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Systematic search for wide periodic windows and bounds for the set of regular parameters for the quadratic map

Zbigniew Galias[a)]
Department of Electrical Engineering, AGH University of Science and Technology, al. Mickiewicza 30, 30–059 Kraków, Poland

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An efficient method to find positions of periodic windows for the quadratic map $f(x) = ax(1-x)$ and a heuristic algorithm to locate the majority of wide periodic windows are proposed. Accurate rigorous bounds of positions of all periodic windows with periods below 37 and the majority of wide periodic windows with longer periods are found. Based on these results, we prove that the measure of the set of regular parameters in the interval $[3,4]$ is above 0.613960137. The properties of periodic windows are studied numerically. The results of the analysis are used to estimate that the true value of the measure of the set of regular parameters is close to 0.6139603. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4983172]

The quadratic map is a classical example of a one-dimensional map displaying complex behavior. One of the open problems regarding this map is what are the measures of sets of regular and stochastic parameters. To compute an accurate rigorous lower bound for the measure of the set of regular parameters, it is necessary to find many (billions) of periodic windows, some of them with very long periods (several hundred thousands of iterations and more). In this paper, we propose a very efficient algorithm to locate periodic windows with specific symbol sequences and use it to compute an accurate lower bound for the measure of the set of regular parameters, thus providing an upper bound for the measure of the set of stochastic parameters.

I. INTRODUCTION

The *quadratic map* $f_a(x) = ax(1-x)$, where $x \in [0,1]$ and $a \in \Omega = [0,4]$, is a classical example of a simple nonlinear map with complex dynamics.[1] It is known that for any $a \in \Omega$, the map f_a has at most one attractor.[2] The *set* Ω^- *of regular parameters* contains parameter values for which there exists a periodic attractor. By $\Omega^+ \subset \Omega$, we denote the set of stochastic parameters for which the map f_a admits an ergodic invariant probability measure which is absolutely continuous with respect to the Lebesgue measure. It is known that measures of these two sets are positive[3–6] and that their union has the full measure.[2]

In this work, we numerically study the set Ω^-. In particular, we are interested in computing a rigorous lower bound for its measure $\mu(\Omega^-)$ and estimating its true value. In Ref. 7, the authors prove that $\mu(\Omega^- \cap [3,4]) > 0.6139421$. They also report the unpublished non-rigorous estimate by Simo that the measure of the set $\Omega^- \cap [3,4]$ is close to 0.6155 (compare also Ref. 8). In Ref. 5, it is shown that the measure of the set Ω^+ is above 10^{-5000}. As the authors state, this bound is not in any way "optimal" or "sharp." Continuing effort is undertaken to improve this bound.[9,10]

The set Ω^- is a union of intervals, which are called periodic windows. Therefore, to obtain a good lower bound of $\mu(\Omega^-)$ we need to find many wide periodic windows. In Ref. 7, the authors propose a method to find periodic windows scanning the parameter space. In this method, test points $a_k \in \Omega$ are selected using the bisection technique; the Newton method is applied to find points $\tilde{a}_k$ close to a_k where superstable periodic orbits exist and various interval arithmetic tools are used to prove the existence of stable periodic orbits in intervals containing points $\tilde{a}_k$. The resulting intervals are subsets of periodic windows. By construction, the method finds only lower bounds of periodic windows' widths. In Ref. 11, a method to find very accurate enclosures of endpoints of periodic windows is presented. In this method, the interval Newton operator is applied to prove the existence of bifurcation points for periodic orbits with specific symbol sequences. All periodic windows with periods $p \le 32$ have been found. The results were used to find very accurate lower and upper bounds of measures $\mu(\Omega_p^-)$ for $p \le 32$, where Ω_p^- denotes the set of parameters belonging to periodic windows with period p.

In this work, we extend the method presented in Ref. 11 to work for large periods. We find all periodic windows with periods $p \le 36$. To find wide periodic windows with longer periods, we consider three types of periodic windows: primary, period-doubling, and period-tupling windows (to be defined in Sec. II). We propose a heuristic method to find wide primary windows and use it to find the majority of primary windows with widths above $w_{\min} = 10^{-15}$. Next, we generate wide period-tupling and period-doubling windows. The results obtained are used to compute a rigorous lower bound for the measure of the set Ω^- and to estimate its true value.

The layout of the paper is as follows. In Sec. II, several properties of periodic windows for the quadratic map are recalled. In Sec. III, the search method is described in detail, and in Sec. IV, the results of applying this method to analyse

a)Electronic mail: galias@agh.edu.pl.

wide periodic windows for the quadratic map are presented. Throughout the paper, we use bold face to denote intervals, interval vectors, and matrices and math italic to denote "real" quantities. The interval with endpoints $\underline{x} \le \bar{x}$ is defined as $\mathbf{x} = [\underline{x}, \bar{x}]$. The diameter and the middle point of $\mathbf{x}$ are denoted as $\mathrm{diam}(\mathbf{x}) = \bar{x} - \underline{x}$ and $\mathrm{mid}(\mathbf{x}) = 0.5(\underline{x} + \bar{x})$, respectively.

II. PRELIMINARIES

The quadratic map is a one-parameter map of the interval $I = [0, 1]$ into itself defined by

$$f_a(x) = ax(1-x), \tag{1}$$

where $a \in \Omega = [0, 4]$.

We use the notation $f_a^0(x) = x$ and $f_a^{k+1}(x) = f_a\big(f_a^k(x)\big)$ for $k \ge 0$. The trajectory of f_a with the initial point x_0 is $(x_0, x_1, x_2, \ldots)$, where $x_k = f_a^k(x_0)$.

We say that x_0 is a *period–p point* of f_a if $f_a^p(x_0) = x_0$, and $f_a^k(x_0) \ne x_0$ for $1 \le k < p$. The corresponding trajectory $x_k = f_a^p(x)$, $k \ge 0$ is called a *period-p orbit*. We say that the periodic orbit $(x_0, x_1, \ldots, x_{p-1})$ is *stable* if $|\lambda_p(a, x_0)| < 1$ where the *multiplier* $\lambda_p(a, x_0)$ of the periodic orbit is defined as the derivative of f_a^p at x_0, i.e.

$$\lambda_p(a, x_0) = \big(f_a^p\big)'(x_0) = \prod_{k=0}^{p-1} f_a{}'(x_k) = \prod_{k=0}^{p-1} a(1-2x_k). \tag{2}$$

Let us recall some well known results on periodic windows for the quadratic map. For details, the reader is referred to introductory books on deterministic chaos.[12,13]

We say that an interval $(a_{\mathrm{left}}, a_{\mathrm{right}}) \subset \Omega$ is a *period-p window* for the family $\{f_a : a \in \Omega\}$ if for all $a \in (a_{\mathrm{left}}, a_{\mathrm{right}})$ there exists a stable period-p orbit of f_a, and $(a_{\mathrm{left}}, a_{\mathrm{right}})$ is a maximal interval with this property. Endpoints of periodic windows are bifurcation points of corresponding periodic orbits. Periodic windows for which at the left endpoint there is a saddle-node/period-doubling bifurcation are referred to as *saddle-node* windows and *period-doubling* windows, respectively. For the quadratic map, there is a period-doubling bifurcation at the right endpoint of each periodic window, and at this point another periodic window (of period-doubling type) starts. Thus, each saddle-node window generates an infinite sequence of periodic windows with common endpoints. Such a sequence is called a *period-doubling cascade*.

Let us consider a fixed value of a. With the point $x \in [0, 1]$, we associate the *symbol sequence* $s(x) = s = (s_0, s_1, \ldots)$ in such a way that $s_k = 0$ if $x_k < 0.5$ and $s_k = 1$ if $x_k \ge 0.5$, where $(x_0, x_1, x_2, \ldots)$ is the trajectory of f_a with the initial point $x_0 = x$. If x is a period-p point, then s is also periodic. In this case, we write $s = (s_0, s_1, \ldots, s_{p-1})$.

Let us introduce an ordering on the set of symbol sequences $\Sigma = \{s = (s_0, s_1, \ldots) : s_k \in \{0, 1\}\}$. We say that $s \prec \hat{s}$ if $s_k < \hat{s}_k$ and $\sum_{i=0}^{k-1} s_i \equiv 0 \pmod 2$ or $s_k > \hat{s}_k$ and $\sum_{i=0}^{k-1} s_i \equiv 1 \pmod 2$, where k is the smallest non-negative integer such that $s_k \ne \hat{s}_k$. This ordering is closely related to the ordering of right/left (RL) patterns introduced in Ref. 14.

