NEW METHOD FOR STABILIZATION OF UNSTABLE PERIODIC ORBITS IN CHAOTIC SYSTEMS

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Abstract

In this paper we present a new method of controlling periodic orbits in chaotic systems. This method can be applied in situations when the chaotic system depends on one system parameter, which can be changed over a continuous interval or over a discrete, two-element set. We compare the new method to other ones, discuss its properties, and illustrate our approach with a numerical example.

1 Introduction

It is well known that even seemingly simple nonlinear systems can behave chaotically. In practical situations however we would prefere to avoid it. There are several possible approaches to the problem of suppressing chaotic behaviour [Ogorzałek, 1993]. In this paper we follow the approach, introduced in [Ott *et al.*, 1990] and studied in other papers [Dressler & Nitsche, 1992, Dąbrowski *et al.*, 1993a], in which control of chaos is understood as stabilization of one of the unstable periodic orbits existing within the strange attractor. In the method presented in [Ott *et al.*, 1990] (referred to as the OGY method), one of the system parameters is changed only when the trajectory intersects the chosen plane and the modification is such that the next intersection of this plane by the system trajectory will fall onto the stable manifold of the periodic orbit. In the case where there are many available parameters one can choose any of them as the control one, but certainly some choices will make the successful control easier to obtain.

Here we present a different approach. The control formula is derived from the condition that the next intersection will be as close to the periodic orbit as possible. We also present the modification of this method for a control parameter with two discrete values only. An example of a system in which such a modification could be applied is an electronic circuit with a switch changing one of the circuit parameters. The switch-control is much easier to implement than the continuous-value (or multilevel) control. In Sec. 2 we recall the notion of Poincaré map and as the motivation for the new method we give an example of a system possessing a periodic orbit attracting along a part of its unstable manifold. In sec. 3 we present the new control method an we discuss the influence of the maximal value of control parameter and the frequency of application of the control signal upon the operation of the method. In Sec. 4 we describe the computer simulations. We present the results of stabilization of unstable periodic orbits in the double-scroll attractor of Chua's circuit.

2 Basic Notions

Let us consider a three-dimensional continuous-time dynamical system, which depends on one system parameter, denoted by p:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{F}(\boldsymbol{x}(t), p). \tag{1}$$

where $\boldsymbol{F}$ is a continuous vector field. We say that the system (1) generates a flow $\phi_t : U \longrightarrow I\!R^n$, where U is some open set in $I\!R^n$, if ϕ satisfies (1) in the sense that

$$\frac{d}{dt}(\phi_t(\boldsymbol{x}))|_{t=\tau} = \boldsymbol{F}(\phi_\tau(\boldsymbol{x}), p)$$

for all $\boldsymbol{x} \in U$ and $\tau \in I = (a, b) \subset I\!R$. $\phi_t(\boldsymbol{x}_0)$ defines a trajectory of the differential equation (1) based at $\boldsymbol{x}_0$ (we will also write this solution as $\boldsymbol{x}(\boldsymbol{x}_0, t)$ or simply $\boldsymbol{x}(t)$).

2.1 Poincaré map, generalized Poincaré map

In this subsection we recall the notion of Poincaré map [Guckenheimer & Holmes, 1983]. Let ϕ_t be a flow arising from the system (1). Let γ be a periodic orbit of the flow. First we choose a local cross section Σ. We assume that Σ is a hyperplane and is everywhere transverse to the flow. We also assume that the intersection of the orbit γ with the hyperplane Σ is one point, denoted by $\boldsymbol{x}_F$ (if this intersection is multipoint we must shrink Σ appropriately). The *Poincaré map* $\boldsymbol{P} : U \longrightarrow \Sigma$ is defined by:

$$\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}_\Sigma(\boldsymbol{x}) = \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}),$$

where U is some neighbourhood of $\boldsymbol{x}_F$ in Σ and $\tau(\boldsymbol{x})$ is the time needed for the trajectory $\phi_t(\boldsymbol{x})$ to return to Σ. The existence of Poincaré maps follows from the following Theorem:

Theorem 1 *([Parker & Chua, 1989]) There exist an open set U with $\boldsymbol{x}_F \in U$, and a unique C^1 map $\tau : U \longrightarrow I\!R$, such that, for all $\boldsymbol{x} \in U$, $\phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}) \in \Sigma$ and $\tau(\boldsymbol{x}_F)$ is the period of γ.*

It is clear that $\boldsymbol{x}_F$ is a fixed point for the map $\boldsymbol{P}$ and the stability of $\boldsymbol{x}_F$ reflects the stability of γ for the flow. The most important property of Poincaré maps is stated in the following theorem:

Theorem 2 *([Parker & Chua, 1989]) Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be any two points on a periodic orbit. Let Σ_1 be a hyperplane passing through $\boldsymbol{x}_1$ transversally to the flow. Likewise, define Σ_2 with respect to $\boldsymbol{x}_2$. Then $D\boldsymbol{P}_{\Sigma_1}(\boldsymbol{x}_1)$ is similar to $D\boldsymbol{P}_{\Sigma_2}(\boldsymbol{x}_2)$.*

From the above theorem the immidiate conclusion is that the eigenvalues of the Jacobian of a Poincaré map are uniquely determined (they do not depend on the choice of a point $\boldsymbol{x}$ on the periodic orbit or the choice of a transversal section).

Let us now choose two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, such that $\boldsymbol{x}_2 = \phi_\tau(\boldsymbol{x}_1)$. Let Σ_1 and Σ_2 be local transversal cross sections such that $\boldsymbol{x}_1 \in \Sigma_1$ and $\boldsymbol{x}_2 \in \Sigma_2$. The *generalized Poincaré map* $\boldsymbol{P}_{\Sigma_1\Sigma_2} : U \longrightarrow \Sigma_2$ is defined by:

$$\boldsymbol{P}_{\Sigma_1\Sigma_2}(\boldsymbol{x}) = \phi_{\tau(\boldsymbol{x})}(\boldsymbol{x}),$$

where U is some neighbourhood of $\boldsymbol{x}_1$ and $\tau(\boldsymbol{x})$ is the time needed for the trajectory based at $\boldsymbol{x}$ to reach Σ_2. For the generalized Poincaré map the property of the eigenvalues of the Jacobian is not preserved. As it will be shown in the next subsection the eigenvalues can vary significantly with the change of points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ on the periodic orbit. Also qualitative change of eigenvalues (change of their position with respect to the unit circle) is possible.

2.2 Example

We will show an example of a two-dimensional dynamical system possessing a periodic orbit, which is unstable and attracting trajectories in a part of it. By a small modification we will produce a three-dimensional system with a periodic saddle-type orbit, which locally repells in both directions. Let us consider the system defined in polar coordinates by the following state equations:

$$\left\{ \begin{array}{rcl} \dot{r} & = & (r-1)(1+c\sin\theta) \\ \dot{\theta} & = & 1 \end{array} \right. \tag{2}$$

By solving Eqs. (2) the global flow can be obtained

$$\phi_t(r_0, \theta_0) = (1 + (r_0 - 1)e^{c\cos\theta_0 - c\cos(t+\theta_0)+t}, t + \theta_0).$$

Let us choose the following local transversal planes:

$$\Sigma_1 = \Sigma = \{(r,\theta) \in I\!R^+ \times S^1 : r > 0, \theta = 0\},$$

$$\Sigma_2 = \{(r,\theta) \in I\!R^+ \times S^1 : r > 0, \theta = \pi\}.$$

Let us define Poincaré map as $\boldsymbol{P} = \boldsymbol{P}_{\Sigma\Sigma}$. The time of flight $\tau(\boldsymbol{x})$ for any point $\boldsymbol{x} \in \Sigma$ is $\tau = 2\pi$ and hence the Poincaré map is given by

$$\boldsymbol{P}(r) = 1 + (r-1)e^{2\pi}.$$

Clearly $r = 1$ is a fixed point of $\boldsymbol{P}$, reflecting the circular closed orbit of radius 1 of the system (2). The Jacobian of this orbit is $D\boldsymbol{P}(1) = \left.\frac{d\boldsymbol{P}}{dr}\right|_{r=1} = e^{2\pi}$, hence the periodic orbit is unstable. Let us define two generalized Poincaré maps: $\boldsymbol{P}_1 = \boldsymbol{P}_{\Sigma_1\Sigma_2}$ and $\boldsymbol{P}_2 = \boldsymbol{P}_{\Sigma_2\Sigma_1}$:

$$\boldsymbol{P}_1(r) = 1 + (r-1)e^{2c+\pi},$$

$$\boldsymbol{P}_2(r) = 1 + (r-1)e^{-2c+\pi}.$$

For $c = \pi$ the Jacobians of generalized Poincaré maps are $D\boldsymbol{P}_1(1) = e^{2c+\pi} = e^{3\pi} > 1$ and $D\boldsymbol{P}_2(1) = e^{-2c+\pi} = e^{-\pi} < 1$. Thus the map $\boldsymbol{P}_2$ is attracting but as the repelling action in the first part ($\boldsymbol{P}_1$) is dominant – the periodic orbit is unstable ($D\boldsymbol{P}(1) = D\boldsymbol{P}_2(1) \cdot D\boldsymbol{P}_1(1)$). Several trajectories of this system based at points close to $(r, \theta) = (1, 0)$ are shown in Fig.1. One can notice that trajectories are globally repelled from the periodic orbit, but locally in the lower half-plane they are attracted.

