# ANALYSIS OF SYNCHRONIZATION OF CHAOTIC SYSTEMS BASED ON LOCAL CONDITIONAL LYAPUNOV EXPONENTS

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## ABSTRACT

In this paper we consider the problem of synchronization of coupled chaotic systems. First we show the limitations of existing techniques for studying of the synchronization. Then we introduce the notion of local conditional (transversal) Lyapunov exponents. We show that they can be successfully used in investigations of synchronization properties. We develop a new criterion for synchronization based on local conditional Lyapunov exponents. The discussion is supported by numerical examples.

## 1 INTRODUCTION

It is well known that when chaotic systems are coupled, they may demonstrate identical oscillations associated with the onset of synchronization [1]. The source of this synchronization is additional dissipation introduced when the variables are not following the same trajectories.

The synchronization has possible applications in communications where in order to extract the information from transmitted chaotic signal a response system must be synchronized with the signal. Therefore the problem of synchronization of chaotic systems is of very high importance.

In this paper we consider the synchronization of uni–directionally coupled discrete chaotic systems.

$$\mathbf{x}(k+1) = \mathbf{F}(\mathbf{x}(k)), \quad (1)$$

$$\mathbf{y}(k+1) = \mathbf{F}(\mathbf{y}(k) + \mathbf{d}(\mathbf{x}(k) - \mathbf{y}(k))), \quad (2)$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ are the state variables of the drive and response systems and $\mathbf{d}$ is a diagonal matrix with diagonal elements $d_1, \ldots, d_n$ being the coupling coefficients.

We say that the systems synchronize if $||\mathbf{y}(k) - \mathbf{x}(k)|| \to 0$ as $k \to \infty$ (the trajectory of the system (1), (2) converges to the synchronization subspace $\mathbf{x} = \mathbf{y}$).

There are several methods useful for studying of synchronization of chaotic systems. The first criterion for synchronization, introduced in [1], is based on conditional Lyapunov exponents calculated along a typical trajectory of the system. When all conditional Lyapunov exponents of the system (2) driven by the signal $\mathbf{x}(k)$ are negative then one expects that the systems synchronize. This may not be true especially for real systems. It may happen that in the neighborhood of an unstable periodic orbit there exist a region where the trajectories are pushed away from the synchronization subspace [2]. Such a situation occurs when one of the Lyapunov exponents associated with the measure supported by the periodic orbit is positive. Small noise inevitable in real systems could force the trajectory to enter such region. This in turn could lead to desynchronization bursts. This is observed in many computer and laboratory experiments [3], [4], [5]. When there is no noise in the system (or the noise level is very small) one observes the synchronization behavior. But when the noise level is increased then desynchronization bursts occur.

Using the above argument one could develop another criterion of successful synchronization based on all conditional (transversal) Lyapunov exponents. In order to ensure synchronization one should evaluate the conditional Lyapunov exponents for all periodic orbits (apart from the "natural" conditional Lyapunov exponents) and check whether they are negative. This is rather difficult and computationally expensive task. There is also another drawback of this method. Even if the periodic orbit attracts the trajectory to the synchronization space globally it is possible that it repels trajectories locally. Examples of such orbits for continuous–time systems are given in [6].

In order to avoid these problems we propose another criterion for characterization of the synchronization behavior. It is based on local conditional Lyapunov exponents.

## 2 CONDITIONAL LOCAL LYAPUNOV EXPONENTS

First we will briefly recall the notions of local Lyapunov exponents and conditional Lyapunov exponents.

*Lyapunov exponents* $\lambda_i(\mathbf{x})$ of a trajectory based at $\mathbf{x}$ are the eigenvalues of the matrix

$$\Lambda(\mathbf{x}) = \lim_{L\to\infty} \left( \left[\mathbf{T}^L(\mathbf{x})\right]^T \mathbf{T}^L(\mathbf{x}) \right)^{\frac{1}{2L}}, \qquad (3)$$

where

$$T^L(\mathbf{x}) = \mathbf{DF}(\mathbf{F}^{L-1}(\mathbf{x}))\ldots\mathbf{DF}(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x}) \qquad (4)$$

is the composition of $L$ Jacobians. The multiplicative ergodic theorem of Oseledec [7] states that for a typical point on the chaotic attractor (for almost all points with respect to the natural measure) the Lyapunov exponents are well defined and do not depend of the initial point.

