Finding periodic windows very close to a selected point in the parameter space

Let us select a parameter value a^* (for example $a^* = 3.9$). We would like to verify whether this point is regular (belongs to a periodic window) or not. If a^* is regular then we may be able to prove it by finding a periodic window containing a^*. Otherwise, since periodic windows are dense in the parameter space, we should be able to find periodic windows arbitrarily close to a^*. In this section we present a method to either prove that a selected value a^* is regular or to find a sequence of periodic windows with positions converging to a^*.

The procedure starts with two symbol sequences $\hat{s}$ and $\tilde{s}$ and corresponding periodic windows $\hat{\mathbf{a}}$ and $\tilde{\mathbf{a}}$ lying on the opposite side of a^*, i.e. $\hat{\mathbf{a}} < a^* < \tilde{\mathbf{a}}$. As a lower bound we may always select the periodic window $\hat{\mathbf{a}} = (3, 1 + \sqrt{6})$ with the symbol sequence $\hat{s} = (01)$. As an upper bound we may select a periodic window with the sequence $\tilde{s} = (0^k 1)$ with a sufficiently large k. Positions of these periodic windows converge to $a = 4$ when k goes to infinity. For example, for the sequences $s = (0^5 1)$ and $s = (0^{10} 1)$ the corresponding periodic windows are $\mathbf{a} = (a_l, a_r) \approx 3.99758^{490}_{252}$ and $\mathbf{a} = (a_l, a_r) \approx 3.9999976468^{908}_{885}$, respectively.

In the kth step of the search procedure, we find the minimal odd-parity symbol sequence s with the smallest length such that $\tilde{s} \prec s \prec \hat{s}$ and the corresponding periodic window $\mathbf{a}$. If the selected parameter value a^* belongs to $\mathbf{a}$ then the computations are stopped—it has been shown that a^* is a regular parameter and that f_a supports a sink. In the opposite case we verify the condition $\mathbf{a} < a^*$. If this condition holds then we assign $\hat{s} = s$ and $\hat{\mathbf{a}} = \mathbf{a}$. Otherwise, we assign $\tilde{s} = s$ and $\tilde{\mathbf{a}} = \mathbf{a}$. In the following section we show several examples that this procedure may be applied to prove that a given parameter is regular (belongs to a periodic window) or find periodic windows extremely close to this point.

E. Filling the parameter space by periodic windows

Methods presented above may be used to find periodic windows filling the parameter space in such a way that the maximum gap between periodic windows is minimized. The following procedure is proposed. First, we select two periodic windows which define bounds of the region in which we search for periodic windows and compute the gap between these two windows. In order to fill the parameter range $a \in [3, 4]$ one may select as a lower bound the periodic window $(3, 1 + \sqrt{6})$ with the symbol sequence (01) and as an upper bound a periodic window with the sequence $(0^k 1)$ for some large k. In this study, the symbol sequence $(0^{39} 1)$ is selected. The corresponding periodic window lies closer than 10^{-20} from the point $a = 4$. During the procedure we store the list of gaps and for each gap periodic windows between which this gap exists along with the corresponding symbol sequences. The list is sorted according to decreasing gap. In each step of the algorithm we pop the first element in the list (the largest gap) and find a periodic window with the smallest period enclosed in this gap. Adding this new periodic window creates two gaps (if the periodic window is of a saddle-node type) or a single gap (if the periodic window is of a period-doubling type). Created gaps are added to the list of gaps unless the gap is smaller than some predefined value $g_{\max}$. The procedure is continued until the list of gaps is not empty. The procedure is designed in such a way that first low-period windows are found. Therefore, the proposed procedure may be used to compute the minimum value of $p_{\max}$ such that it is impossible to reduce the maximum gap to $g_{\max}$ without considering periodic windows with the period $p_{\max}$. Examples are given in the next section.

IV. NUMERICAL RESULTS

In this section, methods presented in Section III are used to carry out a numerical study of the density of periodic windows in the parameter space. The parameter space is $a \in [0, 4]$. As mentioned before for $a \leq 1$ all trajectories converge $x = 0$, while for $a \in (1, 3)$ all trajectories converge to the fixed point $x^* = 1 - 1/a$ apart from trajectories starting at $x = 0$ and $x = 1$. In the following, we consider the parameter range $a \in [3, 4]$ and periodic windows with periods $p \geq 2$. It is estimated that the total width or periodic windows in the interval $a \in [3, 4]$ is $W^* \approx 0.613960301$ (see Ref. 5).

A. Low-period windows

Let us first consider the problem what maximum gap is obtained by finding all periodic windows with short periods. To analyze this problem all periodic windows with periods smaller than 24 are found. The results are presented in Table I. For $p_{\max} = 2, 3, \ldots, 24$, we report the number of periodic windows with periods $p \in \{2, 3, \ldots, p_{\max}\}$, their total width W, the minimum width $w_{\min}$ and the the maximum gap $g_{\max}$ between them. In the last row we show results obtained by considering all periodic windows (with infinitely large periods). The number of all periodic windows is infinite, their minimum width goes to zero and the maximum gap between them also goes to zero. This is the consequence of density of periodic windows in the whole parameter range[2]. The number of periodic windows grows exponentially with $p_{\max}$ approximately as fast as $p_{\max}^{-1} 2^{p_{\max}}$. From the results presented in Table I it follows that the minimum width decreases faster than $10^{-p_{\max}}$. On the other hand the convergence of the total width and the maximum gap is slow. For $p_{\max} = 24$ there are 732699 periodic windows and the maximum gap is slightly below 0.005. Also note that in several cases increasing $p_{\max}$ does not result in decreasing $g_{\max}$ (there is no change in $g_{\max}$ when $p_{\max}$ is increased from 20 to 23.. It follows that finding all short periodic windows is not an effective way to densely cover the parameter space by periodic windows. In the following section, we show that it is sufficient to consider much fewer periodic windows to obtain the required dense covering. For example, only 134 periodic windows are sufficient to obtain the maximum gap $g_{\max} < 0.0048$.