We say that a periodic sequence s of length p is *minimal* if its period is p and it is equal to its smallest cyclic permutation according to the ordering "$\prec$." For example, the order of cyclic permutations of (001) is $(001) \prec (010) \prec (100)$ and hence the sequence (001) is minimal. Points along a periodic orbit have symbol sequences being cyclic permutations of each other. It follows that there is a one-to-one relation between periodic orbits and minimal sequences.

The total number of symbol sequences of length p is 2^p. The number $\mathrm{P}(p)$ of minimal period-p sequences can be computed by subtracting from 2^p the number of symbol sequences having periods being proper divisors of p and dividing the result by p

$$\mathrm{P}(p) = p^{-1}\left(2^p - \sum_{k=1, p \bmod k = 0}^{p-1} k \times \mathrm{P}(k)\right). \tag{3}$$

To set up a relation between minimal sequences and periodic windows, let us define specific types of minimal sequences. We say that a minimal sequence is an *odd-parity sequence* (*even-parity sequence*) if it has odd (even) number of nonzero symbols. Let s be an odd-parity sequence with period p. The sequence $s' = (s_0, s_1, \ldots, s_{p-3}, 1 - s_{p-2}, s_{p-1})$ obtained by flipping the second to last symbol of s is called *the even-parity partner of s*. We say that an odd-parity sequence s with period p is a *saddle-node sequence* if its even-parity partner has period p. Otherwise, we call it a *period-doubling sequence*.

Let us denote by $\mathrm{W}(p)$ the number of odd-parity period-p sequences. There is one odd-parity period-2 sequence: (01) and hence $\mathrm{W}(2) = 1$. For odd $p \ge 3$, each odd-parity sequence has an even-parity partner. For example, there are two period-3 minimal sequences: (001) and (011). The sequence (011) is the even-parity partner of (001). It follows that for odd p half of the minimal period-p sequences are of odd-parity, i.e., $\mathrm{W}(p) = \mathrm{P}(p)/2$. For even $p \ge 4$, there are $\mathrm{W}(p/2)$ odd-parity sequences for which there is no corresponding odd-parity sequence with period p. For example, there are three period-4 minimal sequences (0001), (0011), and (0111). The sequence (0011) is the even-parity partner of (0001). The sequence (0111) is a period-doubling sequence because flipping the second to last symbol gives the sequence (0101) with period 2. Hence, in this case the number of odd-parity sequences is $\big(\mathrm{P}(p) + \mathrm{W}(p/2)\big)/2$ out of which there are $\mathrm{W}(p/2)$ period-doubling sequences. Summarizing, the formula for the number $\mathrm{W}(p)$ of odd-parity period-p sequences reads

$$\mathrm{W}(p) = \begin{cases} 1 & \text{if } p = 2, \\ 0.5 \times \mathrm{P}(p) & \text{if } p \text{ is odd}, p \ge 3, \\ 0.5 \times \big(\mathrm{P}(p) + \mathrm{W}(0.5 \times p)\big) & \text{if } p \text{ is even}, p \ge 4. \end{cases} \tag{4}$$

Let us now discuss the relation between minimal sequences, periodic orbits, and periodic windows. Each minimal period-p symbol sequence corresponds to a single period-p orbit of $f_{4.0}$. All of them are unstable. Let us denote by $x(s, a) = (x_k(s, a))_{k=0}^{p-1}$ the position of the periodic orbit of

f_a with the symbol sequence s, if it exists. Let s be an odd-parity sequence. The multiplier of the periodic orbit $x(s,4.0)$ is $\lambda_p(4.0,x_0(4.0)) = -2^p$. When a is decreased, the position $x(s,a)$ of the periodic orbit changes until at the point a_{right} we have $\lambda_p(a_{\text{right}},x_0(a_{\text{right}})) = -1$. When a is further decreased, we reach the point a_{left} where $\lambda_p(a,x_0(a)) = 1$. The interval $(a_{\text{left}},a_{\text{right}})$ is the periodic window corresponding to the odd-parity sequence s. There exists a point $a_{\text{middle}} \in (a_{\text{left}},a_{\text{right}})$ where $\lambda_p(a_{\text{middle}},x_0(a_{\text{middle}})) = 0$. At this point, the second to last symbol in s flips. In Fig. 1, we show examples how the multipliers λ_p change with a. Curves corresponding to saddle-node sequences (with $\lambda_p(a) < 0$) and their even-parity partners (with $\lambda_p(a) > 0$) are plotted in blue and cyan, respectively. For $a = 4.0$, we have $\lambda_p = -2^p$ for odd-parity sequences and $\lambda_p = +2^p$ for even-parity sequences. When a is decreased, λ_p decreases in absolute value until $\lambda_p = \pm 1$ is reached, which corresponds to a periodic window endpoint.

If s is a saddle-node sequence, then a_{left} is a saddle-node bifurcation point at which the stable periodic orbit and the unstable periodic orbit with the sequence being the even-parity partner of s are born. If s is a period-doubling sequence of length $p = 2k$, then a_{left} is a period-doubling bifurcation point where a period-$2k$ orbit is born from a period-k orbit. For example, the period-doubling window with the sequence (001011) starts at a point where the periodic window with the sequence (001) ends (compare Fig. 1).

From the discussion presented above, it follows that there is a one-to-one correspondence between periodic windows and odd-parity sequences and that saddle-node and period-doubling sequences correspond to saddle-node and period-doubling windows, respectively.

Let us recall the notion of primary sequences and period-tupling sequences (compare Refs. 11, 15, and 16). Let s be an odd-parity sequence with period $k \geq 2$ and s' its even-parity partner. A *period-l-tupling sequence* is obtained by concatenating $l_1 > 0$ copies of s and $l - l_1 > 0$ copies of s' where l_1 is odd to preserve the odd-parity of the final sequence. For example, (s,s') is a period-doubling sequence, (s,s',s') is a period-tripling sequence, while (s,s',s',s') and (s,s,s,s') are period-quadrupling sequences generated from s. The sequence $s = (01)$ is also considered a period-tupling sequence. An odd-parity sequence which is not a period-tupling sequence is called a *primary sequence*. Periodic windows corresponding to primary and period-tupling sequences are called *primary windows* and *period-tupling windows*, respectively.

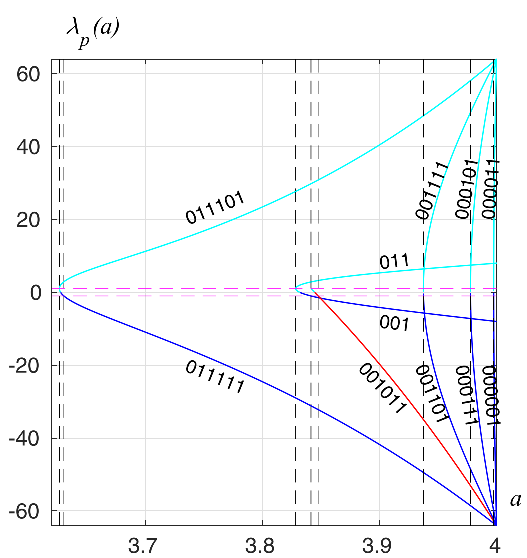

FIG. 1. Multipliers of periodic orbits associated with minimal sequences of length 3 and 6 versus parameter a. Periodic orbits with even-parity, saddle-node, and period-doubling sequences are plotted in cyan, blue, and red, respectively.

III. SYSTEMATIC SEARCH FOR PERIODIC WINDOWS

In this section, we present methods to find rigorous bounds of endpoints of periodic windows. Since we are also going to handle periodic windows with periods above 10^6 the implementations need to be very efficient. Let us start with a description of a method to find approximations of periodic windows' endpoints for a selected odd-parity sequence.

