Exploiting the Concept of Conditional Transversal Lyapunov Exponents for Study of Synchronization of Chaotic Circuits

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Abstract— **The problem of synchronization of coupled chaotic systems is considered. The notion of local transversal Lyapunov exponents is introduced. We show that they can be successfully used in investigations of the synchronization properties. The technique is illustrated with computer simulations.**

Keywords— **Chaotic systems, Lyapunov exponents, synchronization.**

I. Introduction

RECENTLY there has been a considerable interest in using the concept of synchronization of chaos to develop spread spectrum communication systems. In applications in order to extract the information from transmitted chaotic signal a response system must be synchronized with the signal.

In this paper we consider the problem of synchronization of uni-directionally coupled chaotic systems:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) + \mathbf{d}(\mathbf{x} - \mathbf{y}), \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ are the state variables of the drive and response systems and $\mathbf{d}$ is a diagonal matrix with diagonal elements $d_1, \ldots, d_n$ being coupling coefficients.

We say that the systems synchronize if $\mathbf{y}(k) - \mathbf{x}(k) \to 0$ as $k \to \infty$ (the trajectory of the system (1), (2) converges to the synchronization subspace $\mathbf{x} = \mathbf{y}$). It is clear that if the coupling coefficients are big enough the systems will synchronize. The source of this synchronization is additional dissipation introduced when the variables are not following the same trajectories. For communication tasks we look for systems where only one of the coupling coefficients is non-zero (otherwise one needs to send more signals in order to extract the information).

There are several methods for investigating the synchronization of chaotic systems. The first criterion for successful synchronization, introduced in [2], is based on conditional Lyapunov exponents calculated along a typical trajectory of the system. When all conditional Lyapunov exponents of the response system driven by the signal $\mathbf{x}$ are negative then one expects that the systems synchronize. This may not be true especially in the presence of noise [3], [4]. It may happen that in the neighborhood of a periodic orbit there exist a region where the trajectories are pushed away from the synchronization subspace. Such a situation occurs when not all conditional Lyapunov exponents associated with the measure supported by the periodic orbit are negative. In this case small noise could force the trajectory to enter such a region. This in turn could lead to desynchronization bursts [4].

Hence in order to ensure synchronization one should evaluate the transversal Lyapunov exponents for all periodic orbits and check that they are all negative. This is rather difficult and computationally expensive task. There is also another drawback of this method. Even if the periodic orbit attracts the trajectory to the synchronization space globally it is possible that it repels trajectories locally. This may also cause desynchronization bursts.

In this paper we propose another criterion for characterizing the synchronization behavior. It is based on local transversal Lyapunov exponents.

II. Synchronization of hyperchaotic circuits in the presence of noise

The dynamics of the circuit [5] considered in this paper is defined by the following state equation:

$$\begin{aligned} C_1\dot{v}_1 &= f(v_2 - v_1) - i_1, \\ C_2\dot{v}_2 &= -f(v_2 - v_1) - i_2, \\ L_1\dot{i}_1 &= v_1 + Ri_1, \\ L_2\dot{i}_2 &= v_2, \end{aligned} \tag{3}$$

where f is given by:

$$f(x) = m_0 x + 0.5(m_1 - m_0)(|x+1| - |x-1|). \tag{4}$$

Equations (3) and (4) define a hyperchaotic system with two positive Lyapunov exponents: $\lambda_1 \approx 0.25$, $\lambda_2 \approx 0.07$, $\lambda_3 = 0$ and $\lambda_4 \approx -53.2$.

