Study of Synchronization of Linearly Coupled Hyperchaotic Systems

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Abstract

In this paper we report results on synchronization of two simple four-dimensional hyperchaotic electronic circuit by means of the linear coupling method. We prove analytically that using appropriate coupling these hyperchaotic systems synchronizes for all initial conditions. In simulations we observe that synchronization is also possible when the systems are coupled using only one variable. This result is confirmed by computation of the conditional Lyapunov exponents of the response system.

1 Introduction

Since the demonstration of the ability of chaotic systems to synchronize [1] great activity has been induced for analysis of this nonlinear phenomenon [2], [3], [4], [5]. This synchronization has possible applications to communications and control [2].

In this paper we consider the problem of synchronization of two linearly coupled systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + \mathbf{K}(\mathbf{x} - \mathbf{y}), \tag{2}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{K} \in \mathbf{M}(n, n)$ is a diagonal matrix. This scheme is also called the error-feedback synchronization method.

Such a configuration was used by Chen and Dong [6] to force a chaotic system to follow an unstable periodic orbit. This technique was also used by Pyragas [7] to synchronize the Lorenz and Rössler systems.

The synchronization of coupled systems is a concept different from the one introduced by Pecora and Carroll [1], where in the response system some dynamical variables are set identically equal to the variables in the driving system.

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Due to unidirectional connection between the systems (1) and (2) the dynamics of the first system does not depend on the dynamics of the second one. The systems (1) and (2) will be called a *driving system* a *response system* respectively. We say that the systems (1) and (2) synchronize if $||\mathbf{x}(t) - \mathbf{y}(t)|| \to 0$ for $t \to \infty$.

In this paper we consider the case when the system (1) is hyperchaotic (it has at least two positive Lyapunov exponents [8]). In section 2 a hyperchaotic circuit considered in this paper is described. In section 3 we prove using the Lyapunov function method that for appropriate coupling the systems synchronize for all initial conditions. In section 4 we address the question whether the synchronization can be obtained with only one non-zero element of the matrix **K**. This problem can be important for possible aplications of chaotic synchronization in communications. For communication problems the systems (1) and (2) are distant. If only one of the elements k_i is non-zero then in order to synchronize the system (2) one needs to send only a scalar signal through the communication channel.

2 A fourth-order electronic circuit

Let us consider a simple fourth-order electronic circuit [9] shown in Fig. 1(a). The circuit contains the nonlinear resistor N, with characteristics shown in Fig. 1(b). For the implementation of active and nonlinear elements see [9].

The dynamics of the circuit is described by the following set of ordinary differential equations:

$$\begin{aligned} C_1\dot{v}_1 &= f(v_2 - v_1) - i_1, \\ C_2\dot{v}_2 &= -f(v_2 - v_1) - i_2, \\ L_1\dot{i}_1 &= v_1 + Ri_1, \\ L_2\dot{i}_2 &= v_2, \end{aligned} \tag{3}$$

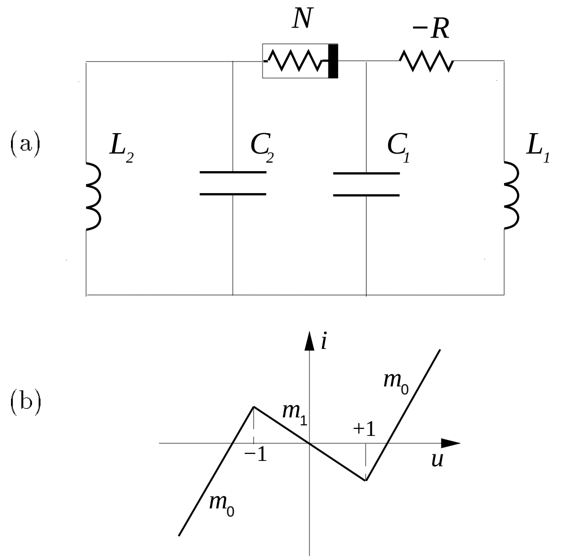

Figure 1: A fourth-order hyperchaotic circuit (a) and characteristics of its nonlinearity (b)

where v_1, v_2, i_1 and i_2 denote the voltage across C_1, the voltage across C_2, the current through L_1 and the current through L_2 respectively, and f is the piecewise linear characteristics given by

$$f(x) = m_0 x + 0.5(m_1 - m_0)(|x+1| - |x-1|). \quad (4)$$

We consider this circuit with the following parameter values: $C_1 = 0.5$, $C_2 = 0.05$, $L_1 = 1$, $L_2 = 2/3$, $R = 1$, $m_0 = 3$ and $m_1 = -0.2$. A typical trajectory of the system is shown in Fig. 2.

