Coexistence of Attractors in the Parallel Inductor-Capacitor-Memristor Circuit

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Plan of the talk

- Memristors.
- Parallel Inductor-Capacitor-Memristor Circuit.
- Bifurcation diagrams.
- Multiple attractors.
- Basins of attraction.
- Confirmation of the existence of multiple attractors.
- Conclusions.

Memristors

- A classical memristor is a circuit element characterized by a nonlinear relation between charge and flux. Its resistance depends on the history of current flowing through the element or the voltage across the element.
- Memristors were theoretically introduced in 1971.
- Memristive behavior was recognized in a nano-scale double-layer $\mathrm{TiO_2}$ film in 2008.
- Memristors have received significant attention due to a number of possible applications including large-capacity nonvolatile memories and neuromorphic systems.
- It has been confirmed that memristors are useful for designing nonlinear oscillators.
- One of the simplest memristor circuits displaying complex behaviors is the parallel inductor-capacitor-memristor circuit.

Current controlled memristor

- For the *ideal charge-controlled memristor* there exist a constitutive relation between the charge q and the magnetic flux φ in the form $\varphi = f_M(q)$. The relation between the voltage and the current is

$$u = R(q)i, \quad R(q) = \frac{df_M(q)}{dq}$$

- The memristance $R(q)$ is a function of the charge q.
- The memristor can be considered as the fourth basic passive circuit element.
- A *generalized voltage controlled memristor*:

$$i = G(z,u)u, \quad \frac{dz}{dt} = f(z,u),$$

where $z \in \mathbb{R}^n$ is the vector of internal state variables and $G(z,u)$ is called the memductance.

The parallel inductor-capacitor-memristor circuit

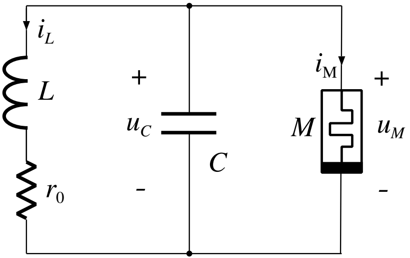
- The parallel inductor-capacitor-memristor circuit: two linear elements (C and L, the inductor series resistance is represented as r_0) and a nonlinear active voltage controlled memristor defined by

$$i_M = \alpha u_M z, \quad \dot{z} = -\beta + u_M^2.$$

- Circuit equations:

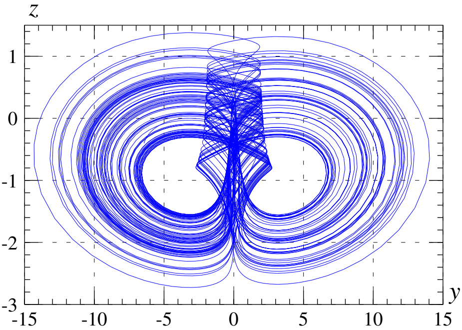
$$\begin{aligned} L\dot{x} &= y - r_0 x, \\ C\dot{y} &= -x - \alpha y z \\ \dot{z} &= -\beta + y^2, \end{aligned}$$

where $x = i_L$ and $y = u_C$.



- Parameter values: $L = 0.025$, $C = 0.025$, $\alpha = 0.825$, $\beta = 10$, r_0 is a bifurcation parameter.

- Parameter values: $L = 0.025$, $C = 0.025$, $\alpha = 0.825$, $\beta = 10$, $r_0 = 1$.



- When $\beta > 0$, $\alpha \neq 0$, $r_0 \neq 0$ the dynamical system has two equilibrium points $(x, y, z) = (\pm\sqrt{\beta} r_0^{-1}, \pm\sqrt{\beta}, -\alpha^{-1} r_0^{-1}) \approx (\pm 3.1623, \pm 3.1623, -1.2121)$.