Two identical systems are connected by means of uni–directional coupling. In the response system

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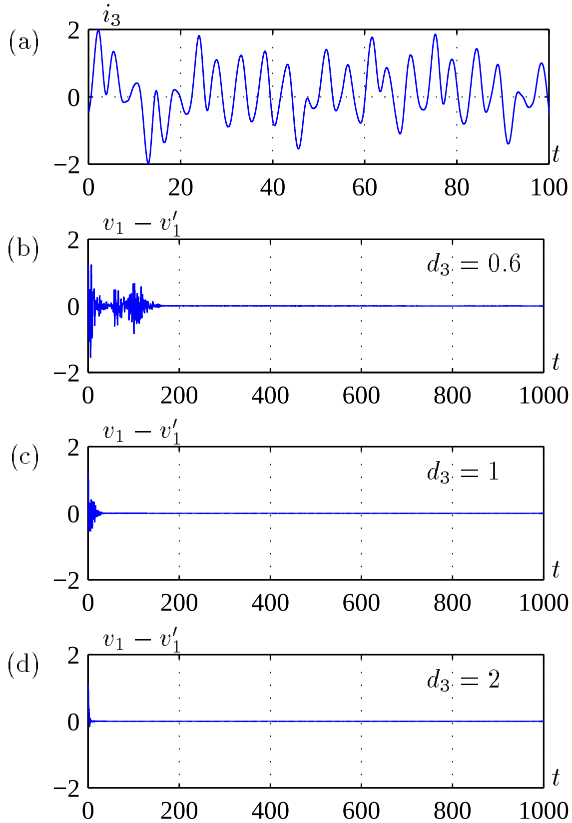

Fig. 1. Synchronization of hyperchaotic circuits: (a) transmitted signal, The synchronization error $v_1 - v_1'$ for $d_3 = 0.6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

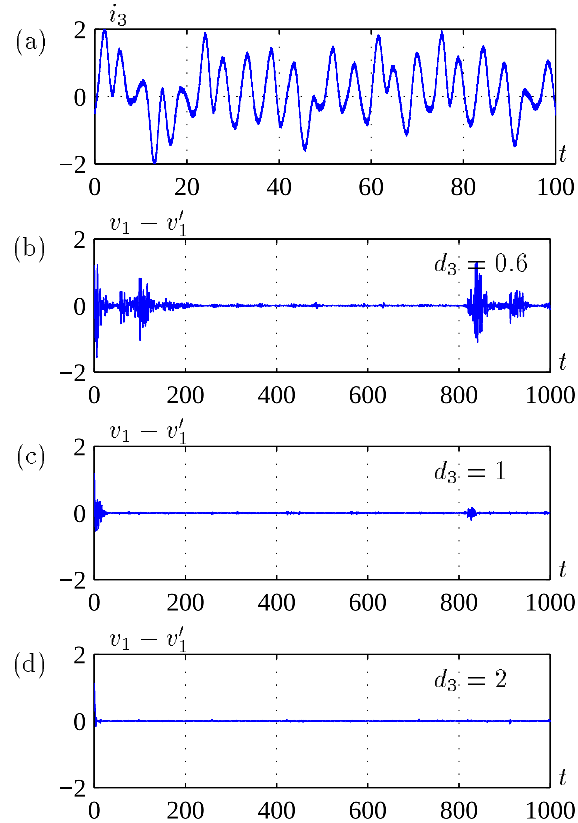

Fig. 2. Synchronization of hyperchaotic circuits in the presence of channel noise: (a) transmitted signal with additive noise of amplitude 0.1, The synchronization error $v_1 - v_1'$ for $d_3 = 0.6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

with the state variables (v_1', v_2', i_1', i_2') in the third equation a linear coupling $d_3(i_1 - i_1')$ is introduced (i_1 is the driving signal).

In this section we investigate the influence of the additive noise added to the transmitted signal on the synchronization behavior.

In the first experiment we drive the response system with the driving signal not corrupted by noise (see Fig. 1a). Synchronization error $v_1 - v_1'$ for different d_3 is shown in Fig. 1b–d. For all values of coupling constant ($d_3 = 0.6, 1, 2$) the synchronization takes place eventually. However for $d_3 = 0.6$ the time necessary to obtain the synchronization is rather long ($t > 150$).