A. Finding accurate approximations of periodic windows' endpoints

Endpoints of periodic windows are saddle-node and period-doubling bifurcation points of corresponding periodic orbits. Let us consider the map $H_{\lambda_0} : \mathbb{R}^{p+1} \ni z \mapsto H_{\lambda_0}(z) \in \mathbb{R}^{p+1}$ defined by

$$H_{\lambda_0}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \\ a \end{pmatrix} = \begin{pmatrix} x_1 - a x_0(1-x_0) \\ x_2 - a x_1(1-x_1) \\ \vdots \\ x_0 - a x_{p-1}(1-x_{p-1}) \\ a^p(1-2x_{p-1})\cdots(1-2x_1)(1-2x_0) - \lambda_0 \end{pmatrix}, \tag{5}$$

where $z = (x_0,x_1,\ldots,x_{p-1},a)^\top$ and $\lambda_0 = \pm 1$. Zeros of H_{λ_0} correspond to bifurcations of period-p orbits. To study period-doubling and saddle-node bifurcation points, we use $\lambda_0 = -1$ and $\lambda_0 = 1$, respectively. Zeros of H_{λ_0} can be found using the Newton method. Let us assume that $z^{(0)}$ is the initial point for the Newton method. The formula for the Newton iteration is $z^{(k+1)} = z^{(k)} - h$ where $h = (h_0,h_1,\ldots,h_p)^\top$ is the solution of the equation

$$H'_{\lambda_0}(z^{(k)})h = H_{\lambda_0}(z^{(k)}), \tag{6}$$

where

$$H'_{\lambda_0}(z^{(k)}) = \begin{pmatrix} -c_0 & 1 & 0 & \ldots & 0 & -b_0 \\ 0 & -c_1 & 1 & \ldots & 0 & -b_1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & -b_{p-2} \\ 1 & 0 & 0 & \ldots & -c_{p-1} & -b_{p-1} \\ -2ad_0 & -2ad_1 & -2ad_2 & \ldots & -2ad_{p-1} & pea^{-1} \end{pmatrix},$$

$H_{\lambda_0}(z^{(k)}) = (g_0, g_1, \ldots, g_{p-1}, e - \lambda_0)^\top$, $b_k = x_k(1 - x_k)$, $c_k = a(1 - 2x_k)$, $d_k = \prod_{i=0, i\neq k}^{p-1} c_i$, $g_k = x_{(k+1) \bmod p} - a x_k(1 - x_k)$, and $e = \prod_{k=0}^{p-1} c_k$.

From (6), it follows that $h_{k+1} = v_k + w_k h_0 + u_k h_p$ for $k = 0, 1, \ldots, p-2$ where $w_k = \prod_{i=0}^{k} c_i$, $u_k = \sum_{j=0}^{k} b_j \prod_{i=j+1}^{k} c_i$, and $v_k = \sum_{j=0}^{k} g_j \prod_{i=j+1}^{k} c_i$. Eliminating $h_1, h_2, \ldots, h_{p-1}$ from (6) yields

$$\begin{pmatrix} q_0 & q_p \\ e-1 & u_{p-1} \end{pmatrix} \begin{pmatrix} h_0 \\ h_p \end{pmatrix} = \begin{pmatrix} r \\ -v_{p-1} \end{pmatrix}, \quad (7)$$

where $q_0 = 2a(d_0 + \sum_{i=1}^{p-1} d_i w_{i-1})$, $q_p = 2a \sum_{i=1}^{p-1} d_i u_{i-1} - pea^{-1}$, and $r = \lambda_0 - e - 2a \sum_{i=1}^{p-1} d_i v_{i-1}$. Hence, the Newton correction h can be computed as

$$h_0 = \frac{r u_{p-1} + v_{p-1} q_p}{q_0 u_{p-1} + (1-e) q_p}, \quad h_p = \frac{-q_0 v_{p-1} + (1-e) r}{q_0 u_{p-1} + (1-e) q_p}, \quad (8)$$

$$h_{k+1} = v_k + w_k h_0 + u_k h_p \quad \text{for} \quad k = 0, 1, \ldots, p-2. \quad (9)$$

The above formulas can be implemented to have a linear complexity versus p both in time and memory.[11]

The algorithm implements a forward shooting version to compute h. It is also possible to implement the backward shooting version[17,18] to solve (6), where instead of computing $\prod_{i=0}^{k} c_i$ one computes the product of inverses. This version should be selected when the product $\prod_{i=0}^{p-1} c_i$ is large.

Let s be a period-p minimal sequence, and a_{right} be the right endpoint of the corresponding periodic window. We discuss the problem how to select an initial point for the Newton method to obtain a fast convergence to $z = (x(s, a_{\text{right}}), a_{\text{right}})^\top$. Zeros of H_{λ_0} with $\lambda_0 = \pm 1$ define bifurcation points of all period-p orbits. Since the number of such orbits grows exponentially with p, it is clear that we have to choose the initial point $z^{(0)}$ for the Newton method very carefully to obtain the convergence to a periodic orbit with a specific symbol sequence. One possibility is to use $z^{(0)} = (x(s, 4.0), 4.0)^\top$ as an initial point. The position $x(s, 4.0)$ of the periodic orbit for $a = 4.0$ can be found using the topological conjugacy between $f_{4.0}$ and the *tent map* $T : [0,1] \mapsto [0,1]$ which is defined as $T(y) = 1 - |2y - 1|$. This version was used in Ref. 11 to find all periodic windows with periods $p \leq 32$. It was shown that the method works fast for periodic windows with low periods. However, it will be shown in Section IV that for larger periods this selection method may lead to very slow convergence or even divergence of the Newton method.

When we know $a \in (a_{\text{right}}, 4.0)$ we may use $z^{(0)} = (\tilde{x}(s, a), a)^\top$ as an initial point for the Newton method, where $\tilde{x}(s, a)$ is an approximation of the position $x(s, a)$ of the periodic orbit of f_a. To approximate $x(s, a)$, we compute backward iterates of the interval $[0, 1]$ under the map f_a. The map f_a is not reversible. However, since the symbol sequence is known, we may easily select the correct branch when computing backward iterates. The numerical procedure is initialized with $\mathbf{x}_0 = [0, 1]$, $k = 0$. In each step, we carry out the following computations: if $k = 0$, define $\mathbf{x}_p = \mathbf{x}_0$, and set $k = p$, calculate $\mathbf{y} = f_a^{-1}(\mathbf{x}_k)$ and assign $\mathbf{x}_{k-1} = \mathbf{y} \cap [0, 0.5]$ if $s_{k-1} = 0$ and $\mathbf{x}_{k-1} = \mathbf{y} \cap [0.5, 1]$ if $s_{k-1} = 1$. Under the assumption that $a > a_{\text{right}}$ these calculations, when carried out in infinite precision, converge to the position of the periodic orbit with the symbol sequence s. In practice, we use finite-precision calculations and the algorithm is stopped when a predefined precision is reached. Middle points of intervals $\mathbf{x}_k$ are selected as approximations $\tilde{x}_k(s, a)$.

If a is not sufficiently close to a_{right}, to speed up computations we may use a bisection step. Let us assume that we have a lower bound and an upper bound a_0 and a_1 for a_{right}, i.e., $a_0 < a_{\text{right}} < a_1$. In the bisection step, we try to compute an approximate position of the orbit for the test point $a = 0.5(a_0 + a_1)$. If computations fail, we conclude that $a > a_{\text{right}}$ and set $a_0 = a$. Otherwise, we set $a_1 = a$. For $p \geq 3$, we may always select $a_1 = 4$ and $a_0 = a_*$, where $a_* \approx 3.56994567187$ is the position of the accumulation point of positions of periodic windows belonging to the first period-doubling cascade (there are no saddle-node windows with period $p \geq 3$ before a_*). Bisection steps are costly and should be made only when the convergence speed of the Newton method is too low.

For saddle-node windows, an approximation of the right endpoint is usually a good starting point for the Newton method applied to the map $H_{\lambda_0=1}$ to find the left endpoint. For period-doubling windows, the left endpoint is found as the right endpoint of the parent window.

B. Interval Newton method to find rigorous bounds for bifurcation points

Once we have an approximate position $\hat{z}$ of the bifurcation point, we may find rigorous bounds for its position. This is achieved by constructing an interval vector $\mathbf{z}$ containing $\hat{z}$ and applying the interval Newton method[19] to prove the existence of a single zero of H_{λ_0} in $\mathbf{z}$. In this method, we have to verify that $\text{N}(\mathbf{z}, \hat{z}) \subset \mathbf{z}$ where $\text{N}(\mathbf{z}, \hat{z}) = \hat{z} - H'_{\lambda_0}(\mathbf{z})^{-1} H_{\lambda_0}(\hat{z})$ is the interval Newton operator for the map H_{λ_0} and $\hat{z} \in \mathbf{z}$. Let $\mathbf{z} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{p-1}, \mathbf{a})^\top$ be an interval vector and $\hat{z} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{p-1}, \hat{a})^\top \in \mathbf{z}$. The following algorithm provides an efficient method to compute the interval vector $\mathbf{h}$ containing solutions of $H'_{\lambda_0}(z) h = H_{\lambda_0}(\hat{z})$ for $z \in \mathbf{z}$.