From the results presented in Table I it follows that there are relatively wide regions in the parameter space free from low-period windows. This phenomenon is discussed in the following sections.

TABLE I. Periodic window with periods $p \in \{2, 3, \ldots, p_{\max}\}$ for $p_{\max} \leq 24$, n is the number of periodic windows, W is their total width, $w_{\min}$ is the minimum width, $g_{\max}$ is the maximum gap between periodic windows.

$p_{\max}$	n	W	$w_{\min}$	$g_{\max}$
2	1	0.4494897	0.4495	0.550510257
3	2	0.4625616	0.01307	0.378937382
4	4	0.5578290	0.000667	0.284336765
5	7	0.5613518	3.89×10^{-5}	0.194082016
6	12	0.5714054	2.38×10^{-6}	0.107783675
7	21	0.5723755	1.47×10^{-7}	0.082462802
8	37	0.5935746	9.20×10^{-9}	0.062145896
9	65	0.5940991	5.75×10^{-10}	0.062145896
10	116	0.5968048	3.59×10^{-11}	0.040800801
11	209	0.5969264	2.24×10^{-12}	0.040800801
12	379	0.6011035	1.40×10^{-13}	0.022396287
13	694	0.6011476	8.77×10^{-15}	0.022396287
14	1279	0.6018930	5.48×10^{-16}	0.017615735
15	2370	0.6020317	3.42×10^{-17}	0.017615735
16	4418	0.6068763	2.14×10^{-18}	0.013263582
17	8273	0.6068860	1.34×10^{-19}	0.013263582
18	15553	0.6073453	8.36×10^{-21}	0.013263582
19	29350	0.6073500	5.22×10^{-22}	0.013263582
20	55564	0.6084532	3.27×10^{-23}	0.008752822
21	105493	0.6084891	2.04×10^{-24}	0.008752822
22	200818	0.6085821	1.28×10^{-25}	0.008752822
23	383179	0.6085834	7.97×10^{-27}	0.008752822
24	732699	0.6099670	4.98×10^{-28}	0.004809769
∞	∞	0.6139603	0	0

TABLE II. The number n of periodic window needed to reduce the maximum gap to $g_{\max}$, W is the total width of periodic windows, $p_{\max}$ is the maximum period of periodic windows.

$g_{\max}$	n	W	$p_{\max}$
10^{-1}	8	0.56566265371	7
10^{-2}	65	0.60371032367	20
10^{-3}	647	0.61068053942	56
10^{-4}	6180	0.61269502716	160
10^{-5}	60445	0.61351094397	448
10^{-6}	598317	0.61379154345	1280
10^{-7}	5952254	0.61389617774	3842
10^{-8}	59202328	0.61393593857	12154
4×10^{-9}	147690838	0.61394355010	19216

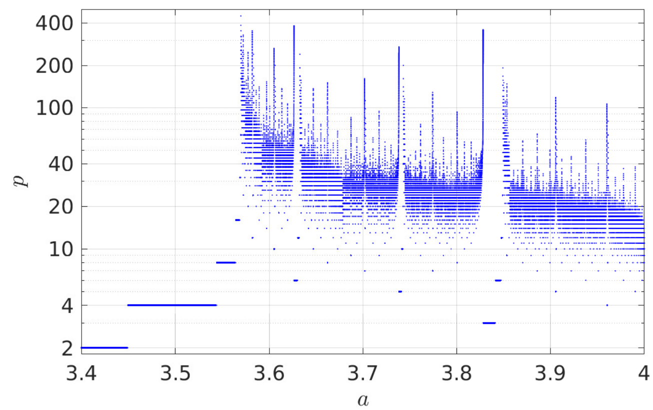

FIG. 1. Periods of periodic windows needed to reduce the maximum gap to $g_{\max} = 10^{-5}$ versus their positions in the parameter space.

B. Density of periodic windows

Search for periodic windows to minimize gaps between them is carried out using the procedure presented in Section III E. Results obtained for various values of the maximum gap $g_{\max}$ are presented in Table II. For each case, we report the number n, the total width W, and the maximum period $p_{\max}$ of periodic windows needed to reduce the maximum gap to $g_{\max}$. The total width of periodic windows in the interval $a \in [3, 4]$ is $W^* \approx 0.613960301$ (see Ref. 5). It follows that one needs at least $(1 - W^*)/g_{\max}$ periodic windows to obtain the maximum gap $g_{\max}$. Note that the number of periodic windows needed to obtain $g_{\max} = 4 \times 10^{-9}$ is $n = 147690838$, which is approximately 50% more than $(1 - W^*)/4 \times 10^{-9} \approx 9.65 \times 10^7$. The maximum period of periodic windows found is $p_{\max} = 19216$.

The total computation time to find 147690838 periodic windows is 1.416×10^7 seconds. The average time to find a periodic window is approximately 0.096 s.

Periods of the set of 60445 periodic windows with the maximum gap $g_{\max} < 10^{-5}$ versus their positions in the parameter space are plotted in Fig. 1. One may see several regions containing periodic windows with high periods. This phenomenon is explained in detail in Section IV C.