Return map

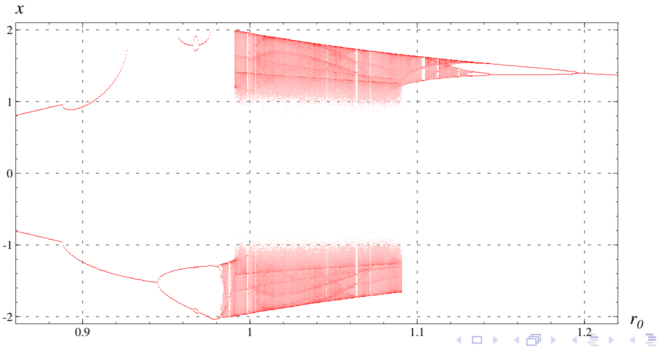
- Equilibria $(x, y, z) = (\pm\sqrt{\beta} r_0^{-1}, \pm\sqrt{\beta}, -\alpha^{-1} r_0^{-1})$, $\sqrt{\beta} = \sqrt{10} \approx 3.16227766$.
- To reduce the continuous problem to the discrete one we use the method of the return map (Poincaré map).
- $\Sigma = \Sigma_1 \cup \Sigma_2$ — union of half-planes,
  $\Sigma_1 = \{w = (x, y, z) \colon y = +3.16227766, \dot{y} < 0\}$,
  $\Sigma_2 = \{w = (x, y, z) \colon y = -3.16227766, \dot{y} > 0\}$.
- The return map $P \colon \Sigma \mapsto \Sigma$

$$P(x) = \varphi(\tau(x), x),$$

$\varphi(t, x)$ is the trajectory with the initial point x, and $\tau(x) > 0$ is the time needed for the trajectory to reach Σ.

Bifurcation diagram for the return map P

- 2001 equidistant values in the interval $r_0 \in [0.5, 1.5]$,
- 500 iterations skipped, 5000 iterations plotted,
- initial point $(x, y, z) = (-0.1, -0.1, 0.1)$ for $r_0 = 0.5$,
- *continuation method*: endpoint of the trajectory is used as an initial point for the next parameter value,
- results obtained for $r_0 \in [0.86, 1.22]$:

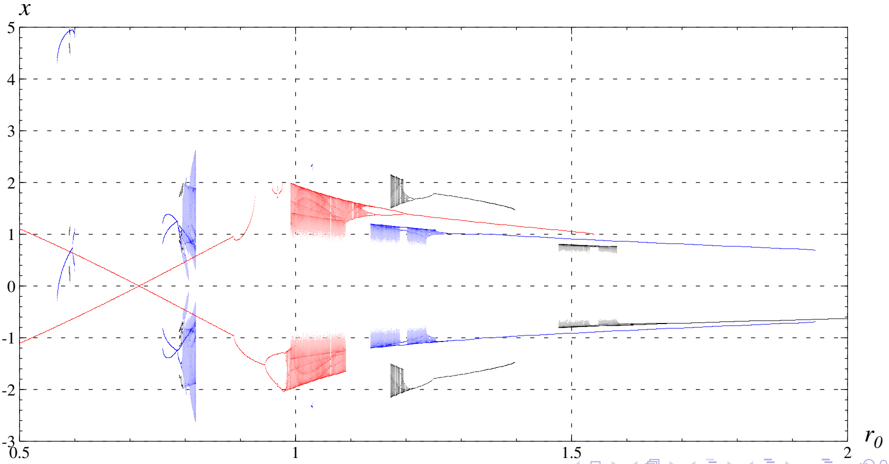


Finding multiple attractors

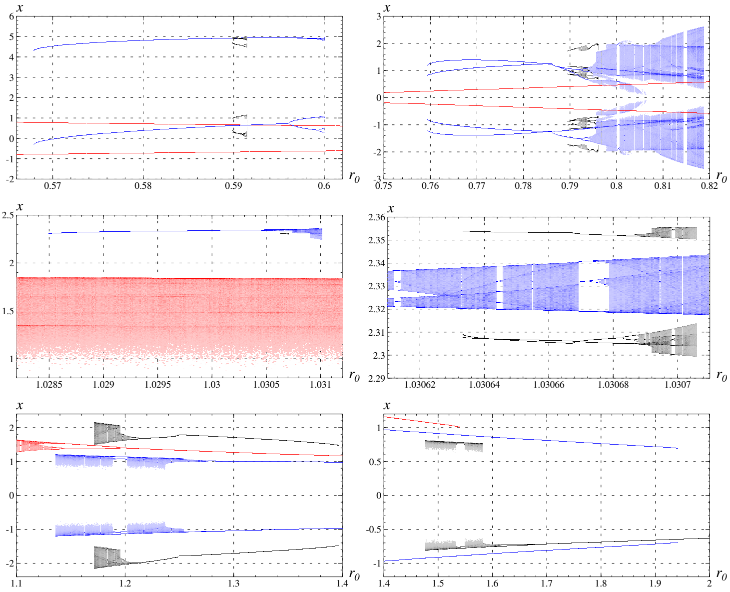
- For nonlinear systems there may exist multiple attractors.
- Finding multiple attractors
  - For fixed parameter values we find steady states for 1001×501 initial points filling uniformly the rectangle $(x_0, y_0, z_0) \in [-10, 10] \times [3.16227766] \times [-5, 5] \subset \Sigma_1$.
  - In each case a new attractor is found, we verify whether it belongs to an existing bifurcation plot.
  - If not, a new bifurcation plot containing this attractor is constructed using the continuation method.
- Computations are carried out for 151 equidistant parameter values in the range $r_0 \in [0.5, 2.0]$.
- Ten disconnected bifurcation plots have been found.

A complete bifurcation diagram $r_0 \in [0.5, 2.0]$

- Ten disconnected bifurcation plots have been found.
- Some regions are very narrow — very fine sampling of the parameter space is necessary to find attractors.

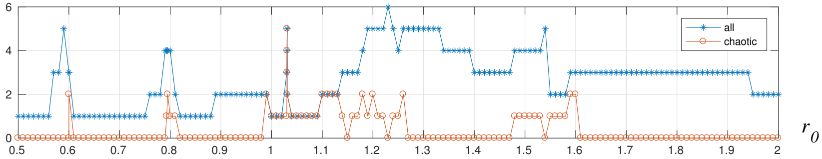


Regions of coexisting attractors



The number of attractors found versus r_0

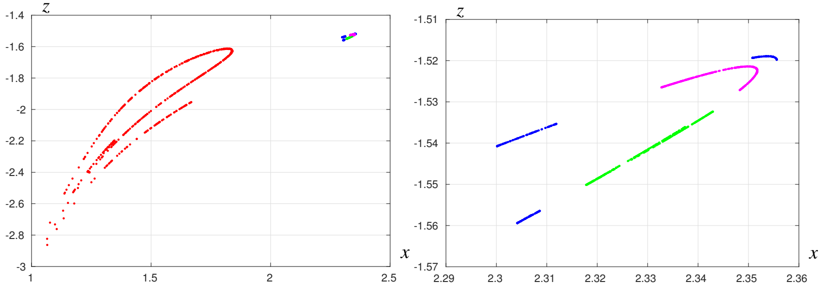
- The total number of attractors and the number of chaotic attractors found versus r_0.



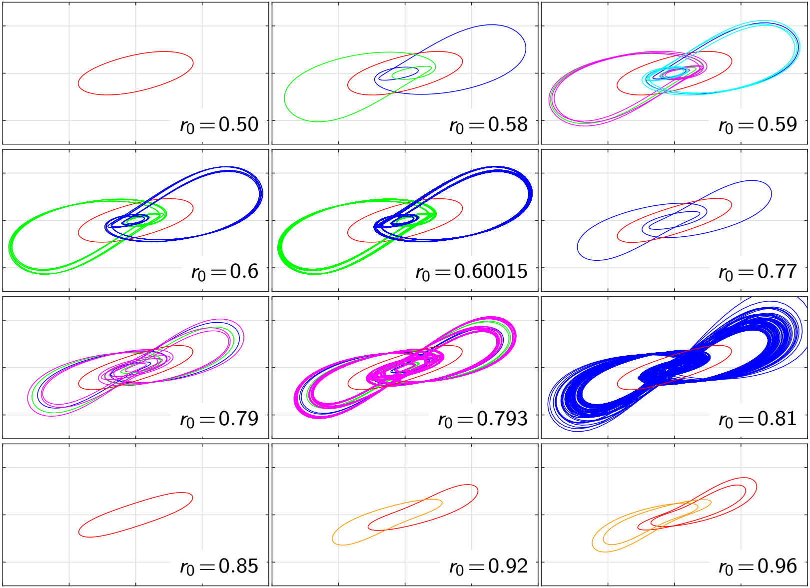
- The maximum number of six (periodic) attractors is observed for $r_0 = 1.23$.
- Four chaotic attractors are observed for $r_0 = 1.0307$.
- Five chaotic attractors are observed for $r_0 = 1.030689$.