To evaluate $\mathbf{h}$, the following computations are carried out in interval arithmetic:

$$\mathbf{c}_k := \mathbf{a}(1 - 2\mathbf{x}_k), \quad \mathbf{b}_k := \mathbf{x}_k(1 - \mathbf{x}_k), \quad \mathbf{d}_k := \prod_{i=0, i\neq k}^{p-1} \mathbf{c}_i,$$

$$\mathbf{g}_k := \hat{x}_{(k+1) \bmod p} - \hat{a}\hat{x}_k(1 - \hat{x}_k), \quad \mathbf{e} := \prod_{i=0}^{p-1} \hat{a}(1 - 2\hat{x}_i),$$

$$\mathbf{u}_k := \sum_{j=0}^{k} \mathbf{b}_j \prod_{i=j+1}^{k} \mathbf{c}_i, \quad \mathbf{v}_k := \sum_{j=0}^{k} \mathbf{g}_j \prod_{i=j+1}^{k} \mathbf{c}_i, \quad \mathbf{w}_k := \prod_{i=0}^{k} \mathbf{c}_i,$$

$$\mathbf{q}_0 := 2\mathbf{a}\left(\mathbf{d}_0 + \sum_{i=1}^{p-1} \mathbf{d}_i \mathbf{w}_{i-1}\right), \quad \mathbf{q}_p := 2\mathbf{a}\sum_{i=1}^{p-1} \mathbf{d}_i \mathbf{u}_{i-1} - \frac{p\mathbf{w}_{p-1}}{\mathbf{a}},$$

$$\mathbf{r} := \lambda_0 - \mathbf{e} - 2\mathbf{a}\sum_{i=1}^{p-1} \mathbf{d}_i \mathbf{v}_{i-1}, \quad \mathbf{h}_0 := \frac{\mathbf{r}\mathbf{u}_{p-1} + \mathbf{v}_{p-1}\mathbf{q}_p}{\mathbf{q}_0 \mathbf{u}_{p-1} + (1 - \mathbf{w}_{p-1})\mathbf{q}_p},$$

$$\mathbf{h}_p := \frac{-\mathbf{q}_0\mathbf{v}_{p-1} + (1-\mathbf{w}_{p-1})\mathbf{r}}{\mathbf{q}_0\mathbf{u}_{p-1} + (1-\mathbf{w}_{p-1})\mathbf{q}_p}$$

$$\mathbf{h}_k := \mathbf{v}_{k-1} + \mathbf{w}_{k-1}\mathbf{h}_0 + \mathbf{u}_{k-1}\mathbf{h}_p, \quad k = 1,2,\ldots,p-1.$$

Due to properties of interval arithmetic, the inclusions between the right-hand side expressions and the evaluated intervals $\mathbf{c}_k$, $\mathbf{b}_k$, $\mathbf{g}_k$, $\mathbf{d}_k$, $\mathbf{e}$, $\mathbf{u}_k$, $\mathbf{v}_k$, $\mathbf{w}_k$, $\mathbf{q}_0$, $\mathbf{q}_p$, $\mathbf{r}$, and $\mathbf{h}_k$ are automatically satisfied and the interval Newton operator can be evaluated as $\mathrm{N}(\mathbf{z},\hat{z}) \subset \hat{z} - \mathbf{h}$. For more details, see Ref. 11.

C. Locating all periodic windows for a given period

To find all period-p windows, we have to consider all even-parity sequences with period p. Period-doubling sequences are handled in a different way, which will be described later. Here, we consider saddle-node sequences.

For each saddle-node sequence s, we select an initial point a larger than the right endpoint a_{right} of the periodic window, compute an approximate position $\tilde{x}(s,a) = \left(\tilde{x}_k(s,a)\right)_{k=0}^{p-1}$ of the periodic orbit of f_a with the sequence s, and use the Newton method applied to the map H_{-1} with the initial point $z^{(0)} = \left(\tilde{x}(s,a),a\right)^\top$ to find an approximate position $\tilde{a}_{\text{right}}$ of the right endpoint a_{right} and approximate position $\tilde{x}(s,\tilde{a}_{\text{right}})$ of periodic orbit for $\tilde{a}_{\text{right}}$. Next, we construct an interval vector $\mathbf{z}$ containing $\hat{z} = (\tilde{x}(s,\tilde{a}_{\text{right}}),\tilde{a}_{\text{right}})$ and apply the interval Newton operator for the map H_{-1} to prove the existence of a bifurcation point within $\mathbf{z}$, which provides rigorous bounds $[\underline{a}_{\text{right}},\bar{a}_{\text{right}}]$ for the position of the right endpoint.

To obtain bounds for the left endpoint a_{left}, we first apply the Newton method for the map H_{+1} with the initial point $z^{(0)} = (\tilde{x}(s,\tilde{a}_{\text{right}}),\tilde{a}_{\text{right}})$ to find an approximate position $\tilde{a}_{\text{left}}$ of the left endpoint a_{left} and $\tilde{x}(s,\tilde{a}_{\text{left}})$. Finally, we apply the interval Newton operator for the map H_{+1} to prove the existence of a bifurcation point in a neighborhood of $\hat{z} = (\tilde{x}(s,\tilde{a}_{\text{left}}),\tilde{a}_{\text{left}})$ and obtain rigorous lower and upper bounds $[\underline{a}_{\text{left}},\bar{a}_{\text{left}}]$ for the position of the left endpoint. Bounds of the width of the periodic window can be computed as $[\underline{w},\bar{w}] = [\underline{a}_{\text{right}} - \bar{a}_{\text{left}}, \bar{a}_{\text{right}} - \underline{a}_{\text{left}}]$.

Computation time depends on the selection of the initial point a satisfying the condition $a > a_{\text{right}}$. Since all orbits exist for $a = 4.0$, we may always select $a = 4.0$. A faster option is to sort all saddle-node sequences according to the ordering "$\prec$" defined in Section II and process them from the last one to the first one. The ordering "$\prec$" preserves positions of windows in the sense that if $s \prec \hat{s}$ then the periodic window corresponding to s exists for smaller a than the one corresponding to $\hat{s}$. Therefore, as an initial guess for the position of the right endpoint we may use the position of the left endpoint of the previously found window. For the first window, we select $a = 4.0$. It will be shown that this approach significantly reduces computation times especially in the case of longer sequences.

D. Finding wide primary windows

From (3) and (4), it follows that the number of sequences grows approximately as $2^{p-1}/p$. Therefore, finding all period-p windows is feasible only for small p.

Let us assume that we want to find primary periodic windows with width larger than $w_{\min}$ (we will use $w_{\min} = 10^{-15}$). This is done recursively for increasing periods. Below, we present a heuristic procedure to find wide primary periodic windows with period $p+1$ based on wide primary periodic windows with period p. First, we select primary period-p sequences corresponding to periodic windows with widths above $w_{\min}$. Next for each sequence, we generate a number of sequences of length $p+1$. Sequences are generated in two ways. In the first version, we insert symbol 0 at a given position. In the second version, we replace symbol 0 at a given position by a subsequence $(s_0s_1) = (11)$. Both versions ensure that the number of nonzero symbols is odd. For each sequence obtained, we find the corresponding minimal sequence. We remove duplicate copies, sequences with period smaller than p and period-tupling sequences. Finally, we sort the set of sequences in the reverse order and apply the procedure presented in Sec. III C to find corresponding periodic windows. It will be shown that this method finds the majority of wide primary windows with period $p+1$.

For larger p, we use a different method. First, in the set of primary windows found so far, we locate families of wide primary windows with symbol sequences (u^kv), where the number of nonzero symbols in sequences u and v is even and odd, respectively. It will be shown that most wide primary windows belong to such families. Then, for each family, we find periodic windows with symbol sequences of the form (u^kv) for increasing k until periodic window's width drops below $w_{\min}$.

E. Finding wide period-tupling windows

As it has been mentioned before, period-tupling sequences are concatenations of primary sequences and their even-parity partners. Each period-tupling descendant of a primary sequence s has the form $t = s^r$, with r being an arbitrary odd-parity sequence. For a primary sequence s, the notation s^r denotes the sequence created by concatenating m copies of s and its even-parity partner s', where m is the length of r. The kth element is s if $r_k = 1$ and s' if $r_k = 0$. For example, for $s = (001)$, $r = (0111)$ we have $s' = (011)$ and $s^r = (s'sss) = (011\,001\,001\,001)$.

Let $S_{\text{pr,w}}$ denote the set of wide primary sequences. Finding their wide period-tupling descendants is carried out recursively based on wide periodic windows located so far. In the kth step, we find all wide period-tupling windows with sequences of the form s^r, where $s \in S_{\text{pr,w}}$ and $r \in R_k$. Period-tupling sequences corresponding to wide windows found in the kth step form the set R_{k+1} for the next step. The process is initiated with $R_1 = S_{\text{pr,w}} \setminus \{(01)\}$ and it is stopped when R_k is empty. The sequence (01) is excluded to avoid considering period-doubling sequences, which are handled separately, as described in Sec. III F.