Fig. 2(a) shows the number of periodic windows needed to obtain a given maximum gap $g_{\max}$. The plot is almost linear in the logarithmic scale for small gaps. From this plot one may estimate that approximately 6×10^8 and 6×10^9 periodic windows are needed to obtain $g_{\max} = 10^{-9}$ and $g_{\max} = 10^{-10}$, respectively.

The procedure proposed in Section III E gives preferences to periodic windows with low periods. It follows that the results obtained during computations may be used to find what periods are are needed to reduce the maximum gap to a given value. The results are plotted in Fig. 2(b). They are in agreement with the results presented in Table I. For example, some periodic windows with periods 20 and 24 have to be included to reduce the maximum gap to 0.0088 and 0.0048, respectively. The existence of horizontal segments in Fig. 2(b) indicates that including certain additional periodic windows with a given period reduces the maximum gap size. The existence of jumps in $p_{\max}$ larger than one confirms that for some p including all period-p windows does not reduce the maximum gap.

The difference $W^* - W$ between the total width $W^* \approx 0.613960301$ of all periodic windows and the total width W of periodic window found in the process of reducing the maximum gap to $g_{\max}$ is plotted in Fig. 3. From this plot and Fig. 2 one may estimate what is the number of periodic win-

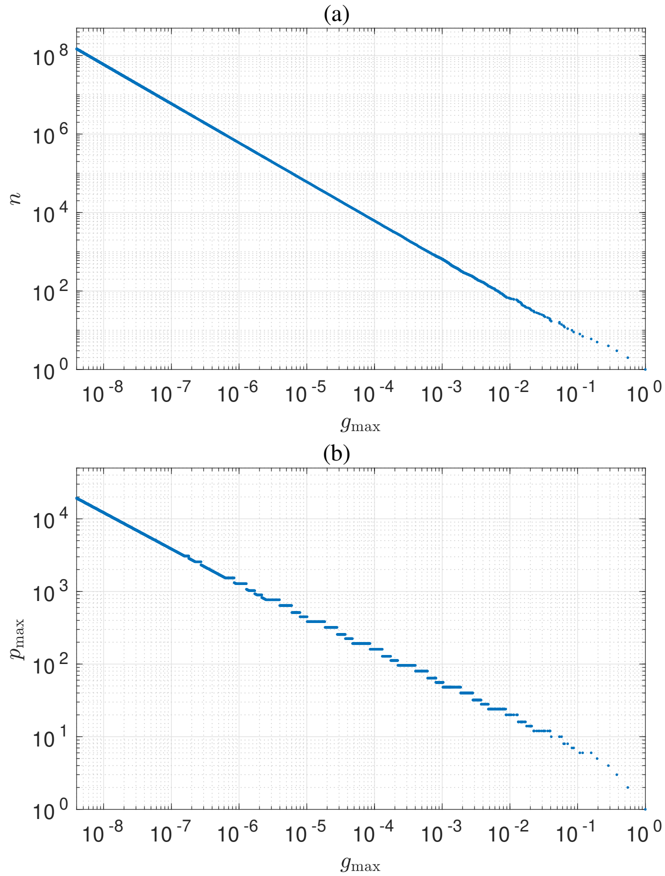

FIG. 2. The number n of periodic windows (a) and the maximum period $p_{\max}$ of periodic windows (b) needed to reduce the maximum gap to $g_{\max}$.

dows which has to be found using the proposed approach to achieve a given accuracy in finding a lower bound for W^*.

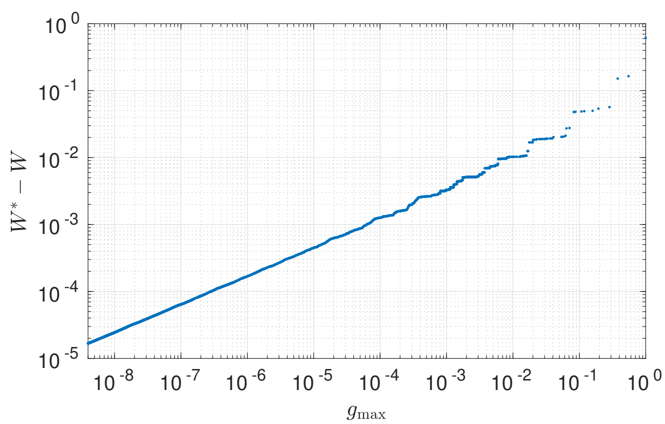

FIG. 3. The difference $W^* - W$ between the total width W^* of periodic windows and the total width W of periodic window found when reducing the maximum gap to $g_{\max}$.

The maximum gap between periodic windows found is smaller than $g_{\max} = 4 \times 10^{-9}$. It follows that all periodic windows wider than 4×10^{-9} are found. There are 13962 such periodic windows. Their total width is $W \approx 0.61388120$. Out of these periodic windows the maximum period $p_{\max} = 8192 = 2^{13}$ is observed for the periodic window with the symbol sequence $\delta((01), 12)$ belonging to the first period-doubling cascade. Results regarding periodic windows wider than 4×10^{-9} are collected in the first row of Table III.

TABLE III. Properties of wide periodic windows, n is the number of periodic windows wider than $w_{\max}$, W is their total width, $p_{\max}$ is the maximum period, the results for $w_{\max} = 4 \times 10^{-9}$ are rigorous.