Chaotic attractors existing for $r_0 = 1.0307$

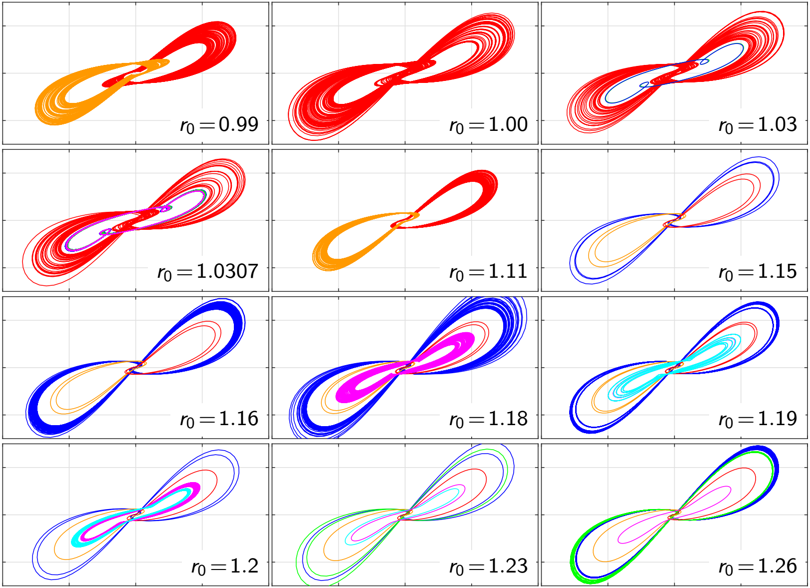
- Four attractors are observed for $r_0 = 1.0307$.
- One of the attractors (red) occupies a large part of the state space,
- Three attractors are located very close to each other (blue, green and magenta).



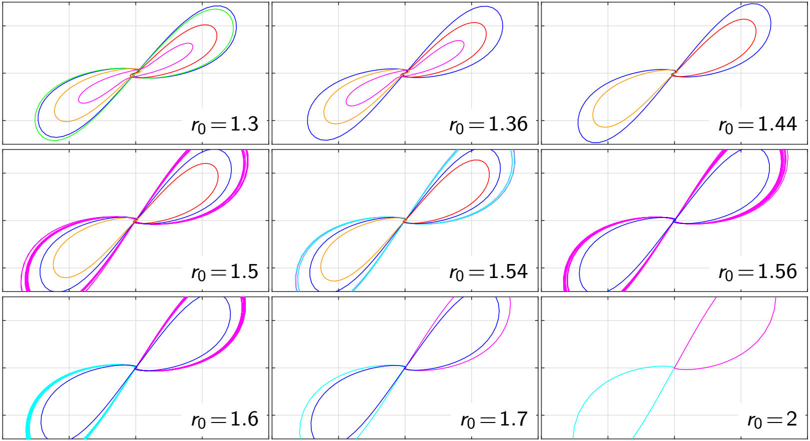
Attractors, variable ranges: $y \in [-10, 10]$, $z \in [-15, 15]$



Attractors, $y \in [-10, 10]$, $z \in [-15, 15]$, cont.