In order to check whether the circuit is hyperchaotic the Lyapunov exponents [8] of the system have been computed: $\lambda_1 \approx 0.25$, $\lambda_2 \approx 0.07$, $\lambda_3 = 0$ and $\lambda_4 \approx -53.2$. Two of them are positive which confirms that the system is hyperchaotic.

3 Synchronization of the hyperchaotic circuit by linear coupling

In this section we prove that the systems can be synchronized by means of linear coupling. The dynamics of the drive system is given by (3) and the response

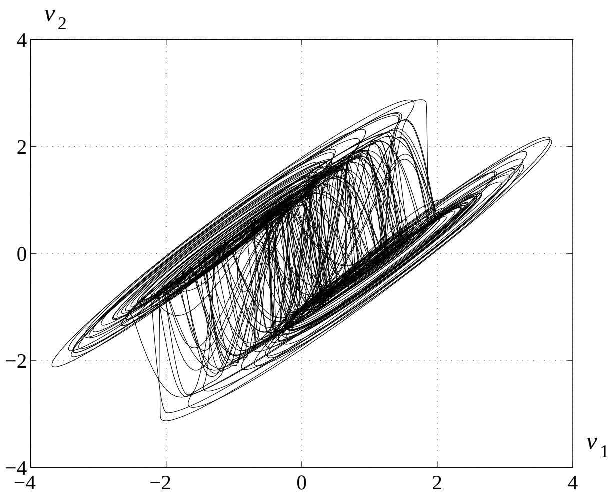

Figure 2: A trajectory of the fourth-order hyperchaotic circuit

system is defined be the following set of equations:

$$\begin{aligned} C_1 \dot{v}_1' &= f(v_2' - v_1') - i_1' + k_1(v_1 - v_1'), \\ C_2 \dot{v}_2' &= -f(v_2' - v_1') - i_2' + k_2(v_2 - v_2'), \\ L_1 \dot{i}_1' &= v_1' + R i_1' + k_3(i_1 - i_1'), \\ L_2 \dot{i}_2' &= v_2' + k_4(i_2 - i_2'), \end{aligned} \quad (5)$$

Let us denote: $e_1 = v_1 - v_1'$, $e_2 = v_2 - v_2'$, $e_3 = i_1 - i_1'$, $e_4 = i_2 - i_2'$. The equations of the "error" system read:

$$\begin{aligned} C_1 \dot{e}_1 &= f(v_2 - v_1) - f(v_2' - v_1') - e_3 - k_1 e_1, \\ C_2 \dot{e}_2 &= -f(v_2 - v_1) + f(v_2' - v_1') - e_4 - k_2 e_2, \\ L_1 \dot{e}_3 &= e_1 + R e_3 - k_3 e_3, \\ L_2 \dot{e}_4 &= e_2 - k_4 e_4, \end{aligned} \quad (6)$$

We expect that when the coefficients k_i are big enough then the systems synchronize.

The following theorem [10] states the existence of matrix $\mathbf{K}$ for which the linearly coupled systems synchronize.

Theorem 1. *If $||\mathbf{x}(0) - \mathbf{y}(0)||$ is sufficiently small then there exist finite values d_i $(i = 1, 2, \ldots, n)$ such that for all matrices $\mathbf{K} = diag(k_1, k_2, \ldots, k_i)$ with $k_i > d_i$ the systems* (1) *and* (2) *synchronize.*

We cannot use the above theorem for our circuit to decide whether the synchronization is possible. This theorem is proved only for the case when $\mathbf{f}$ on the right side of equations (1) and (2) is differentiable (compare proof of this theorem in [10]).

From a similar reason we cannot use directly the general framework for synchronization of dynamical systems presented in [3]. In the proof of the theorem on synchronization by unidirectional linear coupling one uses the fact that the vector field defining a dynamical system is of C^1 class.

In order to find conditions for synchronization of our circuit we will use the global Lyapunov function method [4].