To reduce the number of period-tupling sequences which are considered in a given step, we predict widths of sequences s^r for all $s \in S_{\text{pr,w}}$ based on the results obtained for a single sequence. The prediction is based on the fact that for a fixed r the width of the periodic window with the sequence s^r is proportional to the width of s. In practice, we first find all periodic windows with the sequences $\hat{s}^r$ for $r \in R_k$ for a fixed $\hat{s}$, for example $\hat{s} = (001)$. Then, for a given sequence $s \in S_{\text{pr,w}}$, $s \neq \hat{s}$, we skip sequences s^r, $r \in R_k$ for which the predicted width is below the threshold $w_{\min}$.

F. Finding wide period-doubling windows

To find wide period-doubling windows, we consider all saddle-node windows found in previous steps. For each saddle node window, we find its period-doubling descendants with widths above the threshold $w_{\min}$. This is the final step of the procedure to locate wide periodic windows.

For each period-doubling window in a period-doubling cascade as bounds for the left endpoint, we use bounds for the right endpoint of its parent. To find bounds for the right endpoint, we use the same method as for saddle-node windows. However, this time the selection of the initial guess $a > a_{\text{right}}$ is easy, because widths of windows in a period-doubling cascade decrease almost in the same way in each cascade and in the limit the ratio of widths of subsequent windows is defined by the Feigenbaum constant.[20]

IV. NUMERICAL RESULTS

In this section, we present results of the numerical study of periodic windows for the quadratic map using methods presented in Sec. III. Computations are carried out in multiple precision using the MPFR library.[21] Interval arithmetic support is provided by the CAPD library.[22]

First, we compare different versions of selecting the initial point for the Newton method to find all short periodic orbits. We consider a test problem to find all saddle-node periodic windows with periods $3 \leq p \leq 20$. In the first version, periodic windows are found independently with the initial condition $z^{(0)} = (x(s, 4.0), 4.0)^{\top}$. In this case, the computation time to find all 55 447 periodic windows is 160.25 s using a single core 3.1 GHz processor. In the second version, sequences are reverse sorted according to the ordering "$\prec$" and the results obtained for a given sequence are used to select the initial point for the Newton method for the next sequence. In this version, the total computation time is 144.86 s, which means that the second version is approximately 10% faster for the considered test problem.

To assess the performance of these two versions for longer sequences, let us consider a family of primary sequences $((011)^k 111)$, $k \geq 2$, where the notation $(011)^k$ means that the subsequence (001) is repeated k times. When we apply the first version to find periodic windows for $50 < k \leq 150$, the computation time is 55.77 s. For the second version, the computation time is reduced by 87% to 7.29 s. For all sequences in the family, we have $((011)^k 111) \prec (001)$. Hence, the left endpoint $a_{\text{left}} \approx 3.828427125$ of the periodic window with the sequence (001) can be used as an initial point to find periodic windows in this family. Applying the procedure to compute the periodic window for the sequence $((011)^{200} 111)$ with $a = 4.0$ takes 2 s. When we also use the bisection method, the computation time is reduced to 1.28 s. Using $a = 3.828427125$ reduces computation time to 0.16 s. For the sequence $((011)^{500} 111)$ with the initial point $a = 3.828427125$, the computation time is 0.42 s. To find the periodic window starting at $a = 4.0$, we should use the combination of the Newton method and the bisection method; otherwise, the method fails. In this case, the computation time is 3.74 s.

The examples presented above show that the selection of the initial point $a \geq a_{\text{right}}$ is essential for the fast operation of the algorithm to find periodic windows. When a is sufficiently close to a_{right}, the Newton method converges very fast and using the bisection method is not necessary. The bisection method ensures convergence and helps to reduce the computation time if a is far from a_{right}. When many sequences are considered together, we should first reverse sort them according to the relation "$\prec$" and use results obtained for a given sequence to calculate the initial point for the next sequence. This is especially important for long sequences.

A. All periodic windows with period $p \leq 36$

The algorithm to find bifurcation points presented in Section III C is applied to find all 1 966 957 258 periodic windows with periods $2 \leq p \leq 36$. Computations are carried out using multiple precision interval arithmetic with 256 bits which allows us to find very accurate rigorous bounds of periodic windows' endpoints and widths. Widths of all windows are found with the precision better than 10^{-70}.

Figure 2 shows widths of periodic windows with periods $p \leq 8$ versus their positions in the parameter space. The widest window corresponds to the sequence (01). The next two windows are its period-doubling descendants with sequences (0111) and (01110101). The next widest window is a period-3 window with the sequence (001). It has a common border with its period-doubling descendant with the sequence (001011).

The measure of the set $\bigcup_{p=2}^{36} \Omega_p^-$ is above 0.611834003131. In Ref. 7, the authors found 677 242 periodic windows with periods $2 \leq p \leq 36$ with the total width approximately equal to 0.6118328475, which is smaller by 1.1556×10^{-6} than the true measure of $\bigcup_{p=2}^{36} \Omega_p^-$. Although the results presented in Ref. 7 are based on a very small fraction of the total number of periodic windows, the difference

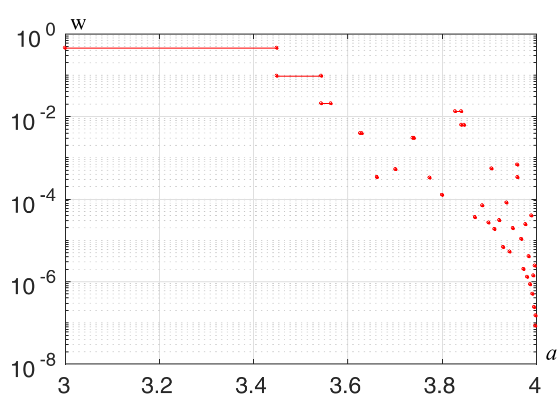

FIG. 2. Widths of periodic windows with periods $p \leq 8$ versus their positions in the parameter space.

in width is small. This means that the approach used in Ref. 7 is successful in locating wide windows and finding a good lower bound of $\mu(\Omega^-)$.

In Fig. 3, we plot the total widths μ_p of period-p windows and the total width of primary period-p windows. These two plots coincide for periods being primary numbers. The total width of primary windows is 0.019185827531, which is approximately 3% of the width of all windows. Let us note that there are only 31 970 period-tupling windows in this set, yet they are responsible for most of the width.

B. Wide primary windows with periods $p \geq 37$

Finding all primary windows for large periods is not feasible due to exponentially growing number of such windows. Therefore, in the search for primary windows we limit ourselves to wide windows only. We refer to windows with the width above $w_{\min} = 10^{-15}$ as *wide windows*. To estimate the impact of skipping narrow windows, let us consider the cases $p = 35$ and 36, for which we know the true results. There are 490 853 349 period-35 primary windows with the total width approximately equal to 7.14584×10^{-8}. Among them, there are 158 388 wide period-35 primary windows and their total width is approximately 7.13328×10^{-8}, which is more than 99.8% of the total width of all period-35 primary windows. For $p = 36$, there are 152 556 wide periodic windows with the total width of 3.00236×10^{-8}, which is more than 99.5% of the total width of 3.01529×10^{-8} of all 954 422 197 period-36 primary windows. It follows that by skipping narrow primary windows we lose only a small fraction of the total width.

To assess the performance of the heuristic procedure for finding wide primary windows presented in Sec. III D, we run it for $p = 35$. Starting with 158 388 wide period-35 primary windows, we generate 2 713 967 test sequences and find 151 854 wide period-36 primary windows with the total width of 3.001943×10^{-8}, which is more than 99.98% of the total width of wide period-36 primary windows. This example shows that the procedure is successful in finding wide primary periodic windows. Applying this procedure to find wide primary windows with periods $37 \leq p \leq 501$ gives 1 460 124 wide primary windows with the total width of 5.1635×10^{-7}.

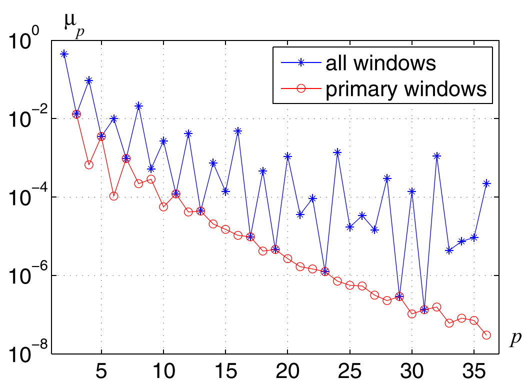

FIG. 3. The total widths $\mu_p = \mu(\Omega_p^-)$ of period-p windows (the "$*$" symbol) and the total width of period-p primary windows (the "◯" symbol).