Next we consider synchronization behavior in a more realistic situation when additive noise is present in the channel. First we set the amplitude of the additive noise at 0.1. The driving signal (containing the noise) is shown in Fig. 2a. Synchronization error for three different values of d_3 is shown in Fig. 2b–d. For $d_3 = 0.6$ large desynchronization bursts are observed (see Fig. 2b). For $d_3 = 1$ we observe almost perfect synchronization. Only one desynchronization burst with a small amplitude is visible. For $d_3 = 2$ the synchronization error remains small for the whole experiment.

Finally we consider synchronization with the driving signal contaminated by the additive noise of amplitude 1 (compare Fig. 3a). For weak coupling $d_3 = 0.6$ one can see frequent desynchronization bursts of large amplitude (Fig. 3b). For $d_3 = 1$ amplitude of bursts is much lower and they are less frequent. For strong coupling $d_3 = 2$ the synchronization error remains small. Non-coherent behavior is damped by strong coupling.

III. Analysis of synchronization based on local transversal Lyapunov exponents

First we will briefly recall the notion of local Lyapunov exponents $\lambda_i(\mathbf{x}, L)$. They are defined as logarithms of the eigenvalues of the matrix [6]:

$$\Lambda(\mathbf{x}, L) = \left(\left[\mathbf{T}^L(\mathbf{x}) \right]^T \mathbf{T}^L(\mathbf{x}) \right)^{\frac{1}{2L}}, \qquad (5)$$

where $\mathbf{T}^L(\mathbf{x})$ is the matrix of partial derivatives of the time-L map induced by the continuous-time system

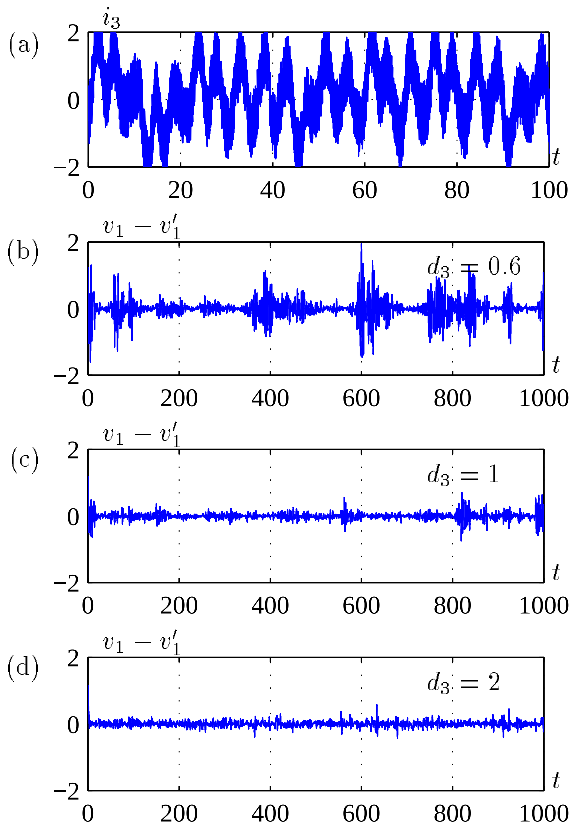

Fig. 3. Synchronization of hyperchaotic circuits in the presence of channel noise: (a) transmitted signal with additive noise of amplitude 1, The synchronization error $v_1 - v_1'$ for $d_3 = 0.6$ (b), $d_3 = 1$ (c), and $d_3 = 2$ (d)

(compare [1]).

Local Lyapunov exponents say how rapidly perturbations of the initial point $\mathbf{x}$ changes after time L from the moment of perturbation. From multiplicative ergodic theorem of Oseledec [1] it follows that local Lyapunov exponents tend to Lyapunov exponents as L goes to infinity.