Theorem 2. *Let the drive and response system be defined by* (3) *and* (5) *respectively. Let us assume that* C_1, C_2, L_1, $L_2 > 0$ *and* $m_1 < m_0$. *Let* $k_1 > -2m_1$, $k_2 > -2m_1$, $k_3 > R$ *and* $k_4 > 0$. *For all initial conditions of the drive and response systems the systems synchronize, i.e.,*

$$\lim_{t\to\infty} ||\mathbf{e}(t)|| = 0. \quad (7)$$

In order to proof the above theorem we need the following lemma concerning the nonlinear function f.

Lemma 1. *Let* f *be defined by* (4). *Let* $m_1 < m_0$.

(a) *If* $x \geq x'$ *then* $m_1(x - x') \leq f(x) - f(x') \leq m_0(x - x')$,

(b) *if* $x \leq x'$ *then* $m_0(x - x') \leq f(x) - f(x') \leq m_1(x - x')$,

(c) $m_1(x - x')^2 \leq (f(x) - f(x'))(x - x') \leq m_0(x - x')^2$.

Proof. The first part can be proved from the definition of f by considering several cases for the position of x and x' with respect to the sets $(-\infty, -1)$, $[-1, 1]$, $(1, \infty)$. (b) is equivalent to (a) (exchange x and x'). Inequalities (c) follows easily from (a) and (b). □

Proof of Theorem 2. $V(\mathbf{e}) = \frac{1}{2}(C_1e_1^2 + C_2e_2^2 + L_1e_3^2 + L_2e_4^2)$ is a Lyapunov function, i.e. the following conditions are satisfied:

$$V(\mathbf{e}) > 0 \qquad \text{for } \mathbf{e} \neq 0, \quad (8)$$

$$V(0) = 0, \quad (9)$$

$$\dot{V}(\mathbf{e}) < 0 \qquad \text{for } \mathbf{e} \neq 0, \quad (10)$$

$$\dot{V}(0) = 0, \quad (11)$$

where $\dot{V}$ is the derivative along the trajectories

$$\dot{V}(\mathbf{e}) = C_1e_1\dot{e}_1 + C_2e_2\dot{e}_2 + L_1e_3\dot{e}_3 + L_2e_4\dot{e}_4. \quad (12)$$

It is clear that the conditions (8), (9) and (11) are fulfilled. Now we prove the condition (10). Substituting in (12) $\dot{e}_i$ from (6) we obtain

$$\begin{aligned}\dot{V}(\mathbf{e}) =& (e_1 - e_2)(f(v_2 - v_1) - f(v_2' - v_1')) \\ & - k_1e_1^2 - k_2e_2^2 - (k_3 - R)e_3^2 - k_4e_4^2.\end{aligned}$$

Using the last part of Lemma 1 for $x = v_2 - v_1$, $x' = v_2' - v_2'$ we get

$$\begin{aligned}(e_1 - e_2)(f(v_2 - v_1) - f(v_2' - v_1')) &\leq -m_1(e_1 - e_2)^2 \\ &= m_1(e_1 + e_2)^2 - 2m_1e_1^2 - 2m_1e_2^2.\end{aligned}$$

Finally we have

$$\begin{aligned}\dot{V}(\mathbf{e}) \leq & m_1(e_1 + e_2)^2 - (k_1 + 2m_1)e_1^2 - (k_2 + 2m_1)e_2^2 \\ & - (k_3 - R)e_3^2 - k_4e_4^2.\end{aligned} \quad (13)$$

As $m_1 < 0$ the first term $m_1(e_1 + e_2)^2$ is never positive. All the coefficients at e_i^2 are negative, and hence the condition (10) holds. Because the function V fulfills the conditions (8–11) the error $\mathbf{e}(t)$ will tend to zero as t goes to infinity. It means that the systems synchronize. □

