Topological Chaos in the Parallel Inductor-Capacitor-Memristor Circuit

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6th September 2016, ICSES, Kraków

Plan of the talk

- Memristors.
- Parallel Inductor-Capacitor-Memristor Circuit.
- Simulation results.
- Rigorous proof that the topological entropy of the circuit is positive.
- Conclusions.

Memristors

- A classical memristor is a circuit element characterized by a nonlinear relation between charge and flux. Its resistance depends on the history of current flowing through the element or the voltage across the element.
- Memristors were theoretically introduced in 1971.
- Memristive behaviour was recognized in a nano-scale double-layer $\mathrm{TiO_2}$ film in 2008.
- Memristors have received significant attention due to a number of possible applications including large-capacity nonvolatile memories and neuromorphic systems.
- It has been confirmed that memristors are useful for designing nonlinear oscillators.
- One of the simplest memristor circuits displaying complex behaviours is the parallel inductor-capacitor-memristor circuit.

Rigorous analysis of memristor oscillators

- Most results concerning memristor oscillators are based on simulations and sometimes computing certain characteristics of nonlinear systems like Lyapunov exponents and simulation based bifurcation analysis.
- Due to rounding errors results obtained by numerical simulations of nonlinear systems may be unreliable.
- Rounding errors accumulate and are propagated in further computations.
- For more complex problems this may result in totally wrong answers.
- In this work rigorous, interval arithmetic based analysis of dynamics of the parallel inductor-capacitor-memristor circuit is carried out.

Current controlled memristor

- For the *ideal charge-controlled memristor* there exist a constitutive relation between the charge q and the magnetic flux φ in the form $\varphi = f_M(q)$. The relation between the voltage and the current is

$$u = R(q)i, \quad R(q) = \frac{df_M(q)}{dq}$$

- The memristance $R(q)$ is a function of the charge q.
- The memristor can be considered as the fourth basic passive circuit element.
- A *generalized voltage controlled memristor*:

$$i = G(z,u)u, \quad \frac{dz}{dt} = f(z,u),$$

where $z \in \mathbb{R}^n$ is the vector of internal state variables and $G(z,u)$ is called the memductance.

The parallel inductor-capacitor-memristor circuit

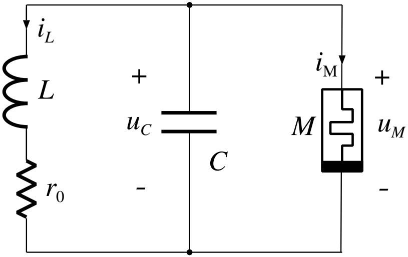
- The parallel inductor-capacitor-memristor circuit: two linear elements (C and L, the inductor series resistance is represented as r_0) and a nonlinear active voltage controlled memristor defined by

$$i_M = \alpha u_M z, \quad \dot{z} = -\beta + u_M^2.$$

- Circuit equations:

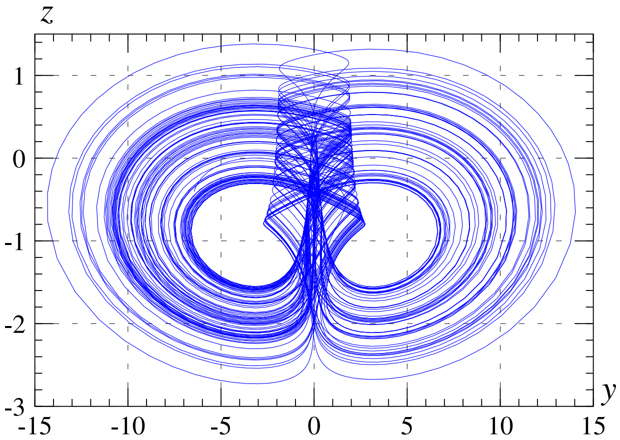
$$L\dot{x} = y - r_0 x,$$
$$C\dot{y} = -x - \alpha yz$$
$$\dot{z} = -\beta + y^2,$$

where $x = i_L$ and $y = u_C$.