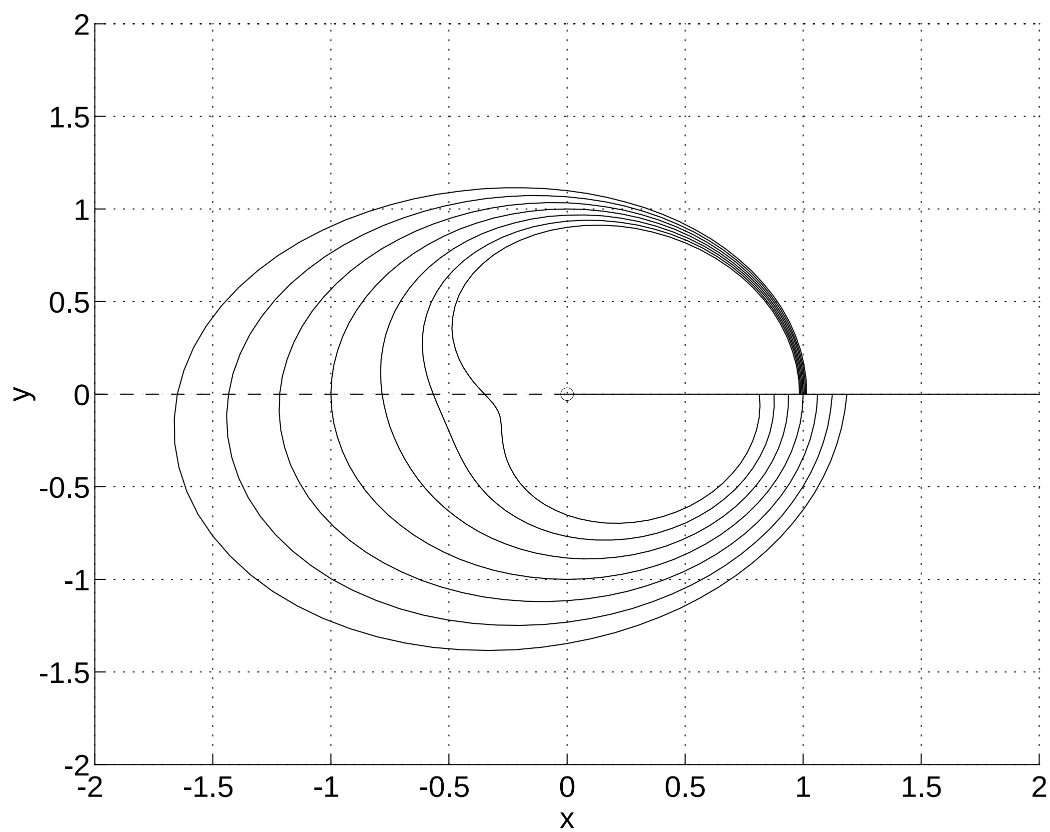

Figure 1: Example of unstable periodic orbit attracting in the lower half-plane

Let us add a third independent variable to the above system:

$$\begin{cases} \dot{r} &= (r-1)(1 + c\sin\theta) \\ \dot{\theta} &= 1 \\ \dot{z} &= -z \end{cases} \tag{3}$$

The global flow is now given by

$$\phi_t(r_0, \theta_0, z_0) = (1 + (r_0 - 1)e^{c\cos\theta_0 - c\cos(t+\theta_0)+t}, t + \theta_0, z_0 e^{-t}).$$

Let us define the following transversal sections:

$$\Sigma_1 = \Sigma = \{(r,\theta,z) \in I\!R^+ \times S^1 \times I\!R : r > 0, \theta = 0\},$$

$$\Sigma_2 = \{(r,\theta,z) \in I\!R^+ \times S^1 \times I\!R : r > 0, \theta = \pi\}.$$

The Poincaré map $\boldsymbol{P} = \boldsymbol{P}_{\Sigma\Sigma}$ is given by

$$\boldsymbol{P}(r,z) = (1 + (r-1)e^{2\pi}, ze^{-2\pi}).$$

Similarly to the two-dimensional system let us define two generalized Poincaré maps:

$$\boldsymbol{P}_1(r,z) = \boldsymbol{P}_{\Sigma_1\Sigma_2}(r,z) = (1 + (r-1)e^{2c+\pi}, ze^{-\pi}),$$

$$\boldsymbol{P}_2(r,z) = \boldsymbol{P}_{\Sigma_2\Sigma_1}(r,z) = (1 + (r-1)e^{-2c-\pi}, ze^{-\pi}).$$

The Jacobian of Poincaré map is

$$D\boldsymbol{P}(1,0) = \begin{bmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix},$$

which means that the fixed point is a saddle. The Jacobians of generalized Poincaré maps are

$$D\boldsymbol{P}_1(1,0) = \begin{bmatrix} e^{2c+\pi} & 0 \\ 0 & e^{-\pi} \end{bmatrix}, \quad D\boldsymbol{P}_2(1,0) = \begin{bmatrix} e^{-2c+\pi} & 0 \\ 0 & e^{-\pi} \end{bmatrix}.$$

The Poincaré map $\boldsymbol{P}$ can be decomposed as $\boldsymbol{P} = \boldsymbol{P}_2 \circ \boldsymbol{P}_1$. Hence the condition $D\boldsymbol{P} = D\boldsymbol{P}_1 \cdot D\boldsymbol{P}_2$ must be fulfilled. For $c = \pi$ the Jacobian $D\boldsymbol{P}_2$ has two eigenvalues $e^{-\pi}$. Thus the stable direction of $D\boldsymbol{P}_2$ does not exist. Similar example can be given for a case where both eigenvalues are unstable. From this example one can see that a decomposition of standard Poincaré map can lead to generalized Poincaré maps with qualitatively different behaviour (they can locally attract in the unstable direction of the periodic orbit or repel in the stable one).

2.3 Comments on the multipoint OGY method

The multipoint OGY formula, as proposed originally in [Ott *et al.*, 1990] does not work properly in all cases. The control formula is derived from the condition that the next intersection will fall onto the stable direction of the next generalized Poincaré map. The silent assumption is that for every generalized Poincaré map $\boldsymbol{P}_j$ there exist stable and unstable directions, which is true for standard Poincaré maps only (singlepoint method). As it has been shown in the previous subsection, in a multipoint case different situations could occur: both real eigenvalues of $D\boldsymbol{P}_j$ can lie outside the unit circle (two unstable directions), both real eigenvalues can lie inside the unit circle (two stable directions) or there could exists two complex eigenvalues. The formula of multipoint OGY method does not say what to do in such situations. The OGY method can be reformulated to be general one. Instead of pushing the trajectory onto the stable direction of the next generalized Poincaré map $\boldsymbol{P}_{j+1}$ one can push it onto the stable direction of periodic orbit, which is the stable direction of the matrix $D\boldsymbol{P}_j \cdot \ldots \cdot D\boldsymbol{P}_1 \cdot D\boldsymbol{P}_n \cdot \ldots \cdot D\boldsymbol{P}_{j+1}$. This is however not a good solution because especially for longer orbits this matrix cannot be computed very accurately. The second reason is that this condition can lead to pushing trajectory onto the stable direction of the periodic orbit also within a region in which the periodic orbit repels locally along the stable direction (compare the example from the previous subsection).

In the next section we propose a new method, which is general (it can be used also when the decomposition of matrices $D\boldsymbol{P}_j$ into stable and unstable directions does not exist). Its idea is very simple. Instead of pushing the trajectory onto the stable manifold, which could repel locally, we try in every step to minimize the distance between the trajectory and the periodic orbit.

3 New Control Method

We consider a dynamical system defined by the state equation (1). We assume that for the nominal value of p, denoted by p_0, the solutions $\boldsymbol{x}(t)$ are chaotic and that there exists an unstable periodic orbit embedded within the attractor. The second assumption is a consequence of one of chaotic attractors' properties, which states that the set of unstable periodic orbits is dense within the attractor. Similarly to the OGY method we assume that the change in the system caused by applying the control parameter in the allowed range is small. By this we mean that for small parameter modifications both the chaotic attractor and the periodic orbit which we want to stabilize do not disappear. In other words we assume that in the considered neighbourhood of the nominal value of the parameter there exists no bifurcation point of this periodic orbit.

3.1 Continuous values of control parameter

First let us consider the case when we can change the parameter p continuously over some interval around its nominal value p_0. Let γ be the unstable periodic orbit, which we want to stabilize. Let us assume that the periodic orbit is parametrized in the following way

$$\gamma = \{\boldsymbol{x}(t) \in I\!R^3 : t \in [0,T]\},$$

where T is the period of the orbit and $\boldsymbol{x}(T) = \boldsymbol{x}(0)$. Let us choose the real values $0 = t_1 < t_2 < \ldots < t_n < T$, which define n points on the periodic orbit ($\boldsymbol{x}(t_j)$; $j = 1 \ldots n$). At each of these points let us choose a plane Σ_j, which is transversal to the periodic orbit γ. For simplicity we assume that these planes are orthogonal to the third axis and are defined by

$$\Sigma_j = \{\boldsymbol{y} = (y_1, y_2, y_3)^T \in R^3 : y_3 = x_3(t_j)).$$

Such an assumption simplifies the implementation of the method. Let $\xi_{Fj} := (x_1(t_j), x_2(t_j))^T$ be a point on the plane Σ_j belonging to the periodic orbit. Without this assumption about transversal planes we would have to define new coordinate systems for each plane, while here we use the coordinates on Σ_j inherited from $I\!R^3$. Let $\boldsymbol{P}_j$ be the generalised Poincaré map between planes Σ_j and Σ_{j+1} associated with the dynamical system considered, namely

$$\xi_{i+1} = \boldsymbol{P}_j(\xi_i, p_i),$$

where $\xi_i \in \Sigma_j$ and ξ_{i+1} is the first intersection with the plane Σ_{j+1} of the trajectory based at ξ_i and p_i is the actual value of control parameter (constant between points ξ_i and ξ_{i+1}). For our method we use the first order approximations of the mappings $\boldsymbol{P}_j$ near ξ_{Fj} and p_0:

$$\delta\xi_{i+1} \approx \boldsymbol{A}_j \delta\xi_i + \boldsymbol{w}_j \delta p_i, \tag{4}$$

where $\delta\xi_i = \xi_i - \xi_{Fj}$, $\delta p_i = p_i - p_0$, $\boldsymbol{A}_j$ is the Jacobian of the map $\boldsymbol{P}_j$ at ξ_{Fj}, p_0 and $\boldsymbol{w}_j = \frac{\partial \boldsymbol{P}_j}{\partial p}(\xi_{Fj}, p_0)$.

We monitor the intersections of the system trajectory with the planes Σ_j. We wait until the intersection ξ_i of the trajectory with the plane Σ_j is close to the point ξ_{Fj}. Then we change p in such a way, that the intersection of the trajectory with the next plane Σ_{j+1} will be as close as possible to the periodic point ξ_{Fj+1}. We will use the Euclidean metric for the derivation of the control formula. We want to choose p such that $||\delta\xi_{i+1}||$ is minimal. In order to find the formula for δp_i we use the following lemma:

Lemma 1 *Let $f(p) = \boldsymbol{A}\xi + \boldsymbol{w}p$, where $p \in I\!R$, $\xi, \boldsymbol{w} \in I\!R^2$ and $\boldsymbol{A}$ is a square 2×2 matrix. For*

$$p = -\frac{\boldsymbol{w}^T \boldsymbol{A}}{||\boldsymbol{w}||^2}\xi$$

$||f(p)||$ is minimal.