$w_{\max}$	n	W	$p_{\max}$
4×10^{-9}	13962	0.61388120	8192
10^{-9}	31484	0.61391590	8192
10^{-10}	121144	0.61394327	32768

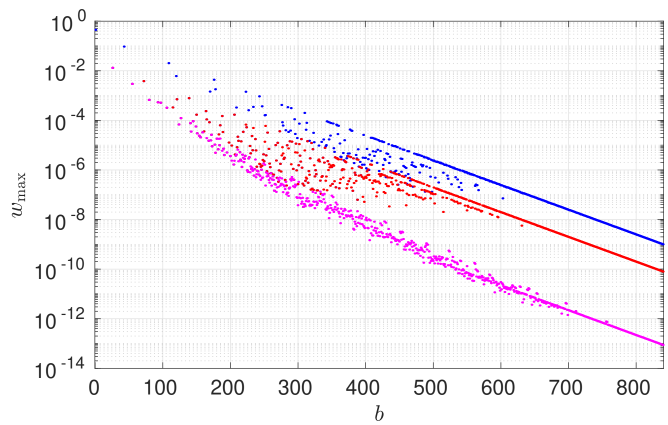

FIG. 4. The maximum width of all periodic windows (blue, upper plot), saddle node periodic windows (red, middle plot), and primary windows (magenta, lower plot) found for gaps $g_{\max} \in (10^{-0.01b}, 10^{-0.01(b-1)}]$, $b \in \{0, 1, 2, \ldots, 840\}$.

The width of a periodic window found by the procedure between two periodic windows is usually only a small fraction of the gap between these windows. Analyzing widths of periodic windows returned by the procedure in subsequent steps one may estimate what is the maximum width of remaining periodic windows. Fig. 4 shows how the maximum width changes during computations when the maximum gap is decreased. The gap range is divided into bins $g_{\max} \in (10^{-0.01b}, 10^{-0.01(b-1)}]$, for $b = 1, 2, \ldots, 840$ and for each bin the maximum width of a periodic window obtained when processing this bin is computed. The results are plotted in Fig. 4 in blue. Similar results for saddle-node windows and primary windows are plotted in Fig. 4 in red and magenta. One can see that for large b the maximum widths of primary windows are approximately 1000 times smaller than the maximum widths of saddle-node windows, and that the maximum widths of saddle-node windows are more than 10 times smaller than the maximum widths of all periodic windows. It follows that period-doubling windows are in general wider than saddle-node windows, and that period-tupling windows are wider than primary windows. For large b all three plots decrease in a linear fashion in the logarithmic scale. For the last bin the bounds are $w_{\max} < 1.025 \times 10^{-9}$ for period-doubling windows, $w_{\max} < 8.186 \times 10^{-11}$ for saddle-node windows, and $w_{\max} < 8.975 \times 10^{-14}$ for primary windows (saddle-node windows which are not period-tupling). These results suggest that in the process of reducing the gap below 4×10^{-9} the majority of periodic windows wider than 10^{-9} are found. Indeed, only a single period-doubling window with the width $w \approx 1.02754 \times 10^{-9}$ is missing. It is found when searching for period-doubling descendants of wide periodic windows (see the next paragraph for more information). Including this window we obtain 31484 periodic windows with the width above 10^{-9}. Their total width is approximately 0.6139159. The results regarding periodic windows wider than 10^{-9} are presented in the second row of Table III.

From the results on widths of saddle-node windows we may claim that all saddle-node windows wider than 10^{-10} are found. We may use these results to find all period-doubling windows wider than 10^{-10}. To this end all windows wider than 10^{-10} not followed by period-doubling descendants are identified and period-doubling windows emerging from these windows are computed until the width of a descendant is wider than 10^{-10}. This way 26257 new period-doubling windows with widths above 10^{-10} are found including a single period-doubling window wider than 10^{-9}. Their total width is approximately 7.127961×10^{-6}. Summarizing, we find 121144 periodic windows wider than 10^{-10} and we claim that all such windows have been found. This statement is in the full agreement with the results presented in Ref. 5, where wide periodic windows are found using a different approach. The total width of 121144 periodic windows wider than 10^{-10} is above 0.61394327. This lower bound of the total width of periodic windows is larger than the lower bound 0.6139421 reported in Ref. 6. The results regarding periodic windows wider than 10^{-10} are presented in the last row of Table III.

To further improve the lower bound 0.61394327 for the measure of periodic windows we search for period-doubling descendants of periodic windows found before. The search is limited to period-doubling windows wider than 10^{-13}. This way we find 1998897 new period-doubling windows with widths $w \in [10^{-10}, 10^{-13}]$. This way we obtain the following lower bound for the total width of all periodic windows

$$W^* > 0.6139565$$

which is close to the lower bound 0.6139601 obtained in Ref. 5, where the search method was optimized to find wide periodic windows.

The results on widths of primary windows found suggest that all primary windows with the width above 10^{-13} are found. This knowledge may be used to generate all period-tupling descendants of these windows to obtain all periodic windows wider than 10^{-13} and to further improve the lower bound for the total width of all periodic windows.

C. Regions free from low-period windows in the parameter space

The results presented in Sections IV A and IV B show that there are relatively wide regions in the parameter space free from low-period windows. For example, the number of periodic windows with periods $p < 40$ is approximately $1.4{\times}10^{10}$. In spite of a huge number of such periodic windows, the maximum gap between them is $g_{\max} \approx 0.00284$. Further analysis of these results reveals three major reasons responsible for this phenomenon. For each case, we provide examples of periodic windows with periods above $p = 16000$, which have to be included to reduce the maximum gap size below 10^{-9}.