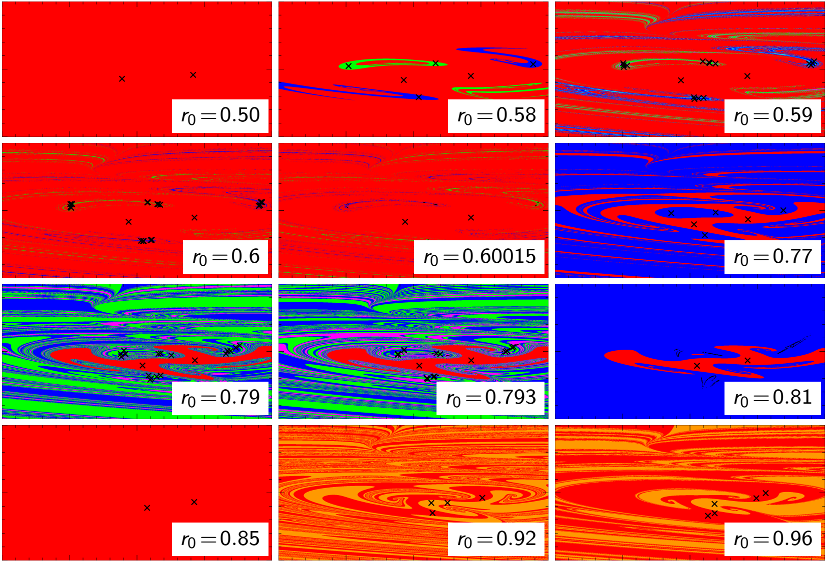
Attractors, $y \in [-10, 10]$, $z \in [-15, 15]$, cont.



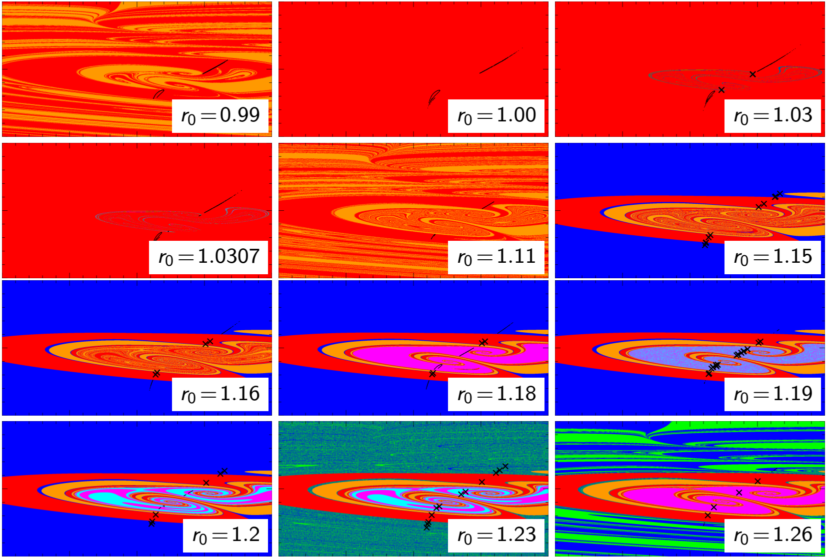
Basins of attraction

- For a given attractor A its *basin of attraction* $\mathrm{B}(A)$ is defined as a set of initial points such that if a trajectory is started in $\mathrm{B}(A)$ then it converges to A.
- Search of basins of attraction for the plane $\Sigma_3 = \{w = (x, y, z) \colon y = 3.16227766\}$.
- We find steady states for 2001×1001 initial points filling uniformly the rectangle $R = (x_0, y_0, z_0) \in [-10, 10] \times [3.16227766] \times [-5, 5] \subset \Sigma_3$.
- Initial points are plotted with colors corresponding to the steady state obtained.
- Attractors are plotted in black: for periodic attractors we use the $\times$ symbol and for chaotic attractors we use dots.