Proof: Let us write the norm of the function f in the form:

$$||f(p)||^2 = ||\boldsymbol{A}\xi + \boldsymbol{w}p||^2 = (\boldsymbol{A}\xi + \boldsymbol{w}p)^T(\boldsymbol{A}\xi + \boldsymbol{w}p) = \xi^T\boldsymbol{A}^T\boldsymbol{A}\xi + \xi^T\boldsymbol{A}^T\boldsymbol{w}p + p\boldsymbol{w}^T\boldsymbol{A}\xi + p\boldsymbol{w}^T\boldsymbol{w}p$$

and then calculate its derivative with respect to p:

$$(||f(p)||^2)' = \xi^T\boldsymbol{A}^T\boldsymbol{w} + \boldsymbol{w}^T\boldsymbol{A}\xi + 2p\boldsymbol{w}^T\boldsymbol{w} = 2\boldsymbol{w}^T\boldsymbol{A}\xi + 2p||\boldsymbol{w}||^2.$$

The minimum can be found from the condition $(||f(p)||^2)' = 0$. The necessary condition for the minimum is obviously fulfilled (function $||f(\cdot)||^2$ is quadratic).
□
In our case for

$$\delta p_i = -\frac{\boldsymbol{w}_j^T\boldsymbol{A}_j}{||\boldsymbol{w}_j||^2}\delta\xi_i =: \boldsymbol{g}_j\delta\xi_i \tag{5}$$

$||\delta\xi_{i+1}||$ is minimal.
Thus we change the value of the parameter p by the amount δp_i ($p_i = p_0 + \delta p_i$) and we expect that when the trajectory intersects the next plane Σ_{j+1} it will pass closer to the unstable periodic orbit.

3.2 Two-level parameter control

In the previous subsection we have dealt with the case of continuous variations of the parameter. Here we consider the case when the control parameter can accept only two values say q_1 and q_2, $q_1 < q_2$. The idea is straightforward. If we want to make the two-level control we just check the sign of the value δp_i and we change the value of parameter p by the amount $sgn(\delta p_i)\delta p_{max}$ from the nominal value $p_0 = (q_1 + q_2)/2$, where $\delta p_{max} = (q_2 - q_1)/2$. In other words, if $\delta p_i \geq 0$, we present parameter q_2 to the system and otherwise q_1. By choosing one of the values q_1, q_2 we make the trajectory to move in the desired direction at the maximal possible speed. If the points $t_1, t_2, \ldots, t_{n+1}$ are close enough to each other then the trajectory will not escape far from the periodic orbit and at the next intersection we will have a chance to keep the error small.

All the parameters necessary for the control can be calculated when we can apply only the two values of the parameter p to the system. Let ξ^1_{Fj} and ξ^2_{Fj} be the intersection points of the periodic orbit with transversal planes Σ_j for q_1 and q_2 respectively. Let $p_0 = (q_1 + q_2)/2$. The positions of the periodic points on Poincaré planes change linearly with the change of control parameter (for small parameter variations), and hence $\xi_{Fj} = (\xi^1_{Fj} + \xi^2_{Fj})/2$ is the approximate position of the periodic orbit for p_0. We can apply the described method for the points ξ_{Fj} and the parameter value p_0. We start to run the system with any of two parameters q_1,q_2. We monitor the intersections of system trajectories with planes Σ_j. We calculate δp_i using formula (5) and we apply the parameter

$$p_i = \begin{cases} q_2 & if \quad \delta p_i \geq 0 \\ q_1 & if \quad \delta p_i < 0 \end{cases}$$

In this case we must apply the control parameter more frequently, because the control signal only "informs" the system in which direction to move. We must check quickly enough if the trajectory does not escape far from periodic orbit. Applying control more frequently is done by choosing more points on the periodic orbit.

3.3 Convergence properties - efectiveness of control

Now we investigate the problem, for which maps $\boldsymbol{P}_j$ and vectors $\boldsymbol{w}_j$ the control is effective. Lemma 1 guaranties only minimization of the distance but does not imply the convergence of the trajectories towards the desired orbit. We say that the control is *effective* if there exists some small $\delta > 0$ and t_0 such that for $t > t_0$ the distance between trajectory and the stabilized periodic orbit is smaller than δ. As it will be shown, the effectiveness of the method depends on the direction of vectors $\boldsymbol{w}$ in relation to the eigenvectors of the Jacobian matrices. Let us consider the period-one case. Let $\boldsymbol{P}(\xi, p)$ be the Poincaré map of the system. Without loss of generality we will assume that $\boldsymbol{O} = [0,0]^T$ is a fixed point of $\boldsymbol{P}$ for $p = 0$ ($\boldsymbol{P}(\boldsymbol{O}, 0) = \boldsymbol{O}$). Let the linear approximation of the map $\boldsymbol{P}$ be of the form

$$f(\xi, p) = \boldsymbol{A}\xi + \boldsymbol{w}p. \tag{6}$$

In order to minimize $||f(\xi, p)||$ we choose $p(\xi) = -\frac{\boldsymbol{w}^T\boldsymbol{A}}{||\boldsymbol{w}||^2}\xi$.

First we derive the condition for $\boldsymbol{w}$ ensuring that in every iteration the distance between the trajecory and the periodic orbit is decreased. We want to find vectors $\boldsymbol{w}$ for which there exists some neighbourhood U of the fixed point $\boldsymbol{O}$ such that for every $\xi \in U$

$$||\boldsymbol{P}(\xi)|| < ||\xi|| \tag{7}$$

The function f can be written as

$$f(\xi) = \boldsymbol{A}\xi + \boldsymbol{w}p(\xi) = \boldsymbol{A}\xi - \frac{\boldsymbol{w}\boldsymbol{w}^T\boldsymbol{A}}{||\boldsymbol{w}||^2}\xi = \left(I - \frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}\right)\boldsymbol{A}\xi =: \boldsymbol{W}\xi. \tag{8}$$

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and $\boldsymbol{w} = ||\boldsymbol{w}|| \cdot (\cos\psi, \sin\psi)^T$.

Lemma 2 *Let f, $\boldsymbol{A}$, $\boldsymbol{w}$ be defined above.*
Then

$$\forall \xi \quad ||f(\xi)|| < ||\xi|| \tag{9}$$

if and only if

$$(a_{11}\sin\psi - a_{21}\cos\psi)^2 + (a_{12}\sin\psi - a_{22}\cos\psi)^2 < 1 \tag{10}$$

Proof: Because the map f is linear it is enough to check the condition (9) for ξ such that $||\xi|| = 1$. Hence it is equivalent to checking if the matrix 2-norm of $\boldsymbol{W}$ is smaller than one. The 2-norm of a matrix $\boldsymbol{B}$ is defined by:

$$||\boldsymbol{B}|| = \max_{||\xi||=1} ||\boldsymbol{B}\xi||$$

where $||\cdot||$ on the right-hand side is the Euclidean norm. It can be proved that the matrix 2-norm is equal to the square root of the largest eigenvalue of the matrix $\boldsymbol{B}^T\boldsymbol{B}$ or equivalently the largest singular value of $\boldsymbol{B}$. Let us write the matrix $\boldsymbol{W}$ in a more convevient form:

$$\boldsymbol{W} = \left(I - \frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}\right)\boldsymbol{A} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos^2\psi & \cos\psi\sin\psi \\ \cos\psi\cos\psi & \sin^2\psi \end{pmatrix}\right) \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \sin^2\psi & -\cos\psi\sin\psi \\ -\cos\psi\cos\psi & \cos^2\psi \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

After simple algebraical manipulations we can write the matrix $\boldsymbol{W}^T\boldsymbol{W}$ as

$$\begin{pmatrix} c_1^2 & c_1c_2 \\ c_1c_2 & c_2^2 \end{pmatrix}$$

where $c_1 = a_{11}\sin\psi - a_{21}\cos\psi$ and $c_2 = a_{12}\sin\psi - a_{22}\cos\psi$. Its eigenvalues are $\mu_1 = 0$ and $\mu_2 = (a_{11}\sin\psi - a_{21}\cos\psi)^2 + (a_{12}\sin\psi - a_{22}\cos\psi)^2$. Hence the 2-norm of $\boldsymbol{W}$ is $||\boldsymbol{W}|| = \sqrt{\mu_2}$. Finally $||\boldsymbol{W}|| < 1$ iff $|\mu_2| < 1$.
□
The condition (10) can be written as

$$\cos 2\psi \cdot A + \sin 2\psi \cdot B > C,$$

where $A = a_{11}^2 + a_{12}^2 - a_{21}^2 - a_{22}^2$, $B = 2(a_{11}a_{21} + a_{12}a_{22})$ and $C = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 - 2$. We can obtain the solution in ψ of this inequality by introducing the new variable θ, such that $\cos 2\theta = A/\sqrt{A^2+B^2}$ and $\sin 2\theta = B/\sqrt{A^2+B^2}$

$$\cos(2\xi - 2\theta) > D = \frac{C}{\sqrt{A^2+B^2}}.$$

One can check that $|D| \leq 1$ iff

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 1 + (det\boldsymbol{A})^2. \tag{11}$$

In this case ψ satisfies inequality (10) iff $\psi - \theta \in [-\frac{1}{2}\arccos D + k\pi, \frac{1}{2}\arccos D + k\pi]$ for some integer k. If (11) does not hold we have two subcases. If $C < 0$ (which corresponds to small elements of matrix $\boldsymbol{A}$) then the condition (10) is fulfilled for all ψ. If all elements of the matrix $\boldsymbol{A}$ are small then trajectories of the original system are attracted by the fixed point, and no control is necessary to obtain demanded behaviour. If $C > 0$ then there exist no ψ satisfying (10).

Let us consider the case when the matrix $\boldsymbol{A}$ has one stable and one unstable eigenvalues. In order to find the optimal position of vector $\boldsymbol{w}$ in relation to the eigenvectors of the Jacobian $\boldsymbol{A}$ let us assume that the matrix $\boldsymbol{A}$ is diagonal. Let f and $\boldsymbol{w}$ be the same as in the previous lemma and

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{12}$$

with $|\lambda_1| > 1 > |\lambda_2|$. In this case $\theta = 0$, $D = \frac{\lambda_1^2+\lambda_2^2-2}{\lambda_1^2-\lambda_2^2}$ and (10) is equivalent to

$$\cos 2\psi > \frac{\lambda_1^2 + \lambda_2^2 - 2}{\lambda_1^2 - \lambda_2^2} \tag{13}$$

It is now clear that the most successful control is achieved when the vector $\boldsymbol{w}$ is parallel to the unstable eigenvector of the Jacobian $\boldsymbol{A}$ (from (13) it follows that ψ should be close to 0 or π which means that $\boldsymbol{w} = \pm||\boldsymbol{w}||(1,0)^T$ while the unstable eigenvector is $e_u = (1,0)^T$). It can be shown that if the eigenvectors of matrix $\boldsymbol{A}$ are orthogonal then θ is the angle between the x-axis and the unstable eigenvector of the matrix $\boldsymbol{A}$.