*Local Lyapunov exponents* $\lambda_i(\mathbf{x}, L)$ are the eigenvalues of the matrix [8]:

$$\Lambda(\mathbf{x}, L) = \left( \left[\mathbf{T}^L(\mathbf{x})\right]^T \mathbf{T}^L(\mathbf{x}) \right)^{\frac{1}{2L}}. \qquad (5)$$

They say how rapidly perturbations of the initial point $\mathbf{x}$ changes in $L$ steps away from the time of perturbation. They tend to global exponents as $L$ goes to infinity.

*Conditional Lyapunov exponents* of two uni–directionally coupled systems are Lyapunov exponents of the response system (2) driven by the driving signal $\mathbf{x}(k)$.

Combining these two concepts we can define *local conditional Lyapunov exponents* as the local Lyapunov exponents of the response system driven by the signal $\mathbf{x}(k)$.

It turns out that they are very useful in studies of synchronization of chaotic systems, especially in the presence of noise.

Conditional Lyapunov exponents are here introduced for the uni–directional synchronization scheme. When the systems are bi–directionally coupled this notion should be replaced by transversal Lyapunov exponents. In the case of uni-directionally coupled systems these two notions coincide. Lyapunov exponents of the master system are responsible for the chaotic behavior of the systems on the synchronization subspace. Conditional Lyapunov exponents of the response system characterize the behavior of the master-slave system in the neighborhood of the synchronization subspace and are responsible for attracting or repelling trajectories to this space. In this paper we consider only the master–slave configuration and hence we will use the notion of conditional Lyapunov exponents.

## 3 SYNCHRONIZATION OF COUPLED HÉNON MAPS

As an example we consider uni-directionally coupled Hénon maps. The drive system is the Hénon map defined by

$$h(x, y) = (1 + y - ax^2, bx), \qquad (6)$$

where $a = 1.4$ and $b = 0.3$ are standard parameter values. The response system is

$$h(x', y') = h(x' + d_1(x - x'), y' + d_2(y - y')), \qquad (7)$$

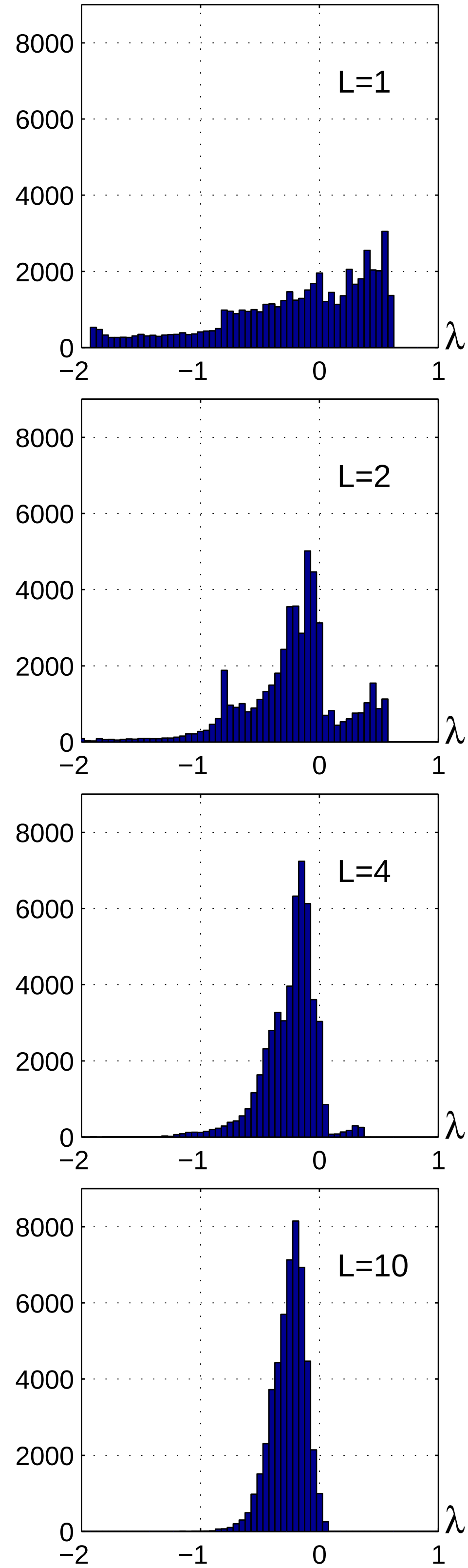


Figure 1: Histogram of maximum local conditional Lyapunov exponent for different time $L$. In all experiments the number of points on the attractor at which the local Lyapunov exponents are computed is 50000. The coupling coefficient is $d_1 = 0.5$.

where $d_1$ and $d_2$ are the coupling coefficients. We will consider the case when only the first coupling coefficient $d_1$ is non–zero.