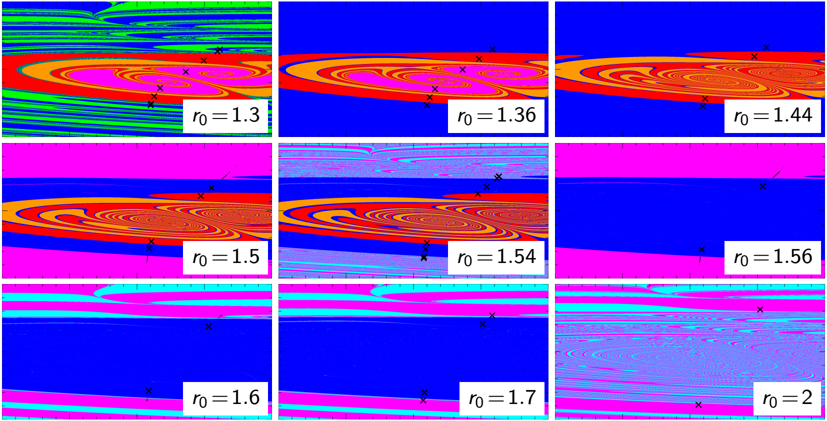
Basins of attraction ($x \in [-10, 10]$, $z \in [-5, 5]$)



Basins of attraction ($x \in [-10, 10]$, $z \in [-5, 5]$), cont.

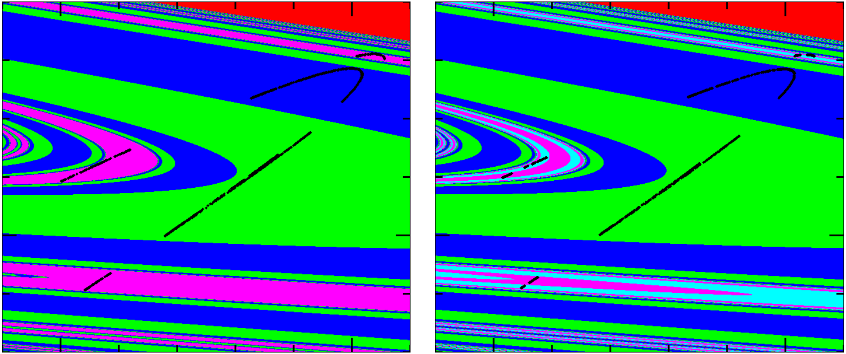


Basins of attraction ($x \in [-10, 10]$, $z \in [-5, 5]$), cont.



- variable ranges: $x \in [2.29, 2.36]$, $z \in [-1.57, -1.51]$.
- four chaotic attractors for $r_0 = 1.0307$,
- five chaotic attractors for $r_0 = 1.030689$.

$r_0 = 1.0307$ $\quad\quad$ $r_0 = 1.030689$



Rigorous study

- Due to rounding errors results obtained by numerical simulations of nonlinear systems may be unreliable.
- Rounding errors accumulate and are propagated in further computations.
- For more complex problems this may result in totally wrong answers.
- Interval arithmetic based computations may be used to obtain rigorous results.
  - The vector field is integrated using the rigorous Taylor integration method of order 30 with automatic time step control.
  - For the evaluation of the return map P and its Jacobian the CAPD library is used.

How to prove the existence of an attractor

- A general technique: construct a trapping region.
  - $\Omega \subset \Sigma$ such that $P(\Omega) \subset P$ is called a *trapping region*.
  - To prove that $P(\Omega) \subset P$ cover Ω by interval vectors $\mathbf{w}_k$, find enclosure $\mathbf{u}_k \supset P(\mathbf{w}_k)$ and verify that $\mathbf{u}_k \subset \Omega$.
  - Each bounded trapping region contains at least one attractor.
  - The existence of non-overlapping trapping regions implies the existence of multiple attractors.
- For periodic attractors use the interval Newton method.
  - The *interval Newton operator* for the map F: $\mathrm{N}(\mathbf{x}) = \hat{x} - F'(\mathbf{x})^{-1}F(\hat{x})$, where $\hat{x} \in \mathbf{x}$ and $F'(\mathbf{x})$ is an interval matrix containing the Jacobian matrices $F'(x)$ for $x \in \mathbf{x}$.
  - if $\mathrm{N}(\mathbf{x}) \subset \mathbf{x}$, then $\mathbf{x}$ contains exactly one zero of f.
  - To prove the existence of period-p orbits of P apply the interval Newton method to the map $F = (F_0, F_1, \ldots, F_{p-1})$ defined by: $F_k(w_0, w_1, \ldots, w_{p-1}) = w_{(k+1) \bmod p} - P(w_k)$, for $k = 0, 1, \ldots, p-1$.