Scanning the results concerning wide primary windows, one may notice that widest primary windows belong to families of the form $(u^k v)$. For example, $((011)^{12}1)$, $((01111)^7 01)$, and $((011)^{11}1101)$ are symbol sequences of the three widest period-37 windows and $((011)^{12}01)$, $((01111)^7 111)$, $((00101)^7 001)$ are symbol sequences of the three widest period-38 primary windows. We identify such families and find wide periodic windows belonging to them. In this way, we find 27 009 wide primary windows with periods above 501 with the total width of 1.6650×10^{-9} and 13 107 new wide primary windows with periods $p \leq 501$ with the total width of 4.9482×10^{-10}. The longest primary sequence corresponding to a wide periodic window found is $((011)^{3716}01)$ with the period $p = 11\,150$.

Figure 4 shows the total width $\mu_{p,\mathrm{PR}}$ of period-p primary windows found versus p. The results for $p \leq 10$ are plotted using the star symbol. One can see that for $p \geq 40$, the measure $\mu_{p,\mathrm{PR}}$ decreases with p in a periodic fashion. This is due to the existence of wide families of primary windows which are responsible for most of the width for large p. Since a family $(u^k v)$ has a non-zero contribution only at periods differing by the length of the sequence u, one observes oscillations with the period being the least common multiple of lengths of sequences u defining wide families. For widest families, we have $u = (011)$, $u = (01111)$, or $u = (00101)$ and hence the oscillations have period 15, which is visible in the middle part of the plot.

Summarizing, we have found 1 990 128 049 primary windows with periods $3 \leq p \leq 11150$ with the total width above 0.01918634753 including 23 202 761 primary windows with periods $p \geq 37$ with the total width of 5.2×10^{-7}. Here, we also report narrow windows.

C. Wide period-tupling windows

Let us now study structures of period-tupling descendants to justify that the method to find wide period-tupling windows proposed in Section III E works properly. Figure 5(a) shows relative widths of period-tupling descendants of the periodic window with the sequence $s = (001)$. We consider period-tupling sequences of the form s^t for all odd-parity sequences t with period $p \leq 8$. For each period-tupling window, we plot its relative width w_r versus its position. The

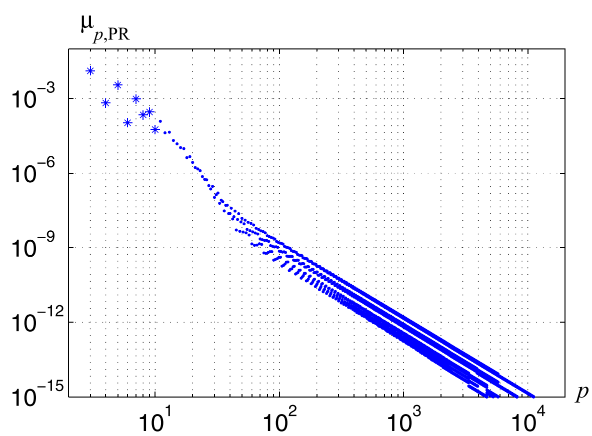

FIG. 4. Total widths $\mu_{p,\mathrm{PR}}$ of primary period-p windows found.

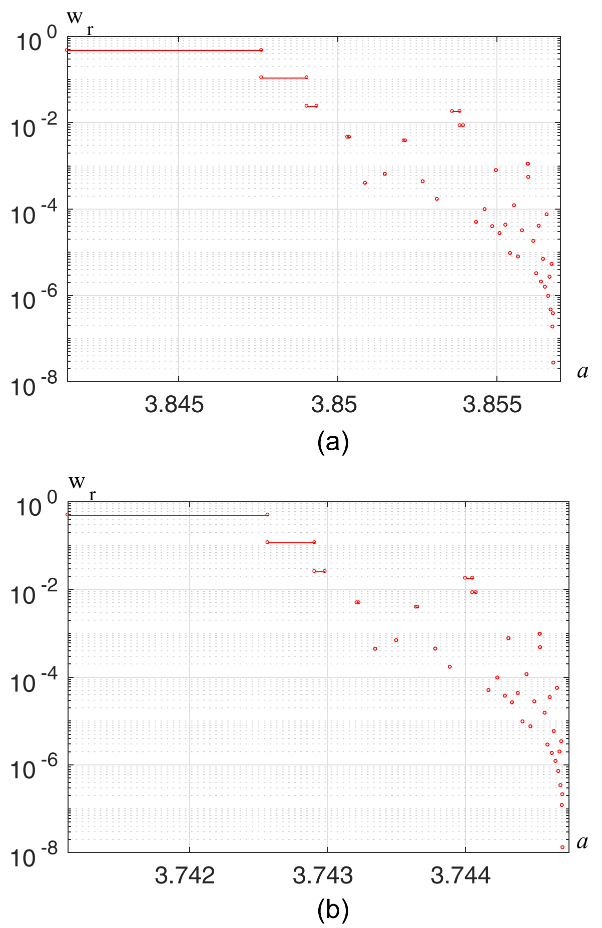

FIG. 5. Relative widths of period-tupling descendants; (a) for the sequence $s = (001)$, (b) for the sequence $s = (00101)$.

relative width w_r is calculated as the ratio of the width of the periodic window with the sequence s^t and the width of the primary window with the sequence s. The results obtained for the sequence $s = (00101)$ are plotted in Fig. 5(b). One can see that these two plots are very similar both in relative positions of period-tupling windows and their relative widths. Similar pictures are observed for other primary sequences. We conclude that it is possible to predict positions and widths of period-tupling windows for all primary sequences based on positions of period-tupling descendants of a single primary sequence. Also note that both plots are similar to the plot of widths of periodic windows with periods $p \leq 8$ (compare Fig. 2). This shows that the structure of periodic windows is self-similar. Period-tupling descendants of any primary window are a reduced copy of the whole structure of periodic windows.

The method described in Section III E is applied to find period-tupling descendants of wide primary windows found in previous steps. We start with $R_1 = S_{\text{pr,w}} \setminus \{(01)\}$. In the kth step for $s = (01)$ and $s = (001)$, we find period-tupling windows with sequences s^r where $r \in R_k$. Then, we find period-tupling descendants of other primary windows. To speed up computation, we consider only those period-tupling descendants for which we predict that their widths are above the threshold $w_{\min} = 10^{-15}$. Predictions are based on results obtained for the sequence $s = (001)$. In this way, we obtain the set R_{k+1} containing new period-tupling sequences. Computations are continued while $R_k \neq \emptyset$. In this way, we find 300 974 482 period-tupling windows. Here, we do not count period-doubling windows. The longest period-tupling sequence corresponding to a wide periodic window found has the length 1 572 864.

In the final step, we sort periodic windows found so far according to increasing parameter values and using the method presented in Section III F, we look for period-doubling windows with widths above the threshold $w_{\min} = 10^{-15}$. In certain cases, to reduce computation times the search is limited to windows with widths above the threshold 10^{-14}. The longest sequence corresponding to wide period-doubling windows is a period-doubling descendant of (01) and has the length 4 194 304. The width of the corresponding period-doubling window is approximately 3.9119×10^{-15}. There are two wide period-3 145 728 windows belonging to a period-doubling cascades of (001) and (011111).

Summarizing, we have found 444 596 181 period-tupling windows including 143 621 699 period-doubling windows.

Widths of periodic windows belonging to period-doubling cascades of saddle-node sequences with periods $3 \leq p \leq 7$ are plotted in Fig. 6. The results for period-doubling cascades starting with sequences (001) and (0001) are plotted in blue and red, respectively. Slopes of plots for large periods are defined by the Feigenbaum constant $\delta = \lim_{k\to\infty} w_k / w_{k+1} \approx 4.669201609$, where w_k denotes the width of the kth window in a period-doubling cascade.[20,23] It is interesting to note that for fixed k the ratios w_k/w_{k+1} do not vary much in different cascades and that they change monotonically towards the limit δ. For example, for the period-doubling cascades shown in Fig. 6 we have $w_1/w_2 \in [1.9998, 2.1389]$, $w_2/w_3 \in [4.2337, 4.3024]$, $w_3/w_4 \in [4.5515, 4.5718]$, $w_4/w_5 \in [4.6458, 4.6470]$, $w_5/w_6 \in [4.6639, 4.6649]$, and $w_k/w_{k+1} \in [4.669201, 4.669202]$ for all $k \geq 11$. It follows that the convergence is quite fast.