In the first experiment we have fixed the coupling coefficient $d_1 = 0.5$. For this coupling the systems (6) and (7) eventually synchronize. However when the driving signal is disturbed by a small additive noise we observe frequent desynchronization bursts.

For the analysis of the synchronization properties we use local conditional Lyapunov exponents. The eigenvalues of matrix (5) have been computed for 50000 points on the attractor. In Fig. 1 we show histograms of maximum local conditional Lyapunov exponent $\lambda_1(\mathbf{x}, L)$ for different number of steps $L$.

In the construction of histograms we have chosen bins of the length 0.05 covering the interval $[-2, 1]$. For each bin the number of points for which the maximum Lyapunov exponent belongs to the bin is plotted. One can clearly see how the spectrum of the maximum local conditional Lyapunov exponent changes with the number of steps $L$. For small $L$ it is rather wide, while for greater $L$ it becomes narrower and much higher. In the limit $L \to \infty$ there should be a very narrow peak at the value of the global Lyapunov exponent.

One can also see that for $L = 1$ there are a lot of points on the attractor with $\lambda_1(\mathbf{x}, 1) > 0$. This is an explanation for the existence of desynchronization bursts observed in the presence of noise.

In Fig. 2 we show how the histogram changes when the coupling coefficient $d_1$ is modified. For $d_1 = 0.6$ large part of the histogram lies above zero. For stronger coupling $d_1 = 0.70$ the part of the histogram above zero is smaller and for coupling $d_1 = 0.75$ the whole histogram lies below zero (compare Fig. 2).

We have performed several simulations of synchronization of two Hénon systems for different values of coupling coefficients, for different initial conditions and for different noise level.

Based on these experiments we propose to use the following synchronization criterion.

**Synchronization criterion**

If for a long enough trajectory all of the local conditional Lyapunov exponents $\lambda(\mathbf{x}, 1)$ are smaller than zero then the synchronization of the chaotic systems will occur. In this case noise of a small amplitude will not influence the synchronization behavior.

In order to find the value of $d_1$ for which all local conditional Lyapunov exponents are negative we have performed the following experiment. For each value of $d_1$ from the interval $[0, 2]$ (with the step 0.01) we computed local conditional Lyapunov exponents at 50000 points along the attractor. At each point we have chosen the maximum local conditional Lyapunov exponent. For each $d_1$ we computed the minimum, average and maximum of these maximum exponent. The results are plotted in Fig. 3. One can clearly see the continuous