The first one is the existence of wide period-doubling windows. An example plotted in Fig. 1 is the period-256 window $\mathbf{a} \approx 3.5699^{432}_{340}$ with the symbol sequence $s = \delta((01), 7)$. Table IV presents widths of periodic windows belonging to period doubling cascades starting at the two widest windows $\mathbf{a}_2 \approx 3.^{4495}_{0000}$ and $\mathbf{a}_3 \approx 3.8^{415}_{284}$ with the symbol sequences (01) and (001), respectively. The periodic window with the symbol sequence $\delta((01), 13)$ belonging to the period-doubling cascade starting at $\mathbf{a}_2 \approx 3.^{4495}_{0000}$ has the period $p = 16384$ and the width $w \approx 8.8375 \times 10^{-10}$. The total width of this periodic window and its period-doubling descendants is above $1.12{\times}10^{-9}$. It is clear that without considering periodic windows with periods above 16000 one cannot reduce the gap after the first period-doubling cascade below 1.1×10^{-9}. Results for the second widest period-doubling cascade are given in the second part of Table IV. These periodic windows are narrower, and hence one needs shorted periods to reduce the gap after this period-doubling cascade below $g_{\max} = 10^{-9}$.

TABLE IV. Periods and widths of periodic windows belonging to period-doubling cascades of period-2 and period-3 windows.

s	p	w
$\delta((01), 10)$	2048	8.9962×10^{-8}
$\delta((01), 11)$	4096	1.9267×10^{-8}
$\delta((01), 12)$	8192	4.1264×10^{-9}
$\delta((01), 13)$	16384	8.8375×10^{-10}
$\delta((01), 14)$	32768	1.8927×10^{-10}
$\delta((001), 9)$	1536	3.0271×10^{-8}
$\delta((001), 10)$	3072	6.4831×10^{-9}
$\delta((001), 11)$	6144	1.3885×10^{-9}
$\delta((001), 11)$	12288	2.9737×10^{-10}

The second reason is involved with narrow periodic windows located in the parameter space before low-period saddle-node windows. Examples in Fig. 1 include peaks located just before saddle node windows with symbol sequences $s =$ (011111), $s =$ (01101), and $s =$ (001). The corresponding periodic windows are $\mathbf{a}_6 \approx 3.6^{304}_{266}$, $\mathbf{a}_5 \approx 3.7^{411}_{382}$, and $\mathbf{a}_3 \approx 3.8^{415}_{284}$. Periods of preceding windows plotted in Fig. 1 are 382, 271, and 358, respectively. Let us first consider the period-3 window $\mathbf{a}_3 \approx 3.8^{415}_{284}$ with the symbol sequence (001). One can show that for each $k \geq 1$ there are no periodic windows with the period $p \leq 3k + 1$ between the periodic window with the symbol sequence $((011)^k 1)$ and $\mathbf{a}_3 \approx 3.8^{415}_{284}$. Properties of periodic windows with symbol sequences $((011)^k 1)$ are collected in Table V. For selected values of k we present the period $p = 3k + 1$, the window's width w and the distance from the periodic window $\mathbf{a}_3 \approx 3.8^{415}_{284}$. From these results it follows that one has to consider periodic windows with periods above 33000 to reduce the gap before $\mathbf{a}_3$ below 1.1×10^{-9}.

TABLE V. Properties of periodic windows with sequences $(011)^k 1$, p is the period, w is the window's width, g is the distance from the periodic window with the sequence (001).

s	p	w	g
$(011)^{20}1$	61	2.878×10^{-9}	3.473×10^{-4}
$(011)^{200}1$	601	2.536×10^{-12}	3.546×10^{-6}
$(011)^{2000}1$	6001	2.546×10^{-15}	3.559×10^{-8}
$(011)^{4000}1$	12001	3.184×10^{-16}	8.900×10^{-9}
$(011)^{8000}1$	24001	3.980×10^{-17}	2.225×10^{-9}
$(011)^{10000}1$	30001	2.038×10^{-17}	1.424×10^{-9}
$(011)^{11000}1$	33001	1.531×10^{-17}	1.177×10^{-9}
$(011)^{12000}1$	36001	1.179×10^{-17}	9.890×10^{-10}

A a second example let us consider the periodic window $\mathbf{a}_6 \approx 3.6^{304}_{266}$ with the symbol sequence (011111). In this case, one can show that the shortest periodic symbol sequence between sequences between (011111) and $((011101)^k 01)$ is $((011101)^k 0111)$, i.e $(011111) \prec ((011101)^k 0111) \prec ((011101)^k 01)$. If follows that there are no periodic windows with periods $p \leq 6k + 2$ between $\mathbf{a}_6$ and the periodic window with the sequence $((011101)^k 01)$. Properties of periodic windows corresponding to symbol sequences $((011101)^k 01)$ are collected in Table VI. From the result obtained for $k = 6000$, it follows that one has to consider periodic windows with periods $p > 36000$ to reduce the gap before $\mathbf{a}_6$ below 1.1×10^{-9}.