The existence of periodic attractors

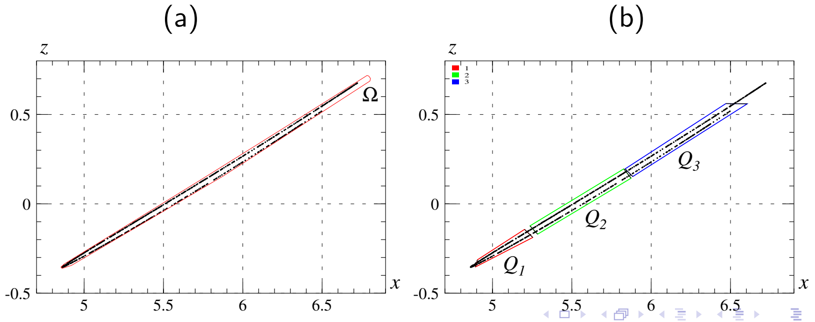
- The existence of all 353 periodic attractors reported in the figure with the number of attractors is proved.
- Example: 6 periodic attractors for $r_0 = 1.23$:
  - p — the period of the orbit of P,
  - T — the flow time (the period of the continuous time system),
  - bounds for (x_0, y_0, z_0).

n	p	T	(x_0, y_0, z_0)
1,2	1	0.326146935489^{28}_{17}	$(\pm 1.35418887855^{96}_{75}, \pm 3.16227766, -1.935338131 81^{76}_{55})$
3,4	2	$0.43047830 60^{4}_{3}$	$(\pm 1.7004761963^{70}_{67}, \pm 3.16227766, -1.41582847815^{74}_{53})$
5,6	3	1.40924841^{71}_{69}	$(\pm 1.105015699 2^{82}_{79}, \pm 3.16227766, -2.462173856270^{96}_{77})$

Two chaotic attractors for $r_0 = 0.99$

(a) the trapping region Ω (red) for the return map P enclosing a numerically observed trajectory (black), the existence of the symmetric attractor follows from the symmetry of the vector field.

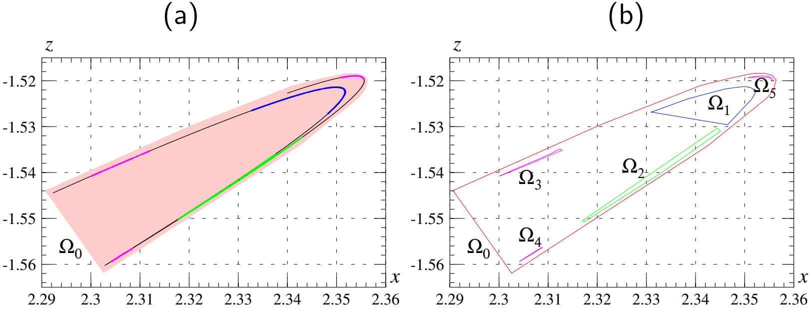
(b) sets Q_1, Q_2, Q_3 with complex symbolic dynamics (implies positive topological entropy, the infinite number of periodic orbits and the topological chaos).



- Four chaotic attractors are observed in simulations: one wide self-symmetric attractor and three tiny attractors (one self-symmetric and a symmetric pair).

(a) The trapping region Ω_0 (light red), its image (black), and three attractors (blue, green, magenta).

(b) The three trapping regions Ω_1, Ω_2, $\Omega_3 \cup \Omega_4 \cup \Omega_5$.



Conclusions

- Dynamical phenomena in the parallel inductor-capacitor-memristor circuit studied numerically.
- A systematic search for coexisting attractors carried out.
- The existence of multiple attractors has been observed and bifurcation diagrams have been constructed.
- Basins of attraction have been computed.
- The coexistence of attractors has been proved using interval analysis tools.
  - The existence of periodic attractors is confirmed by applying the interval Newton method to prove the existence of stable periodic orbits of an associated return map.
  - For numerically observed chaotic attractors the existence of attractors is proved by constructing trapping regions enclosing chaotic trajectories of the return map.
  - The existence of topological chaos is proved using the method of covering relations.