Another sufficient condition ensuring proper behaviour of the method in an ideal case is the existence of some neighbourhood U of the fixed point $\boldsymbol{O}$ such that for all $\xi \in U$

$$||\boldsymbol{P}^n(\xi)|| \longrightarrow \boldsymbol{O} \;\; for \;\; n \longrightarrow \infty \tag{14}$$

Lemma 3 *Let f, $\boldsymbol{A}$, $\boldsymbol{w}$ be the same as in Lemma 2.*
Then for every ξ

$$||f^n(\xi)|| \longrightarrow 0 \;\; for \;\; n \longrightarrow \infty \tag{15}$$

if and only if

$$|a_{11}\sin^2\psi + a_{22}\cos^2\psi - (a_{12}+a_{21})\cos\psi\sin\psi| < 1 \tag{16}$$

Proof: As the map f is linear we must check when all eigenvalues of the matrix $\boldsymbol{W}$ lie inside the unit circle. The eigenvalues of matrix $\boldsymbol{W}$ are $\mu_1 = 0$ and $\mu_2 = a_{11}\sin^2\psi + a_{22}\cos^2\psi - (a_{12}+a_{21})\cos\psi\sin\psi$.
□

The above two lemmas state the properties of the linear approximation of the map $\boldsymbol{P}$. As an immediate conclusion we obtain the following corollary describing the behaviour of the map $\boldsymbol{P}$ in the neighbourhood of the fixed point $\boldsymbol{O}$.

Corollary 1 *From the condition (10) it follows that there exists some neighbourhood U of the fixed point such that for every ξ from this neighbourhood $||\boldsymbol{P}(\xi)|| < ||\xi||$.*
From the condition (16) it follows that there exists some neighbourhood U of the fixed point $\boldsymbol{O}$ such that for every ξ from this neighbourhood $||\boldsymbol{P}^n(\xi)|| \longrightarrow \boldsymbol{O}$ for $n \longrightarrow \infty$.

The conditions (10) and (16) do not assume that a decomposition of the Jacobian into stable and unstable directions exists. Hence they can be used as criteria for the method to work also in a multipoint method. Using this method for the stabilization of periodic orbits with two unstable directions when the condition (16) is fulfilled is however problematic. Such an orbit does not belong to the chaotic attractor and it is very unlikely that the trajectory will ever come inside a small neighbourhood of the periodic orbit allowing us to start the control.

The condition in Lemma 2 is stronger than the one from Lemma 3. The first one ensures that in every iteration the distance from the fixed point of the Poincaré map is decreased, while the second one ensures only convergence of the trajectory to the fixed point. For example let us consider the Poincaré map $\boldsymbol{P}$ with the Jacobian being a diagonal matrix with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0.5$. Then (10) is fulfilled for $\psi \in [-0.464, 0.464] \cup [2.678, 3.605]$ which is about 30% of the 2π interval while (16) is fulfilled for $\psi \in [-0.615, 0.615] \cup [2.526, 3.757]$ which is about 39% of the 2π interval. The condition (16) can sometimes be too weak in real experiments. It is possible that after starting control the trajectory will leave a small neighbourhood limited by the maximal allowed parameter changes and/or by the nonlinear effects.

Now let us study the effect of increasing the number of points along the periodic orbit at which we modify the control parameter. If we decompose the Poincaré map defined above into two maps with equal Jacobian such that $\lambda_1 = \sqrt{2}$ and $\lambda_2 = \sqrt{0.5}$ (which is not always possible but gives us an idea, what happens in the multiplane case) then (10) is fulfilled for $\psi \in [-0.62, 0.62] \cup [2.53, 3.76]$ which is about 39% of 2π interval. From this example one can see that typically increasing the number of Poincaré planes make it easier for $\boldsymbol{w}$ to fulfill the condition (10).

We would like to emphasize, that (10) and (16) are only sufficient conditions for the method to work. In the conditon (7) and (14) the worst case is checked. The trajectory has not to visit the worst case points (directions) or can visit them rarely. Then the fixed point will attract the trajectory on the average and the method could work without satisfying condition (16) or equivalent inequality (14). Another possible criterion would be calculating the average ratio $||\boldsymbol{P}(\xi)||/||\xi||$ over the chaotic trajectory in the neighbourhood of $\boldsymbol{O}$, but this would involve calculation of the invariant measure and is much more complicated.

3.4 Characterization of the two-level control signal

In this subsection we will discuss the problem of choosing p_{max} and the minimal number of points n for a given periodic orbit in the case of the two-level control. First let us consider the case $n = 1$. Let us assume, like in the previous subsection that $\boldsymbol{P}(\xi, p)$ is the Poincaré map of the system, $\boldsymbol{O}$ is the fixed point for $p = 0$ and f defined by (6) is the linear approximation of the map $\boldsymbol{P}$. In this case we choose $p(\xi) = sgn(-\frac{\boldsymbol{w}^T\boldsymbol{A}}{||\boldsymbol{w}||^2}\xi) \cdot p_{max}$ as the control signal. Let us assume that the matrix $\boldsymbol{A}$ has two real eigenvalues λ_1, λ_2 and $|\lambda_1| > 1 > |\lambda_2|$. Let us change the coordinate system in such a way that in this new coordinate system the Jacobian of the periodic orbit is a diagonal matrix defined by (12). Notice that $\boldsymbol{P}(\boldsymbol{O}) = \boldsymbol{P}(\boldsymbol{O}, p(\boldsymbol{O})) = \boldsymbol{P}(\boldsymbol{O}, p_{max}) = \boldsymbol{w}p_{max}$. Hence we are not able to ensure the minimization of distance in subsequent iterations or even converging of trajectories towards the fixed point. We will use the following condition ensuring effectiveness of the method: there exists some small real value δ such that

$$||\xi||_\infty \leq \delta \quad \Rightarrow \quad ||\boldsymbol{P}(\xi)||_\infty < \delta. \tag{17}$$

If this condition is satisfied than any trajectory starting from the ball with radius δ in maximum norm will remain in this ball for ever. This time we have used the maximum norm ($||(x_1, x_2)||_\infty := max(x_1, x_2)$) since for the Euclidean norm the results are difficult to obtain and to interpret (these two norms are equivalent). First we will discuss the problem how to choose the value of p_{max} in order to satisfy (17).

Lemma 4 *Let*

$$f(\xi) = \boldsymbol{A}\xi + \boldsymbol{w}p(\xi) = \boldsymbol{A}\xi + \boldsymbol{w} \cdot p_{max} \cdot sgn\left(-\frac{\boldsymbol{w}^T\boldsymbol{A}}{||\boldsymbol{w}||^2} \cdot \xi\right),$$

where $\boldsymbol{A}$ is the diagonal matrix defined by (12) and $\boldsymbol{w} = (w_1, w_2)^T$. Let δ be a positive real value. Then

$$||\xi||_\infty \leq \delta \quad \Rightarrow \quad ||f(\xi)||_\infty < \delta. \tag{18}$$

if and only if

$$\begin{aligned}
|\lambda_2 w_2| &\leq |\lambda_1 w_1| \\
p_{max} &> \delta \cdot \frac{|\lambda_1| - 1}{|w_1|} \\
p_{max} &< \delta \cdot \frac{1 - |w_2\lambda_2/w_1|}{|w_1|} \\
p_{max} &< \delta \cdot \frac{1 - |\lambda_2|}{|w_2|}.
\end{aligned} \tag{19}$$

Proof: Let us consider the case w_1, w_2, λ_1, $\lambda_2 > 0$. For other cases the proof is similar. In the proof $||\cdot||$ will denote the maximum norm. In order to find the condition equivalent for (18) we calculate the maximum of the norm $||f(\xi)||$ over the ball $\{\xi : ||\xi|| \leq \delta\}$:

$$\begin{aligned}
m &:= \max\{||f(\xi)|| : ||\xi|| \leq \delta\} \\
&= \max\{||\boldsymbol{A}\xi + \boldsymbol{w}p_{max}sgn(-\boldsymbol{w}^T\boldsymbol{A} \cdot \xi)|| : ||\xi|| \leq \delta\} \\
&= \max\{\max\{||\boldsymbol{A}\xi + \boldsymbol{w}p_{max}|| \ : \ ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \geq 0\}, \\
&\qquad \max\{||\boldsymbol{A}\xi - \boldsymbol{w}p_{max}|| \ : \ ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi < 0\}\}.
\end{aligned}$$

It can be checked that

$$\max\{||\boldsymbol{A}\xi + \boldsymbol{w}p_{max}|| : ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \geq 0\} = \max\{||\boldsymbol{A}\xi - \boldsymbol{w}p_{max}|| : ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \leq 0\}$$

and hence

$$m = \max\{||\boldsymbol{A}\xi - \boldsymbol{w}p_{max}|| : ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \geq 0\}.$$

After using the maximum norm definition for the expression $||\boldsymbol{A}\xi - \boldsymbol{w}p_{max}||$ we obtain $m = max\{m_1, m_2\}$ where

$$\begin{aligned}
m_1 &= \max\{|\lambda_1 x_1 + w_1 p_{max}| : ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \geq 0\}, \\
m_2 &= \max\{|\lambda_2 x_2 + w_2 p_{max}| : ||\xi|| \leq \delta, -\boldsymbol{w}^T\boldsymbol{A}\xi \geq 0\}.
\end{aligned}$$

The condition $-\boldsymbol{w}^T \boldsymbol{A}\xi \geq 0$ is equivalent to $\lambda_1\xi_1 w_1 + \lambda_2\xi_2 w_2 \leq 0$, m_1 can be written as

$$\begin{aligned} m_1 &= \max\{|\lambda_1\xi_1 + w_1 p_{max}| : |\xi_1| \leq \delta, |\xi_2| \leq \delta, \lambda_1\xi_1 w_1 + \lambda_2\xi_2 w_2 \leq 0\} \\ &= \max\{|\lambda_1\xi_1 + w_1 p_{max}| : |\xi_1| \leq \delta, \xi_1 \leq \delta \cdot \frac{w_2\lambda_2}{w_1\lambda_1}\}. \end{aligned}$$

Now let us assume that

$$\lambda_1 w_1 \geq \lambda_2 w_2. \tag{20}$$

Then

$$m_1 = \max\{|\lambda_1\xi_1 + w_1 p_{max}| : -\delta \leq \xi_1 \leq \delta \cdot \frac{w_2\lambda_2}{w_1\lambda_1}\}.$$

Because the expression $\lambda_1\xi_1 + w_1 p_{max}$ is linear it follows

$$m_1 = \max\{|-\delta\lambda_1 + w_1 p_{max}|, |\delta \cdot \frac{w_2\lambda_2}{w_1} + w_1 p_{max}|\}.$$

The same argument is true when computing m_2 but this time due to the assumption (20) the condition $\xi_2 \leq \delta \cdot \frac{w_1\lambda_1}{w_2\lambda_2}$ is weaker than $\xi_2 \leq \delta$. Hence:

$$\begin{aligned} m_2 &= \max\{|\lambda_2\xi_2 + w_2 p_{max}| : |\xi_2| \leq \delta, \xi_2 \leq \delta \cdot \frac{w_1\lambda_1}{w_2\lambda_2}\} \\ &= \max\{|\lambda_2\xi_2 + w_2 p_{max}| : |\xi_2| \leq \delta\} \\ &= \max\{|-\delta\lambda_2 + w_2 p_{max}|, |\delta\lambda_2 + w_2 p_{max}|\} \\ &= |\delta\lambda_2 + w_2 p_{max}|. \end{aligned}$$

Finally

$$m = \max\{m_1, m_2\} = \max\{|-\delta\lambda_1 + w_1 p_{max}|, |\frac{\delta w_2\lambda_2}{w_1} + w_1 p_{max}|, |\delta\lambda_2 + w_2 p_{max}|\}.$$

From $m < \delta$ and the additional assumption (20) the conditions (19) can be easily derived.