- Parameter values: $r_0 = 1$, $L = 1/40$, $\alpha = 33/40$, and $\beta = 10$.

A trajectory of the memristor circuit



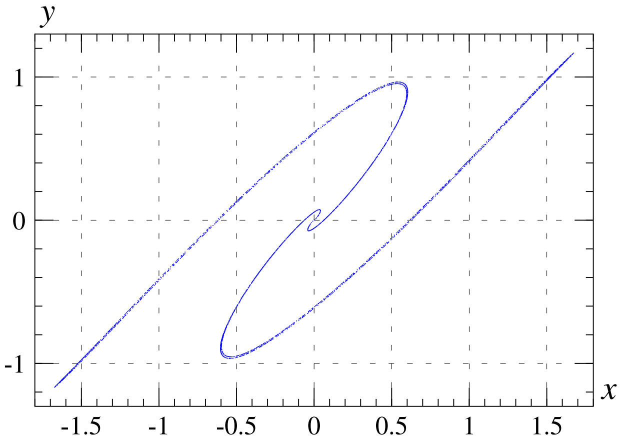
Return map

- To reduce the continuous problem to the discrete one we use the return map (Poincaré map) method.
- Σ — the plane defining the return map.
- Definition of the return map $P\colon \Sigma \mapsto \Sigma$

  $$P(x) = \varphi(\tau(x), x),$$

  $\varphi(t, x)$ is the trajectory with the initial point x, and $\tau(x) > 0$ is the time needed for the trajectory to reach Σ.
- When $\beta > 0$, $\alpha \neq 0$, $r_0 \neq 0$ the dynamical system has two equilibrium points given by $(x, y, z) = (\pm\sqrt{\beta} r_0^{-1}, \pm\sqrt{\beta}, -\alpha^{-1} r_0^{-1}) \approx (\pm 3.1623, \pm 3.1623, -1.2121)$.
- $\Sigma = \{(x, y, z)\colon z = -\alpha^{-1} r_0^{-1}, \dot{z} > 0\}$ is a good choice for the plane defining the return map.

A trajectory of the return map P



- For the evaluation of P we use integration methods based on interval arithmetic,
- The vector field is integrated using the rigorous Taylor integration method of order 30 with automatic time step control,
- The procedure for the rigorous evaluation of the return map P and its Jacobian is written using the CAPD library.

Topological Chaos

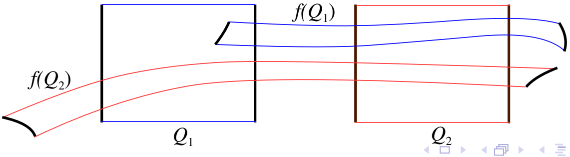
- Definition of topological entropy for $f\colon \mathbb{R}^m \mapsto \mathbb{R}^m$

$$\mathrm{H}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N(n,\varepsilon)}{n}$$

  where $N(n,\varepsilon)$ is the maximum cardinality of an (n,ε)-separated set, a set $E \subset X$ is (n,ε)-separated is for all $x, y \in E$, $x \neq y$ we have $\max_{0\le k<n} \|f^k(x) - f^k(y)\| \geq \varepsilon$.
- Topological entropy measures complexity of trajectories of the system.
- We say that the map f is *topologically chaotic* if $\mathrm{H}(f) > 0$.
- To prove the existence of topological chaos we will use the method of covering relations.