TABLE VI. Properties of periodic windows with sequences $(011101)^k 01$, p is the period, w is the window's width, g is the distance from the periodic window with the sequence (011111).

s	p	w	g
$(011101)^{10}01$	62	1.360×10^{-8}	3.836×10^{-3}
$(011101)^{100}01$	602	8.483×10^{-12}	4.056×10^{-6}
$(011101)^{1000}01$	6002	8.558×10^{-15}	4.099×10^{-8}
$(011101)^{2000}01$	12002	8.558×10^{-15}	4.099×10^{-8}
$(011101)^{4000}01$	24002	1.339×10^{-16}	2.564×10^{-9}
$(011101)^{5000}01$	30002	6.856×10^{-17}	1.641×10^{-9}
$(011101)^{6000}01$	36002	3.968×10^{-17}	1.140×10^{-9}

The third type of periodic windows with large periods needed to reduce the maximum gap are period-tupling descendants of periodic windows from the first period-doubling cascade. An example from Fig. 1 is the period-448 window with the symbol sequence $\delta((01), 5)^{(0111101)}$ being a period tupling descendant of $\delta((01), 5)$ with the period 64. One can show that each periodic sequence lying between two periodic sequences being period-tupling descendants of a single sequence must be a period-tupling descendant of this sequence. In consequence, filling a given interval in the parameter space bounded by periodic windows corresponding to such sequences requires much higher periods. As an example, let us consider two period-tupling sequences generated from the sequence $s = \delta((01), 7)$ with the period 256. The sequence $s^{(10110)}$ with the period $256 \cdot 5 = 1280$ corresponds to the periodic window $\mathbf{a} \approx 3.5699464^{163}_{018}$, and the sequence $s^{(100)}$ with the period $256 \cdot 3 = 768$ corresponds to the periodic window $\mathbf{a} \approx 3.569946^{1394}_{8375}$. The distance between these two windows is $d \approx 4.213 \times 10^{-7}$. When we search for periodic windows between these two periodic windows to reduce the maximum gap below 10^{-9} we find 659 periodic windows with periods divisible by 256. Periods of these periodic windows belong to the set $256 \cdot \{7, 8, \ldots, 86\}$. The maximum period is $p_{\max} = 86 \cdot 256 = 22016$. It follows that filling the interval between two period-tupling descendants of periodic windows from the first period-doubling cascade requires relatively large periods of periodic windows. This observation is also true for period-tupling descendants of other symbol sequences. However, period-tupling descendants of period-doubling windows from the first period-doubling cascade are further away than period-tupling descendants of other symbol sequences and hence more periodic windows are needed to reduce the gap between them to the required value $g_{\max}$.

D. Search for periodic windows close to a given point in the parameter space

The procedure presented in Section III D may be used to find periodic windows close to a given parameter value a^* and to study the problem whether a^* is regular.

Let us first consider 500 equidistant parameter values a_k in the interval $a \in [3.5, 3.999]$, $a_k = 3.5 + 0.001k$ for $k \in \{0, 1, \ldots, 499\}$. The method described in Section III D is applied to find periodic windows close to these parameter values. In each case the search is continued until a periodic orbit is found or the period p of a periodic window found is above $p = 500$. Periodic windows are found in 114 cases presented in Table VII. For each case we report the period p of the periodic window, the number n of parameter values a_k contained in this periodic window, the endpoints $[a_l, a_r]$ and the width w of this periodic window. For period doubling windows, we also report the period p_{SN} of its saddle-node ancestor. Saddle-node windows do not have ancestors, which is denoted by a hyphen. As one could expect in most cases periodic window found are either saddle node periodic windows with low periods ($p \leq 20$) or are period-doubling descendants of such periodic windows. Two exceptional cases are observed for $a = 3.602$ and $a = 3.633$ which belong to period-88 and period-72 windows, respectively. Their widths are below 10^{-6}.

The percentage $q = 114/500 = 0.228$ of cases for which periodic windows in the interval $a \in [3.5, 4.0]$ are found can be used to estimate the total width of periodic windows in $a \in [3.0, 4.0]$ using the formula $W = 0.5 + 0.5 \cdot q = 0.614$. This estimate is very close to the true value $W^* \approx 0.61396$. One can expect that testing more parameter values in the interval

TABLE VII. Periodic windows of the logistic map $f_a(x) = ax(1-x)$ containing points $a_k = 3.5 + 0.001k$ for $k \in \{0, 1, \ldots, 499\}$, p is the period, a is the value of the parameter for which the existence of a sink exists (middle of the periodic window), w is the width of a periodic window.

a_k	n	p	p_{SN}	$[a_l, a_r]$	w
3.500-3.544	45	4	2	$3.^{5441}_{4495}$	9.46×10^{-2}
3.545-3.564	20	8	2	3.5^{644}_{441}	2.03×10^{-2}
3.565-3.568	4	16	2	3.56^{876}_{441}	4.35×10^{-3}
3.569	1	32	2	3.56^{969}_{876}	9.32×10^{-4}
3.583	1	24	12	3.58^{318}_{281}	3.69×10^{-4}
3.602	1	88	22	3.60^{20001}_{19997}	4.09×10^{-7}
3.606	1	20	10	3.60^{627}_{592}	3.49×10^{-4}
3.627-3.630	4	6	–	3.6^{3039}_{2655}	3.84×10^{-3}
3.631-3.632	2	12	6	3.63^{2186}_{0389}	1.80×10^{-3}
3.633	1	72	36	3.63^{3004}_{2995}	8.22×10^{-6}
3.634	1	18	–	3.63^{4007}_{3939}	6.72×10^{-5}
3.656	1	18	–	3.65^{600064}_{599985}	7.92×10^{-7}
3.673	1	10	–	3.67^{30351}_{29993}	3.58×10^{-5}
3.702	1	7	–	3.70^{215}_{164}	5.14×10^{-4}
3.739-3.741	3	5	–	3.7^{4112}_{3817}	2.95×10^{-3}
3.742	1	10	5	3.74^{257}_{112}	1.45×10^{-3}
3.743	1	80	5	3.74^{3001}_{2985}	1.60×10^{-5}
3.829-3.841	13	3	–	3.8^{415}_{284}	1.31×10^{-2}
3.842-3.847	6	6	3	3.84^{761}_{150}	6.11×10^{-3}
3.848-3.849	2	12	3	3.84^{904}_{761}	1.43×10^{-3}
3.855	1	30	15	3.85^{50035}_{49986}	5.00×10^{-6}
3.856	1	12	–	3.85^{60043}_{59901}	1.42×10^{-5}
3.906	1	5	–	3.90^{611}_{557}	5.36×10^{-4}
3.961	1	8	–	3.96^{110}_{077}	3.20×10^{-4}