Let us now consider the second case when the assumption (20) does not hold. Then $\lambda_1 w_1 < \lambda_2 w_2$. In this case we will obtain the same formula for m but with exchanged indices:

$$m = \max\{|-\delta\lambda_2 + w_2 p_{max}|, |\frac{\delta w_1\lambda_1}{w_2} + w_2 p_{max}|, |\delta\lambda_1 + w_1 p_{max}|\}.$$

As $|\lambda_1| > 1$ then the last element $|\delta\lambda_1 + w_1 p_{max}| > \delta$, hence $m > \delta$ and the condition (18) is not fulfilled.
□

From (19) one can see that p_{max} depends linearly on δ. If we want the trajectory to remain closer to the stabilized periodic orbit we must choose smaller p_{max}.

Corollary 2 *If*

$$\begin{aligned} |\lambda_2 w_2| &\leq |\lambda_1 w_1| \\ |\lambda_1| &< 2 - |\frac{w_2\lambda_2}{w_1}| \\ |\lambda_1| &< 1 + (1 - |\lambda_2|)|\frac{w_1}{w_2}| \end{aligned} \tag{21}$$

then there exists p_{max} such that (17) hold.

Proof: We choose δ such that inside the ball $\{\xi : ||\xi||_\infty < \delta\}$ the linear approximation (6) is good. From the previous lemma the value of p_{max} satisfying conditions (19) can be chosen if and only if

$$\begin{aligned} |\lambda_2 w_2| &\leq |\lambda_1 w_1| \\ \delta \cdot \frac{|\lambda_1| - 1}{|w_1|} &< \delta \cdot \frac{1 - |w_2\lambda_2/w_1|}{|w_1|} \\ \delta \cdot \frac{|\lambda_1| - 1}{|w_1|} &< \delta \cdot \frac{1 - |\lambda_2|}{|w_2|} \end{aligned}$$

which is equivalent to conditions (21).
□

The condition (17) is only a sufficient condition for the method to work and hence the conditions for λ_1, λ_2, w_1 and w_2 in the previous two Corollaries are not strictly equivalent to the proper behaviour of the method. But in spite of that we have found them to be good criteria when searching for the best location of points on the periodic orbit used for the control.

Corollary 3 *For $|\lambda_1| \geq 2$ there exists no pair (δ, p_{max}) satisfying (19). If λ_2 could be neglected ($|\lambda_2| \ll 1$) (which is usually true for unstable periodic orbits in chaotic systems) and $|w_1| > |w_2|$ (which should be true for the effective control) then for $|\lambda_1| < 2$ there exist p_{max} and δ satisfying (19) and the effectiveness of control depends on noise in the system.*

Corollary 4 *Let n denote the number of points on the periodic orbit used for the two-level control. Let λ_1 be the unstable eigenvalue of the periodic orbit. If*

$$n \leq \ln_2 |\lambda_1|,$$

then there exists no pair (δ, p_{max}) satisfying (19) for every generalized Poincaré map $\boldsymbol{P}_j$, $j \in \{1, \ldots, n\}$.

Proof: In order to satisfy (19) we must choose so many points on the periodic orbit that all matrices $\boldsymbol{A}_j$ have their eigenvalues smaller than two. Let us assume that $|\lambda_1^j| < 2$ are the unstable eigenvalues of matrices $\boldsymbol{A}_j$, and λ_1 is the unstable eigenvalue of matrix $\boldsymbol{A} = \boldsymbol{A}_n \cdot \ldots \cdot \boldsymbol{A}_1$. Then

$$|\lambda_1| < |\lambda_1^1| \cdot \ldots \cdot |\lambda_1^n| < 2^n$$

and hence the Corollary is true.
□

3.5 Properties of the method

The two methods described are feedback methods. Only one accessible system parameter with only two levels is necessary for stabilization of periodic orbits embedded within the attractor. For the implementation of the method one must calculate 5 parameters for each of the n points (4 in the case of two-level control), namely three coordinates of points $\boldsymbol{x}(t_j)$ and two coefficients of vector $\boldsymbol{g}_j$ defined in (5) necessary for obtaining the control signal δp_i using Eq. (5). All the parameters necessary for the control can be calculated without knowledge of the system equations. Periodic orbits can be found from the data series using the procedures given by Lathrop and Kostelich [Lathrop & Kostelich, 1989], the approximation of the Jacobians $\boldsymbol{A}_j$ could be found using the standard LS algorithm. For the determination of vectors $\boldsymbol{w}_j$ one could use the method presented in [Dressler & Nitsche, 1992]. For the case when only one system variable is measureable the well-known delay coordinate embedding technique is also available.

3.6 Discussion of parameter settings ensuring proper operation of the method

There are several parameters which must be set properly to ensure the desired behaviour of the controlled system. The first parameter is d_{max}, which determines whether to activate control or not. If intersection of the trajectory with one of Poincaré planes is detected then the distance d between the actual intersection and the periodic orbit is calculated. If it is greater than d_{max} then the parameter perturbation is set to zero. If $d \leq d_{max}$ then the parameter change is computed using Eq. (5) and the control is activeted. Another possibility is not to check the distance and apply the control always after detection of an intersection. But this solution does not work properly in most situations because one orbit can intersect a given plane many times. Also the linearisation conditions do not hold in large neighbourhoods. The value of d_{max} could not be chosen to small because chaotic transients would be too long and also the noise could affect the result. During the experiments we have used $d_{max} = 0.01$ and we have calculated distances after normalizing variables to the interval $[0, 1]$.

Another important parameters are: n - the number of points on periodic orbit at which we modify the control signal and the positions of these points. We already know the lower boundary for n in the case of two-level control. It seems that the most effective control should be obtained when the unstable eigenvalues of matrices $\boldsymbol{A}_j$ are equal in absolute value. The same should also be true for continuous-value control. But as will be shown in one of examples this is not always true.

The next parameter which must be chosen carefully in the two-level control is p_{max}. The value of p_{max} should satisfy conditions (19) from Lemma 4, but unfortunately we do not know the value δ for which the linearisation is correct. It cannot be chosen too big due to nonlinear effects and too small because change in the system caused by applying the signal $\pm p_{max}$ must excede the level of noise. Usually it has to be chosen by a trial-and-error. The influence of the number of points n on allowable values of p_{max} is discussed in the next section. Generally for greater n one has more freedom in choosing p_{max}.

4 Simulation Results

For the experiments we have used the canonical Chua's circuit [Chua & Lin, 1990], with the dynamics described by a third-order state equation:

$$\begin{aligned} C_1\dot{x} &= -g(x) + z \\ C_2\dot{y} &= -Gy + z \\ L\dot{z} &= -x - y - Rz \end{aligned} \tag{22}$$

where $g(\cdot)$ has a three-segment piecewise-linear characteristic:

$$g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|) \tag{23}$$

We have used the following parameter values: $C_1 = 1$, $C_2 = -0.632$, $G = -0.0033$, $L = -1.02$, $R = -0.33$, $G_a = -0.419$, $G_b = 0.839$. For this set of parameters the double-scroll attractor exists. The double-scroll attractor is shown in Fig.2. Four of the periodic orbits embedded within the attractor are shown in Fig.3. We use the following coding of unstable periodic orbits: $\gamma_{m,n}$ denotes periodic orbit with m windings around P_+ and n windings around P_-, where P_+ and P_- are unstable equilibria of the system in the regions $x > 1$ and $x < -1$ appropriately.

During the experiments the state equation has been integrated numerically using fourth-order Runge-Kutta method with time step 0.1 and saved as a three-dimensional time series.

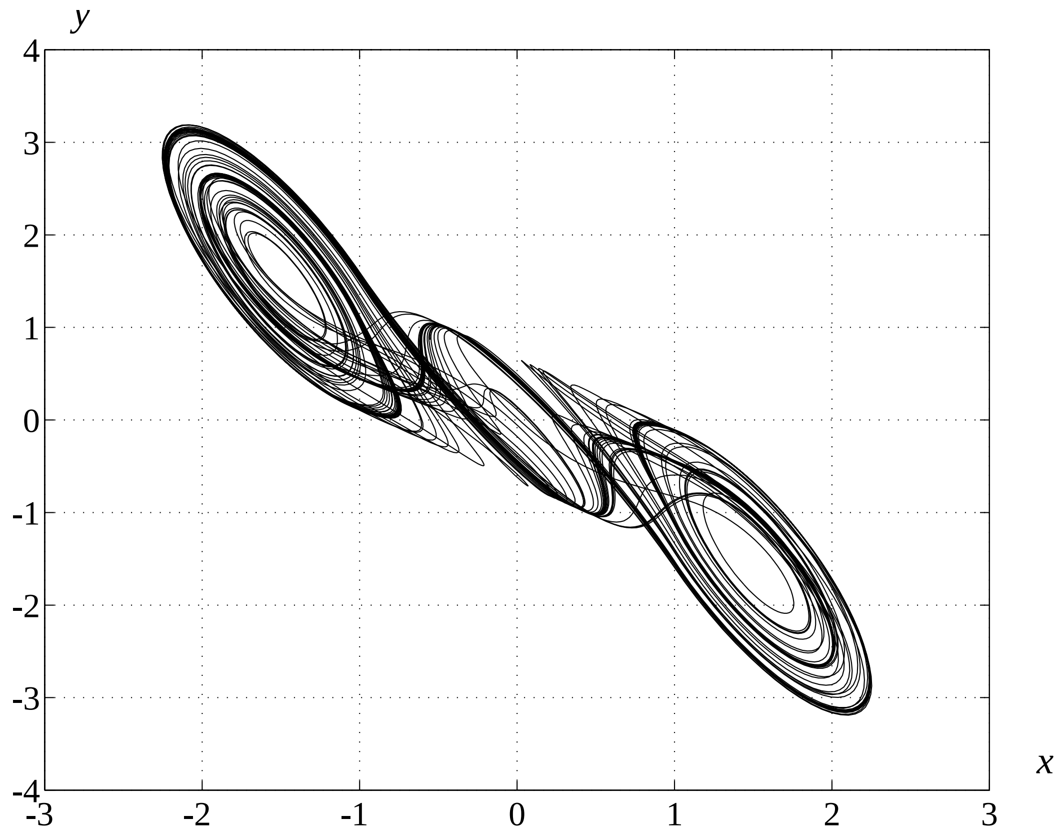

Figure 2: The double scroll Chua's attractor

All the parameters necessary for the control have been calculated from this data series without the knowledge of the system state equation. In all experiments we have used Poincaré planes orthogonal to the first axis. We have chosen C_1 as the control parameter.