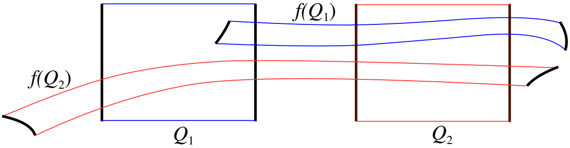
Covering relations for two-dimensional maps

- $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a continuous map.
- $Q_i, Q_j \subset \mathbb{R}^2$ are topological rectangles (sets which can be deformed to rectangles). We select two opposite edges of each topological rectangle and call them *vertical edges*. The other two edges are called *horizontal*.
- Q_i *f-covers* Q_j if the image of Q_i is enclosed in the interior of a topological stripe defined by horizontal edges of Q_j in such a way that images of vertical edges of Q_i lie on the opposite sides of Q_j.
- Example: Q_1 *f-covers* Q_2, Q_2 *f-covers* Q_1 and itself, vertical edges and their images are plotted in black.



How to prove the existence of topological chaos

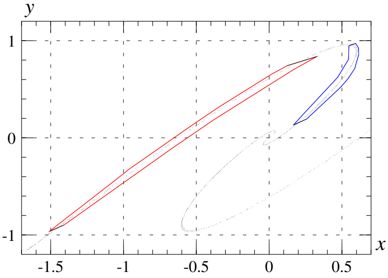
- Select pairwise disjoint topological rectangles $Q_1, Q_2, \ldots, Q_p$.
- Find covering relations between sets Q_i.
- Create the transition matrix $A \in \mathbb{R}^{p \times p}$: $A_{ij} = 1$ if Q_i f-covers Q_j and $A_{ij} = 0$ otherwise.
- Theorem: the topological entropy of f is not less than the logarithm of the spectral radius of A: $\mathrm{H}(f) \geq \log(\varrho(A))$.
- Example:
  - covering relations:



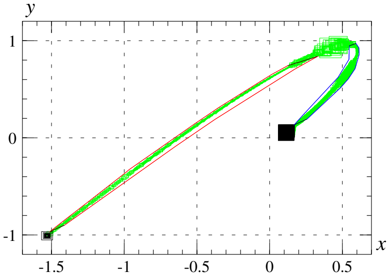
  - The transition matrix: $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\varrho(A) = (1 + \sqrt{5})/2$,
  - The topological entropy: $\mathrm{H}(f) \geq \log(\varrho(A)) > 0.48$.

Symbolic dynamics for P

Candidate sets Q_1 and Q_2



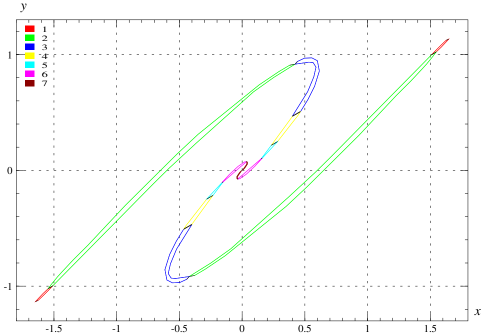
proof that Q_2 covers Q_1 and Q_2



- the proof that Q_1 and Q_2 both P-cover Q_1 and Q_2 requires evaluating P over 9295 boxes covering edges of $Q_1 \cup Q_2$ by 9295 boxes and 114979 boxes covering its interior.
- The transition matrix, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\varrho(A) = 2$.
- $\mathrm{H}(P) \geq \log 2 > 0.693$.

More complex symbolic dynamics

Sets supporting symbolic dynamics



Transition matrix

$$A = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0
\end{pmatrix}$$

- Proof: evaluation of P over 157728 (border) and 3409037 (interior) boxes.
- $\mathrm{H}(P) \geq \log(\varrho(A)) > 1.138$.

Conclusions

- The existence of nontrivial symbolic dynamics for the return map associated with the continuous dynamical system defining the behavior of the parallel inductor-capacitor-memristor circuit has been proved.
- We proved that the topological entropy of the return map is positive and that the continuous time system is topologically chaotic.
- Using very precise selection of sets supporting the symbolic dynamics we have shown that the dynamics of the return map is more complex than the dynamics of the topological horseshoe.
- It follows that there exist infinitely many periodic orbits and that the number of period-p orbits of the return map grows asymptotically faster than $3.12^p/p$.