Local transversal Lyapunov exponents are the local Lyapunov exponents corresponding to eigenvectors transversal to the synchronization subspace. It turns out that local transversal Lyapunov exponents are a very useful tool in studies of synchronization of chaotic systems, especially in the presence of noise.

For the values of the coupling constant d_3 considered in the previous section we have computed global transversal Lyapunov exponents. For $d_3 = 0.6$ the greatest transversal Lyapunov exponent is only slightly negative $\lambda \approx -0.02$. For $d_3 = 1$ and $d_3 = 2$ the greatest transversal Lyapunov exponent is $\lambda \approx -0.15$ and $\lambda \approx -0.37$ respectively. Hence for all the cases transversal Lyapunov exponents are negative and one could expect synchronization. We have already seen that this is true but only when the

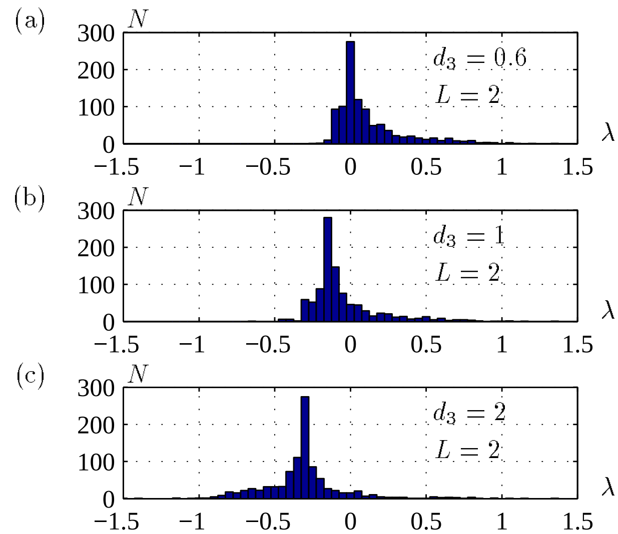

Fig. 4. Histogram of maximum local transversal Lyapunov exponent for $L = 2$

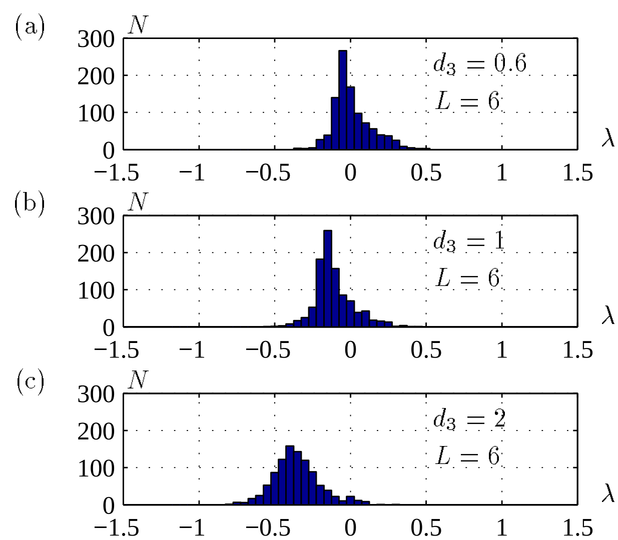

Fig. 5. Histogram of maximum local transversal Lyapunov exponent for $L = 6$

driving signal is not contaminated by noise (compare Fig. 1). In order to explain the behavior of the coupled systems in the presence of noise we will compute local transversal Lyapunov exponents.