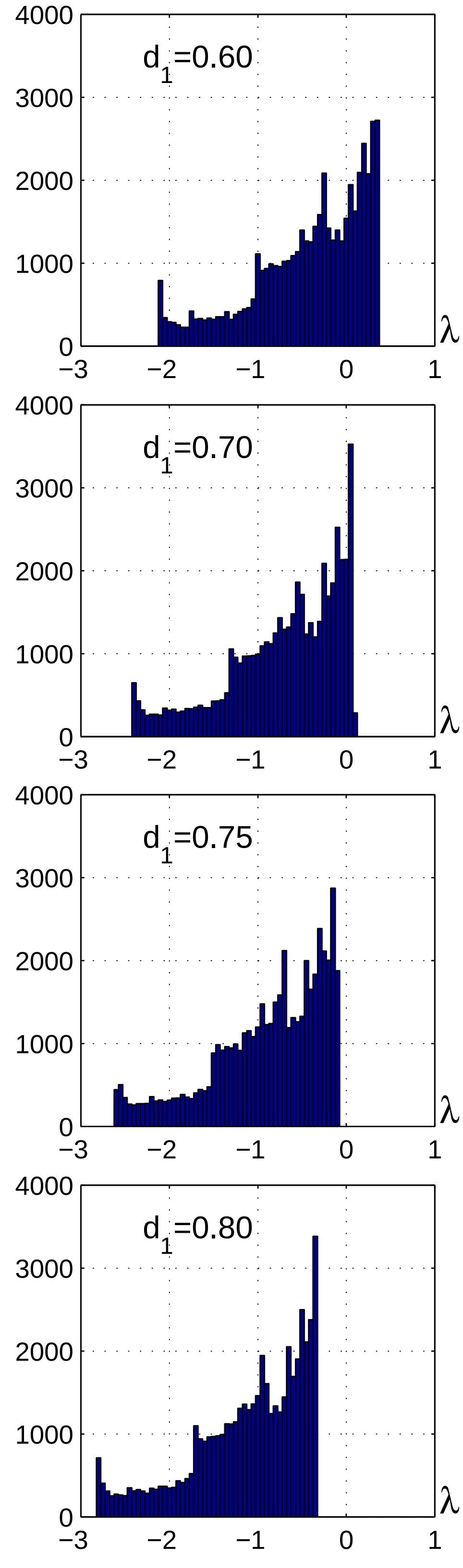


Figure 2: Histogram of maximum local conditional Lyapunov exponent for different coupling coefficient $d_1$. In all experiments the number of points on the attractor at which the local Lyapunov exponents are computed is 50000. The time is $L = 1$.

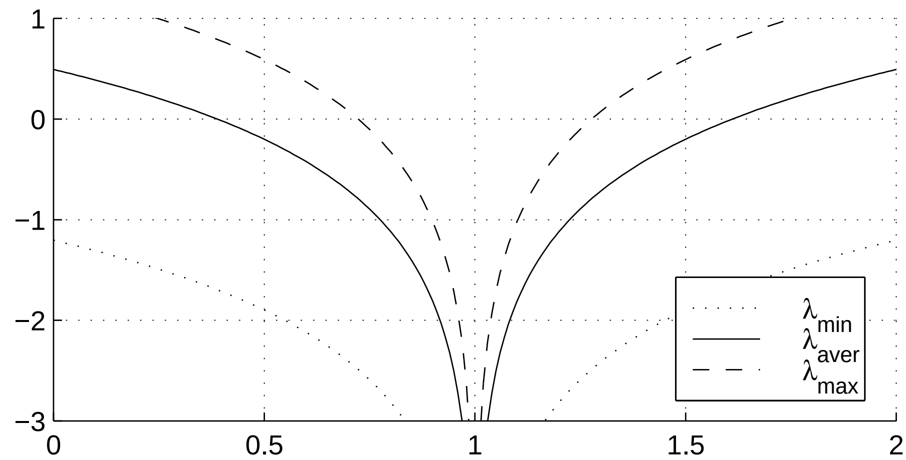


Figure 3: Spectrum of the maximum local conditional Lyapunov exponent. Average, maximum and minimum conditional Lyapunov exponent are plotted with solid, dashed and dotted lines respectively. The number of steps is $L = 1$

change of these values with the change of the coupling coefficient. The average value (solid line) becomes negative at $d_1 = 0.38$ while the maximum value (dashed line) becomes negative for $d_1 = 0.73$. It is clear that when there is no noise in the system one can achieve synchronization with $d_1 > 0.38$. However if due to some noise the trajectory is pushed away from the synchronization subspace and this happens in the region where the local conditional Lyapunov exponent is positive then the trajectory will be repelled from the synchronization subspace and synchronization burst will occur. In order to avoid this possibility we must ensure that local conditional Lyapunov exponents are negative everywhere on the attractor. This condition is true for $d_1 > 0.73$.

## 4 CONCLUSIONS

We have discussed the possibility of using local conditional Lyapunov exponents for characterization of synchronization of coupled chaotic systems. We have shown that they could be effectively used for prediction of synchronization behavior in the presence of noise. We have developed a new criterion for synchronization and confirmed that it is useful in analysis of synchronization.

Local Lyapunov exponents can be used without modifications in the analysis of continuous–time systems (compare [9]). This idea can also be easily generalized for bi-directional coupling. In this case one needs to use local transversal Lyapunov exponents (in case of uni-directional coupling they coincide with the conditional exponents).

## ACKNOWLEDGMENTS

This research was sponsored by the University of Mining and Metallurgy, Kraków.

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