$a \in [3.5, 4.0]$ should improve this estimate.

Let us now consider examples of parameter values a_k for which periodic windows are not found. To observe how the results change with the parameter a we consider four equidistant values of a not belonging to periodic windows found: $a^* \in \{3.6.3.7, 3.8, 3.9\}$.

For each case, we start with symbol sequences $(01) \succ (0001)$ corresponding to periodic windows $\mathbf{a}_2 \approx 3.^{4495}_{0000}$ and $\mathbf{a}_4 \approx 3.960^{769}_{102}$. The search for periodic windows close to a^* is continued until the period of the orbit is below 500. The calculations are carried out using multiple precision arithmetic with 1024 bits of precision, which is required to handle very narrow periodic windows with widths below 10^{-200}. As an example let us consider $a^* = 3.9$. After 388 and 389 iterations we obtain period–498 and period–499 windows lying on the opposite sides of $a = 3.9$, with the distances from $a = 3.9$ smaller than 10^{-105}. Widths of these periodic windows are approximately 8.094×10^{-213} and 9.308×10^{-212}.

Fig. 5 shows periods of windows found versus the iteration number. Note that in all cases periods grow faster than the iteration number, which means that sometimes the period has to be increased by more than one to find the proper symbol sequence. One may see that for $a = 3.6$ we need approximately $n = 160$ iterations to reach $p = 500$, while for $a = 3.9$ the number of iterations needed is $n \approx 390$. This difference is related to the fact that for larger a more symbol sequences are admissible and in consequence shorter periodic sequences may satisfy the required conditions. In the limit case when $a = 4$ all symbol sequences are admissible and the period grows with the same speed as the iteration number (symbol sequences (0^k1) correspond to periodic windows with positions converging to $a = 4$ as k goes to infinity).

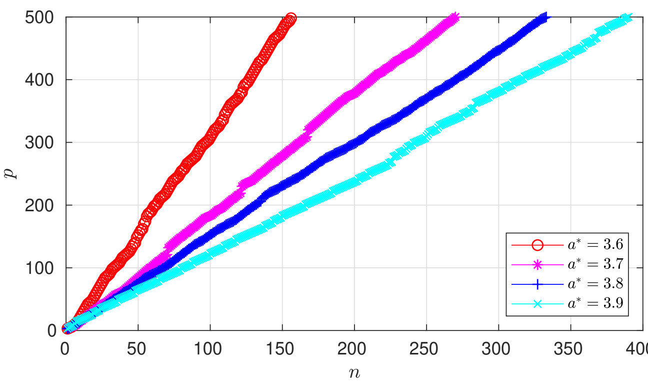

FIG. 5. Search for periodic windows close to $a^* \in \{3.6.3.7, 3.8, 3.9\}$. Periods p of periodic windows found versus the iteration number n.

Fig. 6(a) and 6(b) show the gap (the distance between two closest periodic windows lying on the opposite size of the considered parameter value) versus the current period and the iteration number, respectively. One may see that the gap decreases almost linearly in the logarithmic scale. Reduction versus period is faster for larger a, while the reduction versus the iteration number is almost the same for all cases. This is a consequence of using the bisection method. In the optimal case the bisection method in each step splits the region in half, and after n steps the distance should be reduced by the factor 2^{-n}. For $n = 300$ we have $2^{-300} \approx 4.9 \times 10^{-91}$ which is close to what can be seen in Fig. 6(b).

Fig. 7 shows the distance between the periodic window found and a^* versus the period. This plot is similar to the plot shown in Fig. 6(a). The difference is the existence of outliers—points in the plot lying below the line created by other points. For some iterations the procedure comes closer to a^* than expected for this iteration number.

Fig. 8(a) shows widths of periodic windows found versus the period. One may see that widths decrease much faster than gaps (compare Fig. 6(a)). This is further illustrated in Fig. 8(b) where the ratio $r = w/g$ of the width and the gap is plotted versus the period. The ratio decreases very fast with the period. For $p = 100$ the ratio is below 4×10^{-10} for all cases. For $p = 500$ the ratio is below 10^{-40} for $a = 3.6$ and below 10^{-100} for $a = 3.9$. It means that if a selected point does not belong to a periodic window with a small period then it is very unlikely that it belongs to any periodic window. Thus, with a high level of confidence we may state that all parameter values $a_k = 3.5 + 0.001k$ with $k \in \{0, 1, \ldots, 499\}$ for which periodic windows with periods $p \leq 500$ have not been found are not regular, i.e., the map f_{a_k} is chaotic.