Local Lyapunov exponents have been computed using the method proposed in [6]. First we fix the time delay $L = 2$. Local transversal Lyapunov exponents have been computed at 1000 points along the attractor. Then we have constructed histograms of the greatest of them. In the construction of histograms we have used bins of the length 0.05. The results are presented in Fig. 4. One can clearly see that the spectrum moves to negative values as the

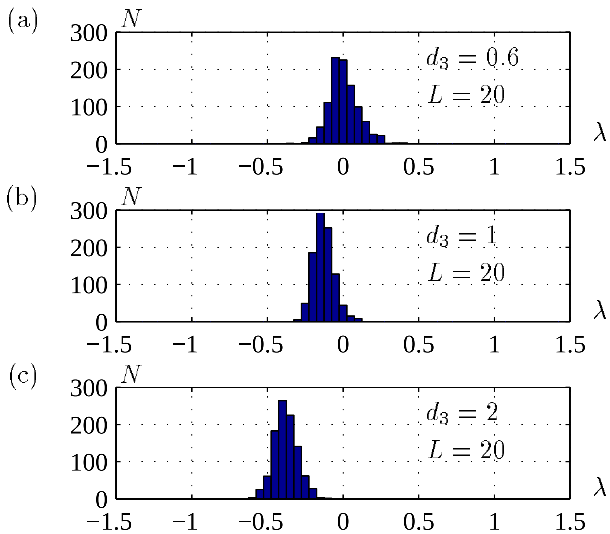

Fig. 6. Histogram of maximum local transversal Lyapunov exponent for $L = 20$

coupling coefficient is increased. For $d_3 = 0.6$ the average value is positive $\lambda_{\text{aver}} \approx 0.11$. For only 39% points on the attractor local transversal Lyapunov exponents are all negative. This explains frequent desynchronization bursts in Fig. 3b. For $d_3 = 2$ the average value is negative $\lambda_{\text{aver}} \approx -0.31$ and more than 90% of the spectrum lies below zero. This is an indication that for this coupling the synchronized behavior is robust and not very sensitive to noise (compare Fig. 3d).

In Fig. 5 we show histograms of maximum local transversal Lyapunov exponents computed for time $L = 6$. Spectrum is narrower (in comparison with $L = 2$) and shifted slightly towards negative values. For $d_3 = 0.6$ the average value is still positive $\lambda_{\text{aver}} \approx 0.012$. Now for approximately 57% points on the attractor all local transversal Lyapunov exponents are negative. For $d_3 = 2$ the average value is $\lambda_{\text{aver}} \approx -0.35$ and 97% of the spectrum lies below zero.

Finally we have computed local transversal Lyapunov exponents for $L = 20$ (see Fig.6). For $d_3 = 0.6$ the average value is still slightly positive $\lambda_{\text{aver}} \approx 0.004$. For strong coupling $d_3 = 2$ the whole spectrum is situated below zero and the average value is $\lambda_{\text{aver}} \approx -0.37$.

If we further increase the value of L we will observe in a histogram a very narrow peak at the value of the "natural" maximum transversal Lyapunov exponent.

For investigations of synchronization properties one should compute average local Lyapunov exponents. Usually they are different from global Lyapunov exponents. In the limit $L \to \infty$ the average values tend to global Lyapunov exponents. The whole spectrum is also very important It tells us how frequently (with respect to the natural measure on the attractor) trajectories are repelled from the synchronization subspace.

In order to use local Lyapunov exponents for the analysis of synchronization one should choose the time L properly. It cannot be too large as this would protect us from obtaining any information about the systems behavior in short time (for $L \to \infty$ local exponents converge to global exponents). On the other hand if we choose very small L we could get too restrictive conditions for the synchronized behaviour. For discrete time systems the problem is much easier. If the whole spectrum of local transversal exponents for $L = 1$ lies below zero than one expects synchronized behavior (see [7]).

IV. Conclusions

In this paper we have discussed the possibility of using local transversal Lyapunov exponents for characterization of the synchronization of chaotic systems. We have shown that there is a strong correlation between local transversal Lyapunov exponents and behaviour of coupled chaotic systems. We have shown that local transversal Lyapunov exponents could be effectively used for studying of synchronization properties, especially in the presence of noise. Their advantage is also that they could be easily computed. It is not necessary to find periodic orbits and compute their transversal Lyapunov exponents to investigate synchronization properties. We have also discussed the problem of choosing the time L for which local Lyapunov exponents are evaluated.

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