The results regarding the number and the total width of periodic windows of a given type are collected in Table I. It follows that the measure of the set of regular parameters satisfies:

$$\mu(\Omega^- \cap [3,4]) > 0.613960137, \tag{10}$$

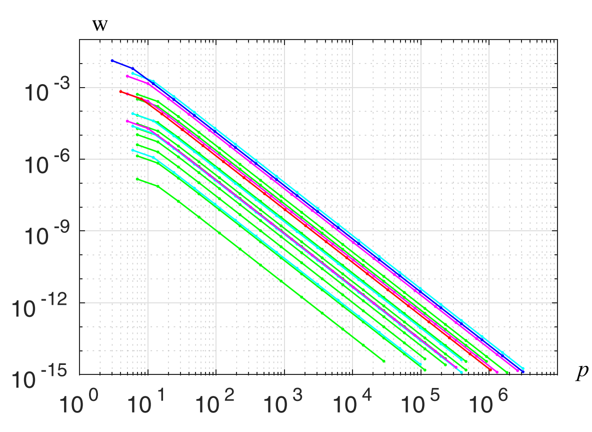

FIG. 6. Widths of periodic windows in period-doubling cascades.

TABLE I. The number and total width of periodic windows of a given type.

Type	Number	Total width
All	2 434 724 230	0.6139601370509258
Primary	1 990 128 049	0.0191863475318510
Period-tupling	444 596 181	0.5947737895190748
Saddle-node	2 291 102 531	0.0271871221799938
Period-doubling	143 621 699	0.5867730148709320
All wide	75 341 957	0.6139601017921689
Primary wide	3 513 394	0.0191863447312755
Period-tupling wide	71 828 563	0.5947737570608933
Saddle-node wide	40 967 731	0.0271870998291728
Period-doubling wide	34 374 226	0.5867730019629961

which is by 1.8028×10^{-5} larger than the bound 0.613942108 reported in Ref. 7.

Recall that periodic windows are classified as primary windows and period-tupling windows (non-primary). We have found much more primary windows than period-tupling windows. This is due to carrying out the exhaustive search for periodic windows with periods $p \leq 36$. These windows have however very little impact on the total width.

The results regarding wide windows (with the width above $w_{\min} = 10^{-15}$) are collected in the bottom part of Table I. Observe that skipping narrow windows causes that the total width drops by less than 3.6×10^{-8}. One can see that there are much fewer wide primary windows than wide period-tupling windows.

Total widths μ_p of period-p windows found are plotted in Fig. 7 in blue. Contribution of period-tupling windows is plotted in red on top of the first plot. Note that when the width associated with primary windows is small compared to the width associated with period-tupling windows only the red dot is visible. For periods being primary numbers, there are no period-tupling windows, and hence in this case, only a blue dot is shown.

D. An estimate of the true measure of $\mu(\Omega^-)$

Inequality (10) provides a rigorous lower bound for the measure $\mu(\Omega^-)$. In this section, we estimate the true value of this measure and find its upper bound. Calculations reported in this section are non-rigorous.

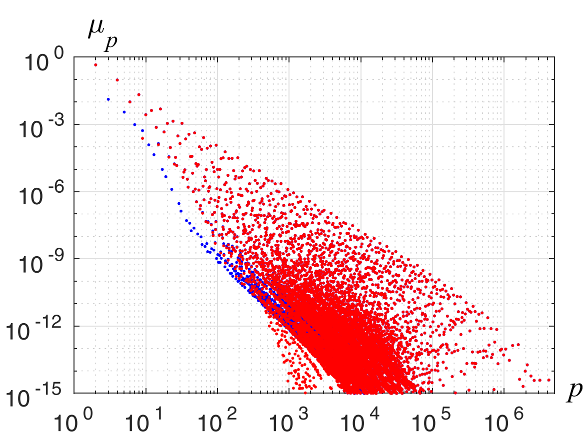

FIG. 7. The total widths of period-p windows found (blue) and the total width of period-p period-tupling windows found (red) versus p.

Let us first estimate the true value of $\mu(\Omega^-)$. Let us denote by b_n the total width of periodic windows found with widths belonging to the bin $\left[2^{-0.1n}, 2^{-0.1(n+1)}\right)$. Figure 8 shows b_n versus bin position. One can see that in the range $\left[10^{-13}, 10^{-8}\right]$ the plot is almost linear on the logarithmic scale. This observation lets us state the hypothesis that the relation $\log(b_n) \approx c_1 n + c_0$ is also true for narrower windows. The drop in the left-hand part of the plot is probably related to the fact that not all windows belonging to the corresponding bins have been found.

The data in the range $\left[10^{-13}, 10^{-10}\right]$ are fitted with the model $\log(b_n) = c_1 n + c_0$ yielding $c_1 \approx -0.028880$, $c_0 \approx -4.917367$. This model is plotted in Fig. 8 as a red dashed line. Using this model, we may approximate the total width of periodic windows as $\sum_{n=1}^{N} b_n + \sum_{n=N+1}^{\infty} \exp(c_1 n + c_0)$, where the first sum involves bins for which we know all corresponding periodic windows. Assuming that all periodic windows with widths above 10^{-13} have been found we obtain $N = 431$ and

$$\mu(\Omega^- \cap [3,4]) \approx 0.613960301, \qquad (11)$$

which is slightly above the rigorous lower bound 0.613960137 presented in Table I.

To find a reliable upper bound of $\mu(\Omega^-)$, let us first study the problem what is the total width of primary windows. Since we know results for periods $p \leq 36$, we need to find an upper bound of the width of primary windows with $p \geq 37$ not found by the procedure. Let us first estimate the width of narrow primary windows belonging to families $(u^k v)$. Widths of periodic windows belonging to the three widest families $((011)^k 01)$, $((011)^k 1)$, and $((01111)^k 01)$ are shown in Fig. 9. One can see that on the logarithmic scale for large p widths change linearly with p. Let us denote by w_p the width of a period-p window belonging to a given family. Linear regression models $\log(w_p) \approx q \log(p) + r$ for each family are computed and shown in Fig. 9 as dashed lines. Models have been computed based on periodic windows with periods $3000 \leq p \leq 4000$. For the considered families, the parameters of the models are $q_1 \approx -2.999145$, $r_1 \approx -6.58848$, $q_2 \approx -2.999877$, $r_2 \approx -7.506251$, and $q_3 \approx -2.999068$, $r_3 \approx -8.5648447$, respectively. Using the model, the width of a period-p window belonging to the

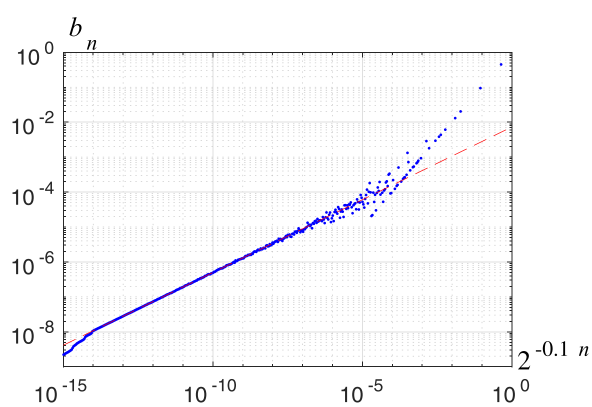

FIG. 8. The total width of periodic windows found with widths belonging to the bin $\left[2^{-0.1n}, 2^{-0.1(n+1)}\right)$ versus the bin position.

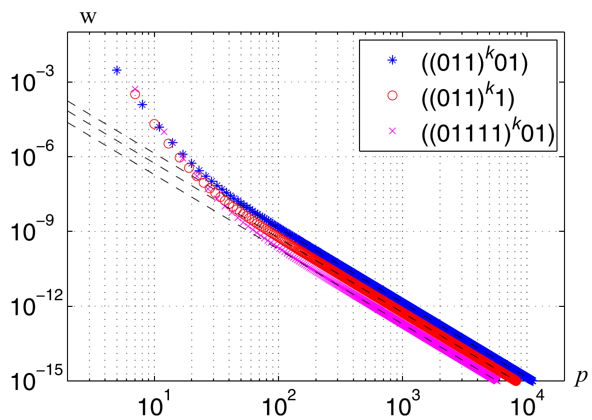

FIG. 9. Widths of periodic windows belonging to families $((011)^k01)$, $((011)^k1)$, and $((01111)^k01)$.