4.1 Continuous-value control of period-one orbit

In the first experiment we have controlled the period-one orbit (compare Fig. 3a) using single-point ($n = 1$) continuous-value control method. We have chosen one point on the orbit defined by: $x = 1, \dot{x} < 0$. The computed eigenvalues and vector $\boldsymbol{w}$ used during the control are presented below.

λ_1	λ_2	w_1	w_2
-2.55	-0.002	0.75	0.21

The condition (10) is satisfied, and hence also the condition (16). The result of the control is shown in Fig. 4.

We would like to stress that Lemma 3 gives sufficient conditions only if we assume that all the parameters in the linear approximation are accurate and the nonlinear effects are negligible. But this is not always true, especially if parameters are found from a data series. We have tried to control the same periodic orbit for a slightly changed nominal value of parameter, $C_1 = 1.02$. This time computed coefficients are

λ_1	λ_2	w_1	w_2
-3.39	-0.002	0.71	0.19

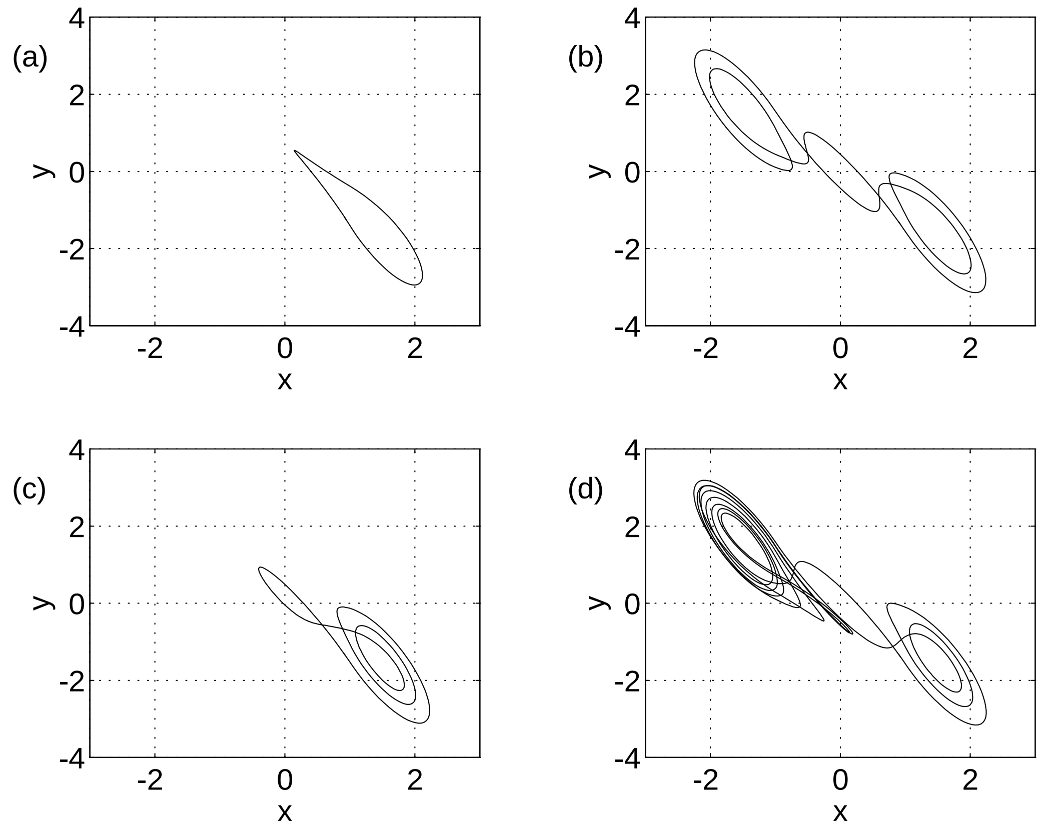

Figure 3:
Examples of unstable periodic orbits within the double scroll strange attractor,
(a) $\gamma_{1,0}$, (b) $\gamma_{2,2}$, (c) $\gamma_{3,0}$, (d) $\gamma_{3,9}$.
The subscripts denotes the number of windings around the equilibria P_+ and P_-

Although for these coefficients the assumptions of Lemma 3 hold we were not able to stabilize this periodic orbit. The reason could be some inaccuracies in calculation of coefficients and/or too large ustable eigenvalue. Such a large value could increase errors of computed parameters exponentially and cause that the control signal is applied too rarely.

4.2 Continuous-value control of longer orbits

As it has been mentioned before applying the control signal once per period could be not frequent enough due to strong repelling action in the unstable direction. Hence we must apply the control signal more frequently (greater n).

In Figs. 5-7 we show the examples of successful control of orbits $\gamma_{2,2}$, $\gamma_{3,0}$, $\gamma_{3,9}$. For the control of symmetric orbit $\gamma_{2,2}$ we have used two points on the orbit defined by: $x = 1$ and $\dot{x} < 0$. One can see (compare Fig. 5) that these points do not divide the orbit equally in time, but this is not necessary for the method to work. We would like to remark that the second Jacobian has no unstable direction, hence one could not use the OGY method here.

We have also tried to stabilize orbit $\gamma_{3,0}$ (Fig. 6). The control was possible with three points on the orbit. We have chosen one point on the plane $x = 1$, and two points on the plane $x = 0$.

In Fig. 7 we show how the method works with an extremely long orbit. For the stabilisation of orbit $\gamma_{3,9}$ with 12 windings around system equilibria P_+ and P_- we needed 10 points on the

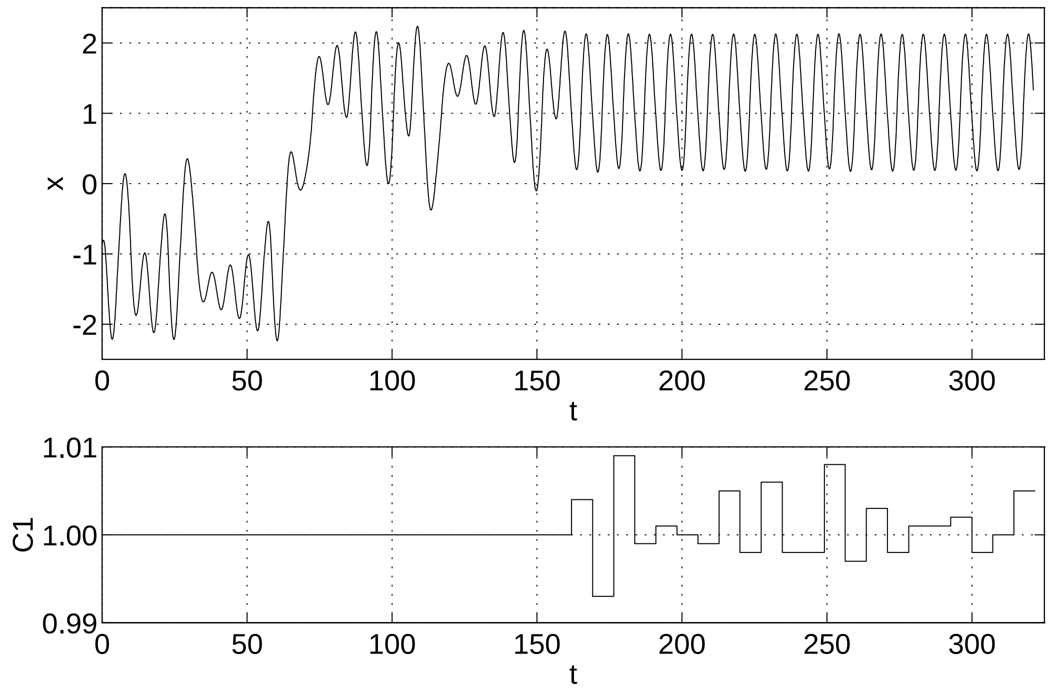

Figure 4:

Control of $\gamma_{1,0}$, period-one orbit. (a) state variable x, (b) control signal C_1

orbit.

4.3 Two-level control

In this section we report the results of implementation of the two-level control method. In Figs. 8..10 we present the stabilized period-one orbit using $n = 10, 6, 4$ points. For each value of n we have tried to choose n points on the periodic orbit in such a way that maximal eigenvalue of matrices $\boldsymbol{A}_j$ is as small as possible. We have noticed that for small n one must choose the value p_{max} more carefully. The values of p_{max} for which the stabilization has been successful are summarized below:

n	p_{max}
10	0.0008..0.10
6	0.0008..0.08
4	0.0018..0.06
3	0.0020..0.03
2	–

For $n = 2$ the stabilization was unsuccessful (compare Fig. 11) but we have obtained long periodic movements breaked by chaotic bursts. For $p_{max} = 0.0024$ we have observed the longest periodic parts. Why is control with two points not possible? This periodic orbit has the unstable eigenvalue $\lambda_1 = -2.55$. Hence from Corollary 4 the minimal value for n is 2. We have chosen two points on the orbit lying on the plane $x = 1$. The linear coefficients of the corresponding Poincaré maps are:

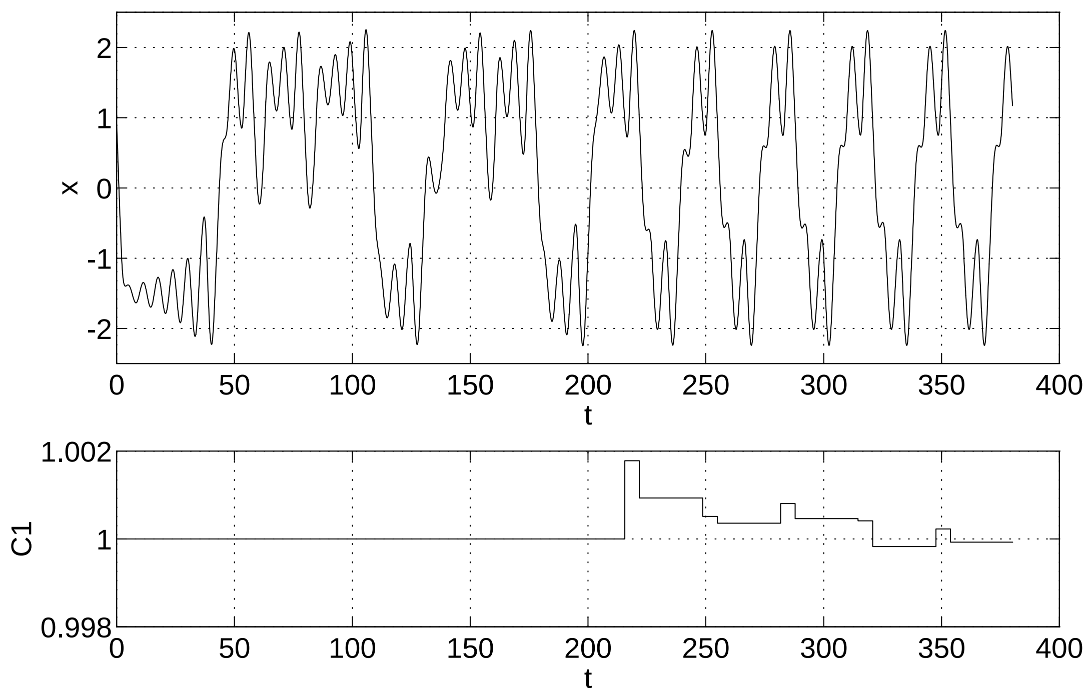

Figure 5:

Control of $\gamma_{2,2}$, symmetric orbit, $n = 2$

j	λ_1^j	λ_2^j	w_1^j	w_2^j
1	1.53	-0.44	-0.13	1.28
2	-1.54	0.02	0.76	2.58

The vectors $\boldsymbol{w}_j$ are evaluated in coordinates defined by eigenvectors of matrices $\boldsymbol{A}_j$. One can see that vectors $\boldsymbol{w}_j$ are badly situated. From Corollary 2 it follows that the following conditions should hold: $|\lambda_1^1| < -2$ and $|\lambda_1^2| < 1.3$. None of them is satisfied and this could be an explanation why control is not possible.

It is interesting to observe for the case $n = 3$ the unstable eigenvalues of matrices $\boldsymbol{A}_j$. For three points on the orbit defined by $(x = 1,\, \dot{x} < 0)$, $(x = 0.4,\, \dot{x} < 0)$, $(x = 1.8,\, \dot{x} > 0)$ the corresponding eigenvalues are $\lambda_1^1 = 2.76$, $\lambda_1^2 = 1.04$, $\lambda_1^3 = -1.16$. These values are far from the optimal case when all unstable eigenvalues are equal in magnitude, with absolute values smaller than two. But for these positions of points $\boldsymbol{x}(t_1)$, $\boldsymbol{x}(t_2)$, $\boldsymbol{x}(t_3)$ the method has worked properly. We have managed to position points $\boldsymbol{x}(t_j)$ in such a way that all unstable eigenvalues have satisfied the condition $|\lambda_1^j| < 1.8$ but then the control has been unsuccessful. The reason could be vectors $\boldsymbol{w}_j$ which in the second case are badly situated.

In general to check if for a specific n the control is possible one has to test several positions of points $\boldsymbol{x}(t_j)$ and several values of p_{max}.

In the last experiment we have controlled the orbit $\gamma_{2,2}$ using 10 points on the orbit (Fig. 12). The method works properly and the conclusion is that the two-level control method can also be used for longer periodic orbits but obviously greater n must be used to obtain successful control.