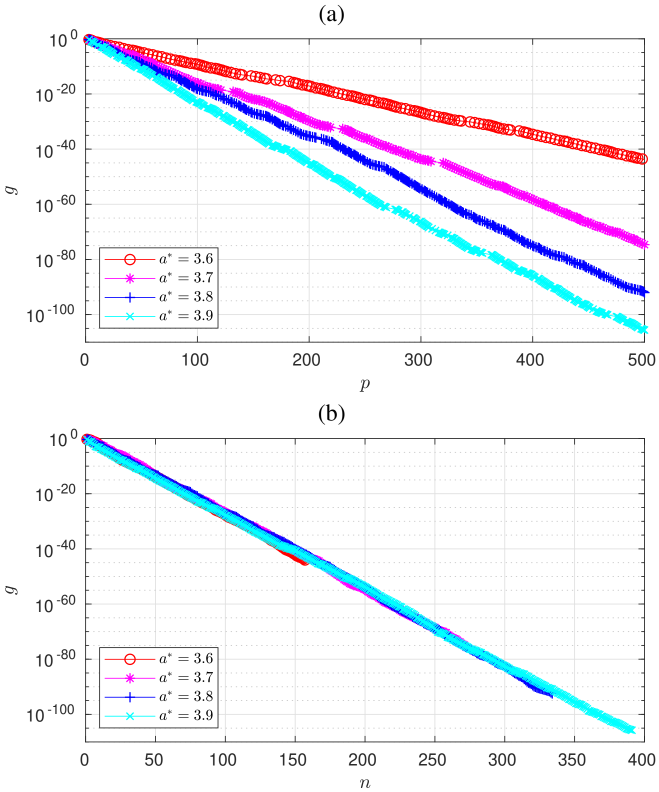

FIG. 6. Search for periodic windows close to $a^* \in \{3.6.3.7, 3.8, 3.9\}$; (a) the gap versus the current period and (b) the gap versus the iteration number.

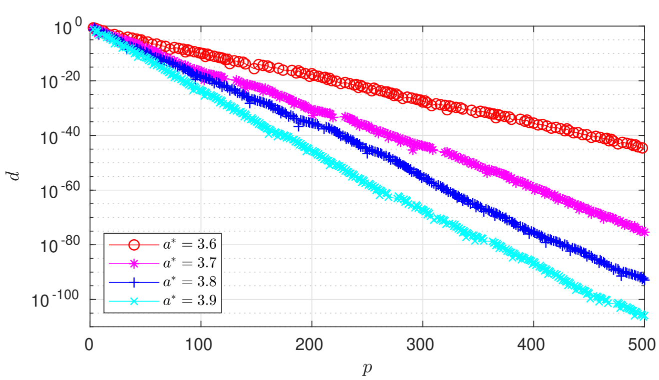

FIG. 7. Search for periodic windows close to $a^* \in \{3.6.3.7, 3.8, 3.9\}$. The distance from periodic windows found and the point a^* versus the period.

V. CONCLUSIONS

An efficient method to find periodic windows for the logistic map close to a selected point in the parameter space was proposed. The method was used to find periodic windows densely filling the parameter space and to find periodic windows extremely close to selected points in the parameter space. The maximum distance between periodic windows found in this study is smaller than 4×10^{-9}. It follows that all periodic windows with the width above 4×10^{-9} were found.

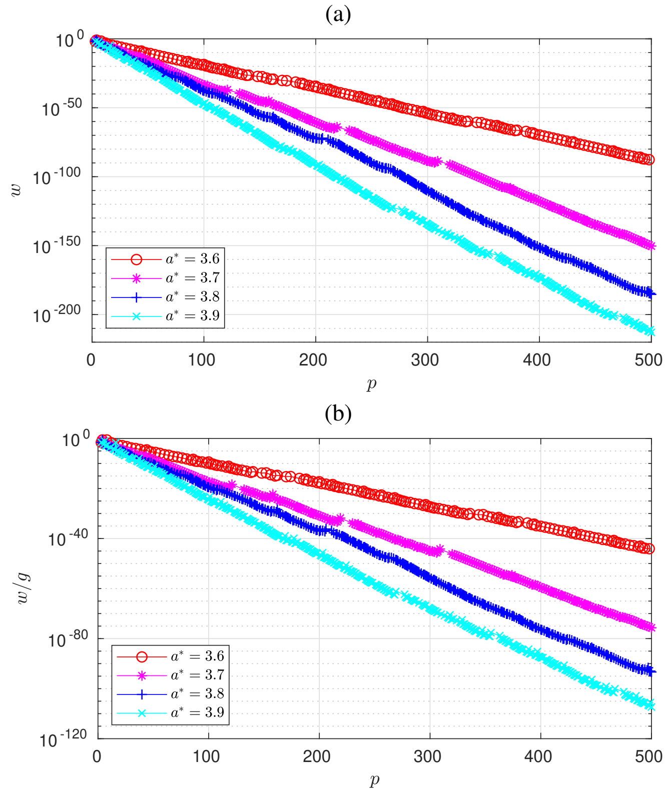

FIG. 8. Search for periodic windows close to $a^* \in \{3.6.3.7, 3.8, 3.9\}$; (a) widths of periodic windows versus the period, (b) the ratio $r = w/g$ of periodic window width and the gap versus the period.

Period-doubling windows were calculated to find all periodic windows wider than 10^{-10}. The reasons for the existence of relatively wide parameter ranges free from low-period orbits were given.

Proposed methods may be applied to study the problem of existence of periodic windows for other one-dimensional maps with a single extremum.

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AUTHOR DECLARATIONS

The author has no conflicts to disclose.

DATA AVAILABILITY

The data supporting the findings of this study are available from the author upon a reasonable request.

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