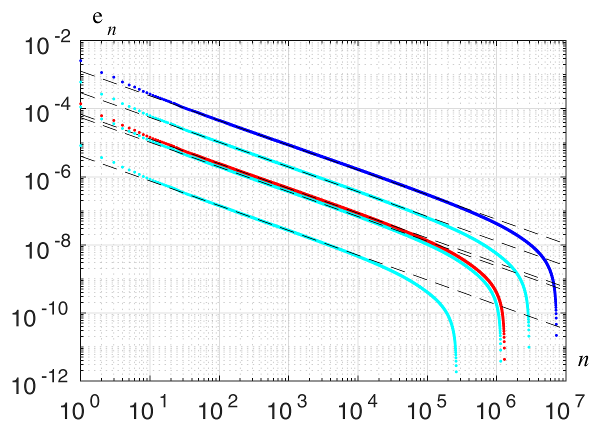

FIG. 10. Error in the computation of the total width of period-tupling descendants by considering n widest period tupling windows only.

family can be approximated as $w_p \approx w_1 p^q$, where $w_1 = \exp(r)$. For example, the relative error between the true width of the periodic window with period 5003 belonging to the first family and its approximate value is less than 5×10^{-5}. This indicates that the models are of a good quality. It is interesting to note that parameters q are almost identical for all three families considered. This observation also holds for other families. Let n_1 and n_2 denote the lengths of the sequences u and v, respectively. For $q < -1$, to estimate the total width of periodic windows in the family with periods $p > p_0 = n_1 k_0 + n_2$, one can use the following formulas obtained by the integral test:

$$\frac{w_1(n_1+p_0)^{q+1}}{n_1(-q-1)} < \sum_{k=1}^{\infty} w_1(p_0+n_1 k)^q < \frac{w_1 p_0^{q+1}}{n_1(-q-1)}. \quad (12)$$

For example, for the family $((011)^k01)$, we obtain bounds $[1.8583 \times 10^{-12}, 1.8594 \times 10^{-12}]$ for the total width of periodic windows with periods larger than 11 150. Similar calculations are applied to the 49 widest families of primary windows. The total width of narrow primary windows is estimated to belong to the interval $[1.42541, 1.42543] \times 10^{-11}$, which is several orders of magnitude less than the total width of primary windows with periods $p \geq 37$. This shows that narrow windows belonging to families of primary windows have a negligible contribution to the total width. Since for large p most of the widths of primary windows is associated with wide families of type $(u^k v)$, it follows that skipping narrow windows for large p introduces a negligible error in estimating the total width.

Based on the discussion in this section and in Sec. IV B, we conclude that the total width of primary windows not found is below 1% of 5.2×10^{-7}, which is the total width of found primary windows with periods $p \geq 37$.

Now, we study the problem what is the total width of period-tupling windows. Figure 10 shows the error e_n between the total width of period-tupling descendants found and the total width obtained when considering only n widest period-tupling descendants. The results obtained for the period-3 window, the period-4 primary window, and period-5 primary windows are plotted in blue, red, and cyan, respectively. In the computations, only wide windows ($w \geq 10^{-15}$) are taken into account. One can see that on the logarithmic scale, the middle part of each plot is almost linear. One can expect that this is also valid for larger n. Drops in plots shown in Fig. 10 are due to considering wide windows only.

Let w_0 be the width of a primary window and w_n (for $n \geq 1$) the width of its nth widest period-tupling descendant. Let us denote by v_n the ratio of the sum $\sum_{k=1}^{n} w_k$ and w_0. We fit the model $v_n = v_\infty - cn^q$ to the data presented in Fig. 10 for $n \in [100, 10\,000]$. For larger n, the results are not reliable, since we consider wide windows only. For example, for the period-3 window, we obtain parameters $v_\infty = 0.66529$, $c = 0.097164$, and $q = -0.72609$. Approximations $w_0(v_\infty - cn^q)$ computed using the obtained models are plotted in Fig. 10 as black dashed lines. Using this model, we obtain an estimate for the total width of period tupling descendants as $w_0 v_\infty$. To obtain an upper bound for the error introduced by considering only the n widest period-tupling descendants, we compute v_∞, c, q, $e_n = v_\infty - w_0^{-1}\sum_{k=1}^{n} w_k$ for several low-period primary windows. The maximum values $v_{\infty,\max} = 0.72$, $c_{\max} = 0.1033$, $q_{\max} = -0.7260$, and $e_{n,\max}$ are used to compute upper bounds for the width of narrow period-tupling descendants of other primary windows. To have an accurate estimate based on the number of n widest period-tupling windows, we need to be sure that no wide windows are missing. We assume that all windows wider than 10^{-13} have been correctly identified. For each primary window, we estimate that the total width of the remaining period-tupling windows is below $w_0 e_{n,\max}$ for $n \leq 1000$ and below $w_0 c_{\max} n^{q_{\max}}$ for $n > 1000$, where w_0 is the width of the primary window considered and n is the number of its period-tupling descendants wider than 10^{-13}.

Taking into account a contribution from period-tupling descendants of primary windows found, a contribution from primary windows not found, and a contribution from their period-tupling descendants, we obtain an upper bound 0.613966 for $\mu(\Omega^-)$. Summarizing, we have the following bounds for the measure of the set of regular parameters

$$0.613960 < \mu(\Omega^- \cap [3,4]) < 0.613966,$$

where the lower bound is rigorous. This is equivalent to the following bounds for the measure of the set of stochastic parameters

$$0.386034 \leq \mu(\Omega^+) \leq 0.386040,$$

where this time the upper bound is rigorous.

V. CONCLUSIONS

A systematic method to find wide periodic windows for the quadratic map has been proposed. Classification of periodic windows has been carried out. A heuristic method to find the majority of wide primary and period-tupling windows has been proposed. We have found all periodic windows with periods $p \leq 36$ and the majority of wide periodic windows with longer periods. Very accurate rigorous bounds of their widths have been calculated. Using these results, we computed a lower bound for the measure of the set of regular parameters better than the existing ones and estimated its true value. The obtained rigorous lower bound is also a non-trivial upper bound for the measure of the set of stochastic parameters. Several properties of primary and period-tupling windows have been revealed. This includes self-similarities in structures of period-tupling descendants, scaling of widths of primary windows belonging to specific families, and properties of period-doubling cascades. Based on these properties, an upper bound of the measure of the set of regular parameters has been computed.

SUPPLEMENTARY MATERIAL

See supplementary material for the data regarding periodic windows wider than 10^{-10}. There are 121 144 such periodic windows and their total width is above 0.61394327. For each periodic window, the following data are provided: the symbol sequence, type of the window (saddle-node, period-doubling, period-tupling), very accurate bounds for both endpoints, and an approximate width.

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[1] R. May, Nature **261**, 459 (1976).
[2] M. Lyubich, Ann. Math. Second Ser. **156**, 1 (2002).
[3] M. Jakobson, Commun. Math. Phys. **81**, 39 (1981).
[4] M. Benedicks and L. Carleson, Ann. Math. **122**, 1 (1985).
[5] S. Luzzatto and H. Takahashi, Nonlinearity **19**, 1657 (2006).
[6] S. Day, H. Kokubu, S. Luzzatto, K. Mischaikow, H. Oka, and P. Pilarczyk, Nonlinearity **21**, 1967 (2008).
[7] W. Tucker and D. Wilczak, Physica D **238**, 1923 (2009).
[8] C. Simó and J. C. Tatjer, in *European Conference on Iteration Theory* (1991), pp. 335–342.
[9] Y. Huang, "Measure of parameters with acim nonadjacent to the Chebyshev value in the quadratic family," Ph.D. thesis (University of Maryland, 2011).
[10] A. Golmakani, S. Luzzatto, and P. Pilarczyk, Exp. Math. **25**, 116 (2016).
[11] Z. Galias, Int. J. Bifurcation Chaos **25**, 1550139 (2015).
[12] K. Alligood, T. Sauer, and J. Yorke, *Chaos: An Introduction to Dynamical Systems* (Springer, 1996).
[13] R. Gilmore and M. Lefranc, *The Topology of Chaos* (Wiley, 2011).
[14] N. Metropolis, M. Stein, and P. Stein, J. Comb. Theory, Ser. A **15**, 25 (1973).
[15] B. Derrida, A. Gervois, and Y. Pomeau, Ann. Inst. Henri Poincare **29**, 305 (1978); available at http://www.numdam.org/item?id=AIHPA_1978__29_3_305_0.
[16] Z. Wan-Zhen, H. Bai-Lin, W. Guang-Rui, and C. Shi-Gang, Commun. Theor. Phys. **3**, 283 (1984).
[17] B. Coomes, H. Koçak, and K. Palmer, in *Six Lectures on Dynamical Systems*, edited by A. Aulbach and F. Colonius (World Scientific, 1996), pp. 163–211.
[18] Z. Galias, Topol. Appl. **124**, 25 (2002).
[19] A. Neumaier, *Interval Methods for Systems of Equations* (Cambridge University Press, Cambridge, UK, 1990).
[20] M. J. Feigenbaum, J. Stat. Phys. **21**, 669 (1979).
[21] MPFR, GNU MPFR library, 2015.
[22] CAPD, CAPD library, 2015.
[23] K. Briggs, Math. Comput. **57**, 435 (1991).