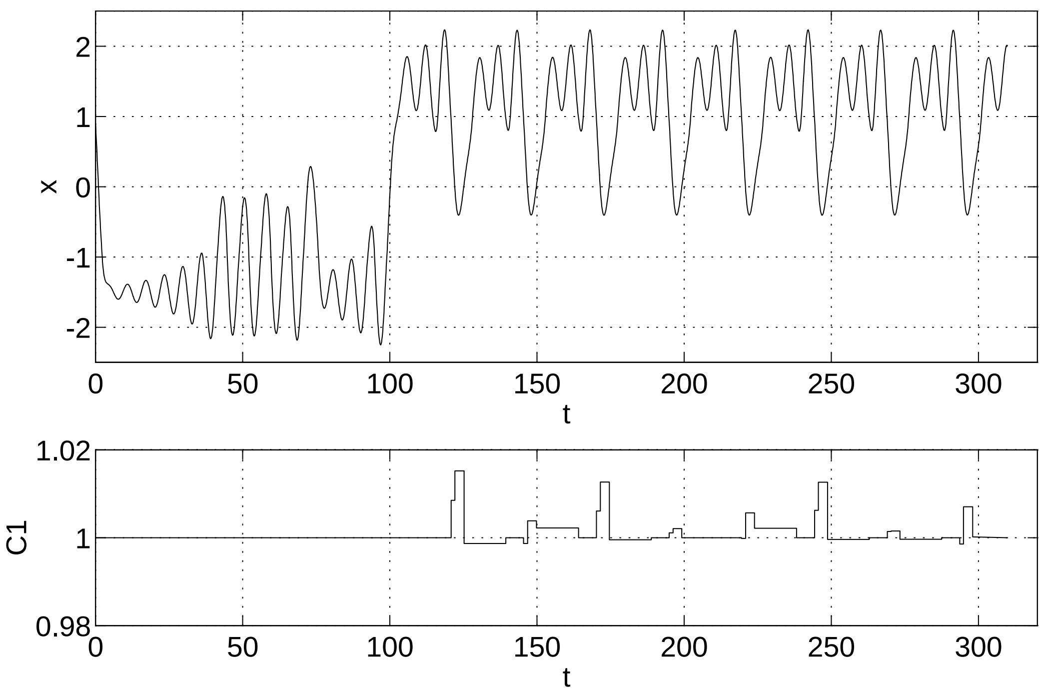

Figure 6:
Control of $\gamma_{3,0}$, period-three orbit, $n = 3$

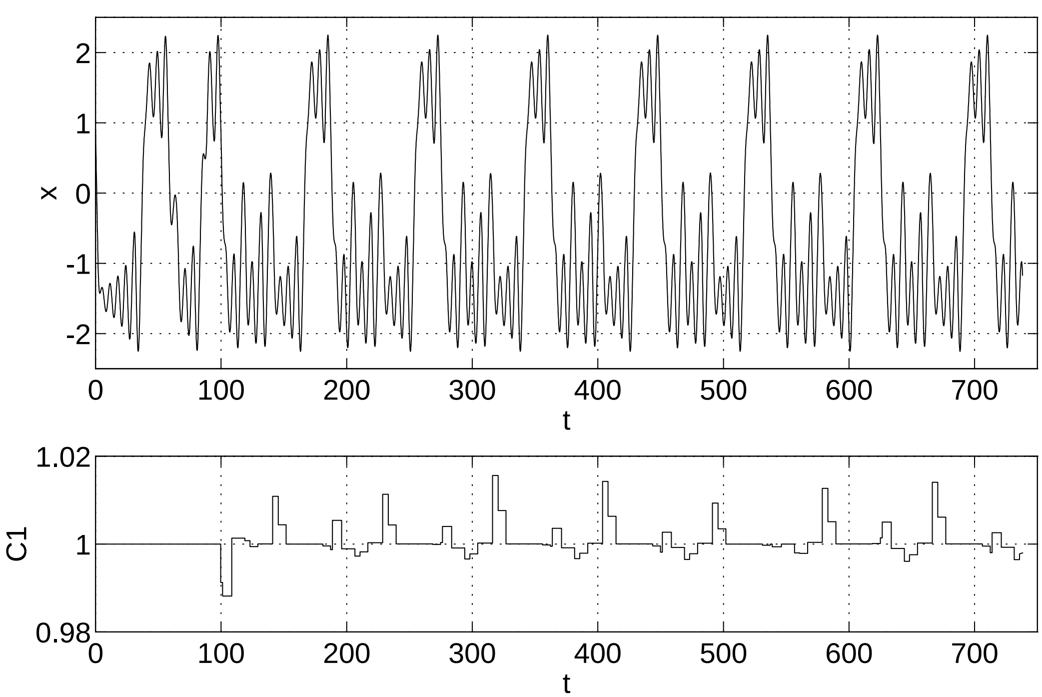

Figure 7:
Control of $\gamma_{3,9}$, long orbit, $n = 10$

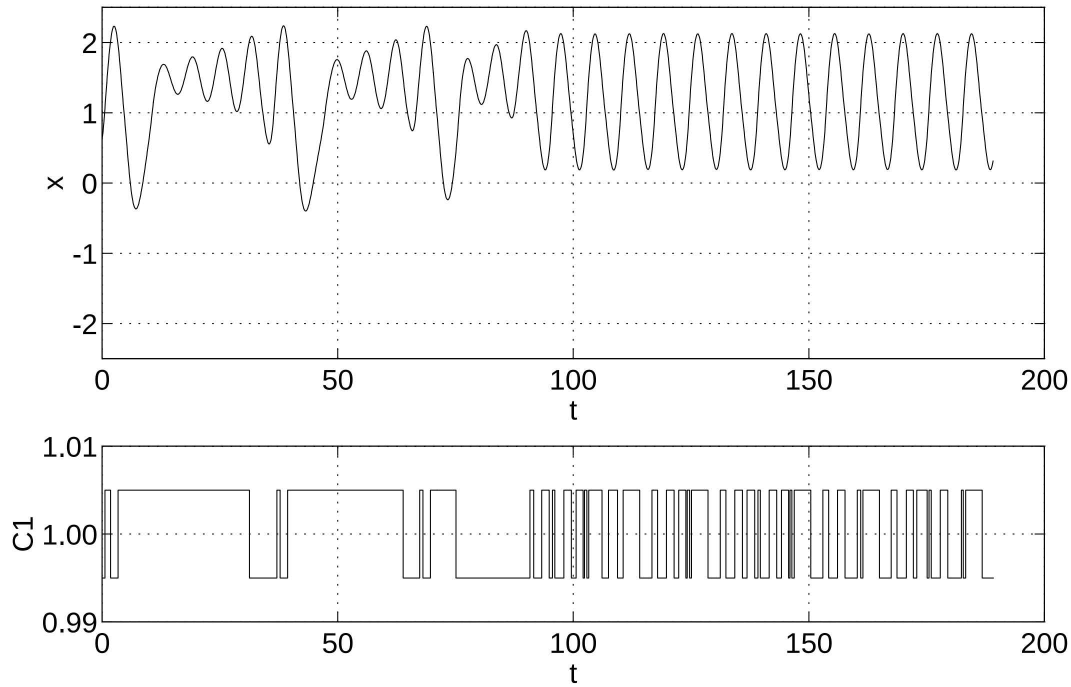

Figure 8:
Two-level control of period-one orbit, n=10

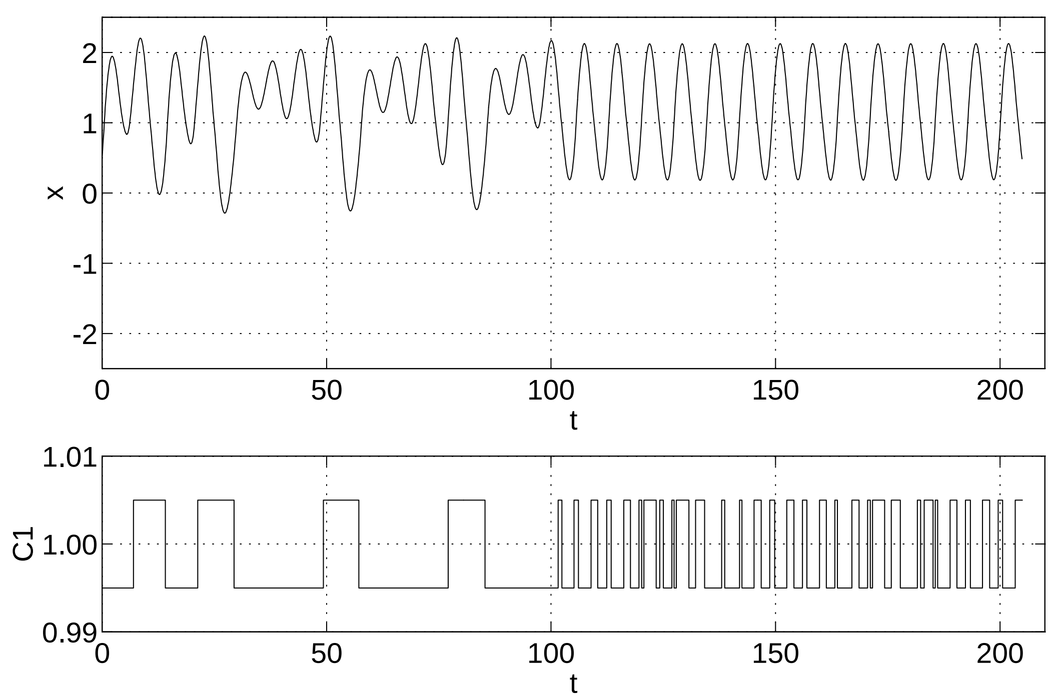

Figure 9:
Two-level control of period-one orbit, n=6

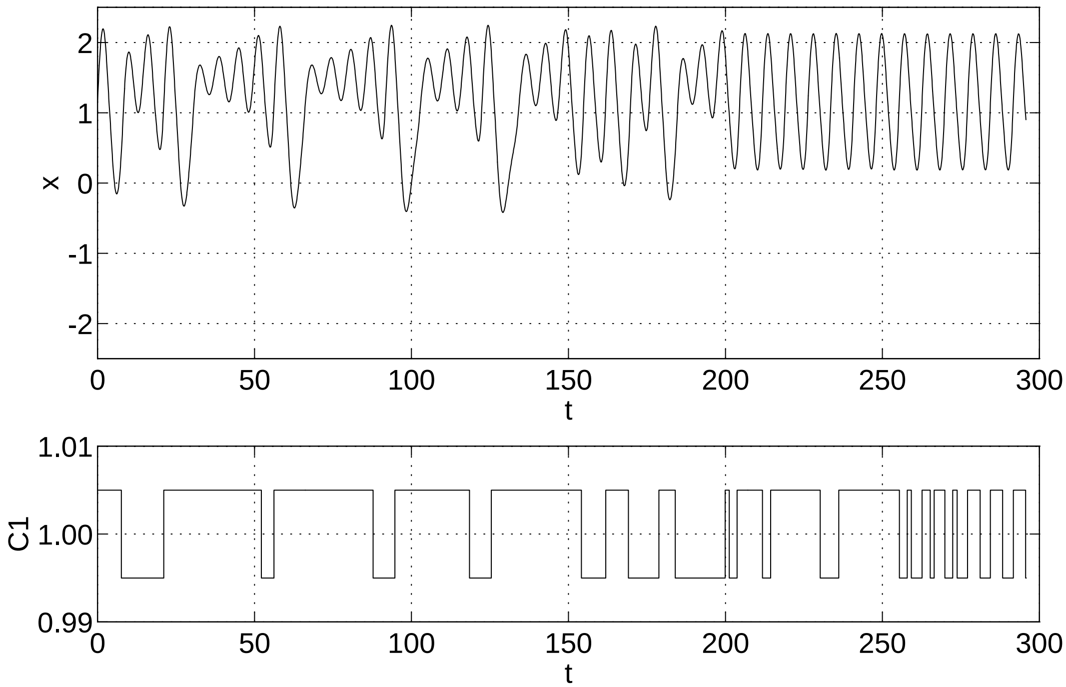

Figure 10:
Two-level control of period-one orbit, n=4

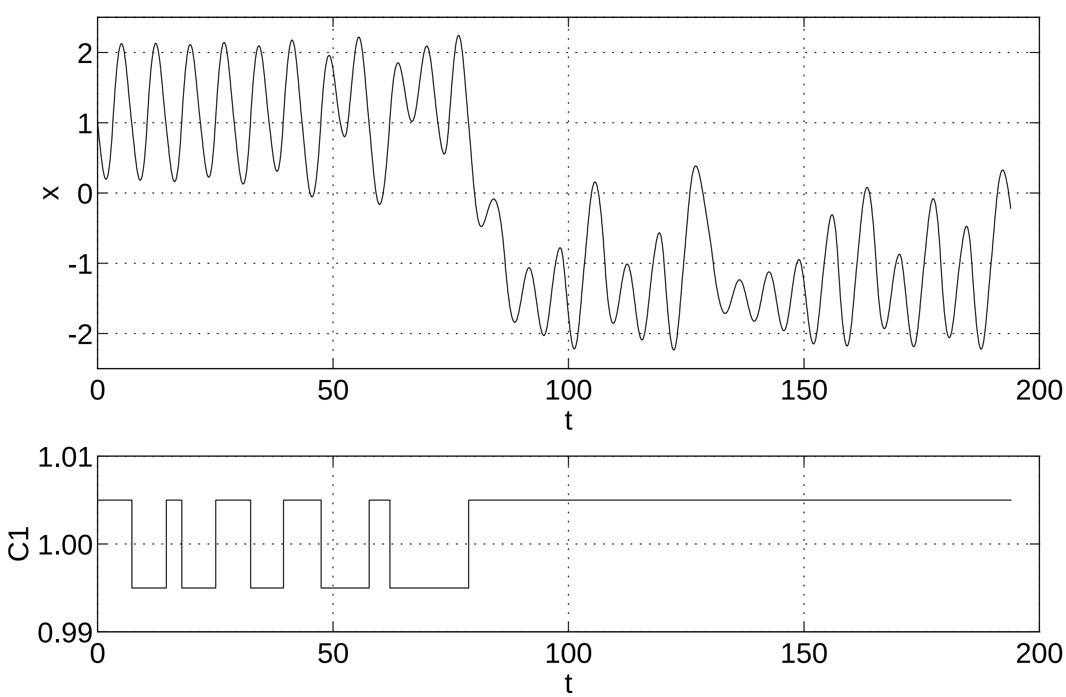

Figure 11:
Attempt of two-level control of period-one orbit, n=2

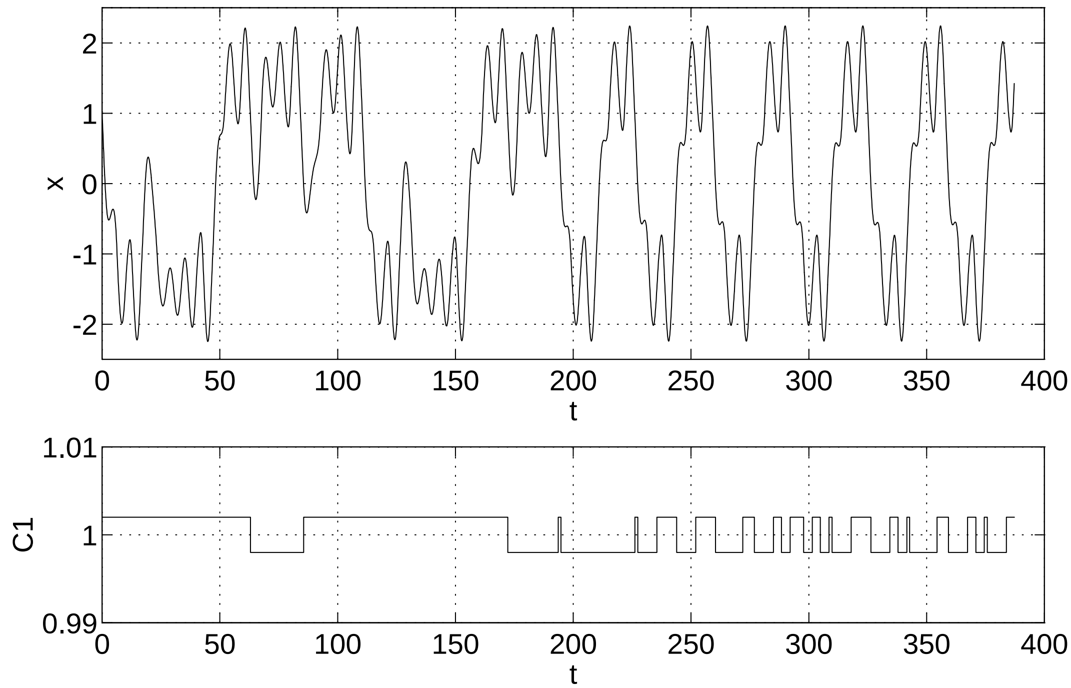

Figure 12:
Two-level control of $\gamma_{2,2}$, symmetric orbit, n=10

5 Conclusions

The new general method of controlling chaotic systems is introduced. It can be used for stabilizing periodic orbits in chaotic systems, when only one system parameter is accessible. Its modification (called the two-level control) for the case, when the allowed range for the control parameter value is a two-element set, is also described. Some theoretical results on the choice of the method's parameters are given. The dependence of the effectiveness of control upon various parameters has been discussed. The control formulas introduced in the paper have been applied in simulations of Chua's circuit. Both methods have been found to work properly in computer simulations. We have managed to stabilize several periodic orbits using both methods.

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