4 Computer simulations

In the previous section we have found conditions for k_i which ensure synchronization. In order to obtain the negative derivative of the Lyapunov function along the trajectory we have to use four positive coupling coefficients (compare Theorem 2). Negative derivative means that the Lyapunov function is continuously decreasing. A weaker condition is based on conditional Lyapunov exponents (CLE's) of the nonautonomous response system [1]. They characterize the exponential rate at which the perturbation of the initial conditions in a response system changes with time. If all of them are negative then the response system is asymptotically stable (i.e., if $||\mathbf{y}_1 - \mathbf{y}_2||$ is small enough then $||\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_1) - \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_2)|| \to 0$ for $t \to \infty$, where $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)$ denotes the trajectory of the response system for initial conditions of the drive and response systems being $\mathbf{x}_0$ and $\mathbf{y}_0$ respectively). CLE's of the response system coincide with CLE's of the "error" system, and hence asymptotic stability of the response system is closely related to the synchronization property [11]. This condition is commonly

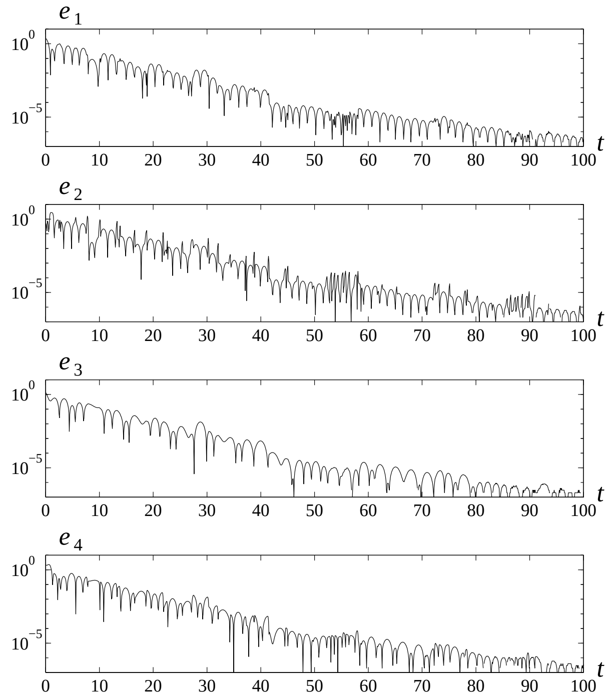

Figure 3: Synchronization of the four-dimensional circuit for $k_3 = 1.0$.

used, in fact a lot of "proofs" of synchronization are based on computation of conditional Lyapunov exponents.

In this section we will consider the special case when only one of elements k_i is non-zero. We will try to answer the question whether it is possible to synchronize the systems in this case. The performance of the synchronization will be measured in terms of conditional Lyapunov exponents.

We say that the systems are v_1 (respectively v_2, i_1, i_2) coupled if only k_1 (respectively k_2, k_3, k_4) is different from zero. In our simulations we observed that the systems synchronize only for the i_1-coupling. An example of synchronization of the i_1-coupled systems with coupling coefficients $k_1 = k_2 = k_4 = 0$, $k_3 = 1.0$ is shown in Fig. 3. One can see that all of the variables of the "error" system tend to zero as time goes to infinity.

In Fig. 4 we show the synchronization process for different values of the coupling coefficient. For $k_3 = 0$ (no coupling) the error $e_1 = v_1 - v_1'$ behaves in a chaotic way. As there is no connection between the systems the signals v_1 and v_1' are uncorrelated (compare Fig. 4(a)). For $k_3 = 0.3$ (Fig. 4(b)) we still do not observe synchronization but there is some correlation between signals. For $k = 0.4$ the correlation is much stronger. One can clearly see the long period of coherent behavior of the systems (Fig. 4(c)). In fact in this case the observation time was extended to $t = 300$, because for $t < 200$ we would not observe the lack of synchronization. For $k_3 = 0.6$ and $k_3 = 1.0$ the synchronization occurs. For greater k_3 the rate of convergence is higher.

In order to analyze the behaviour of the response system the conditional Lyapunov exponents have been computed. The results are presented in Fig. 5. The value of k_3 is chosen from the interval $(0, 1)$ with the step 0.01. For each value the conditional Lyapunov exponents are computed. The greatest of them is plotted. One can see that for weak coupling ($k_3 < 0.55$) at least one of the conditional Lyapunov exponents is positive (we do not observe synchronization), while for $k_3 > 0.6$ all of the Lyapunov exponents are negative (the synchronization occurs). If the systems synchronize then the maximum Lyapunov exponent can be used to estimate the rate of convergence of the "error" system. For $k_3 = 1.0$ the maximum conditional Lyapunov exponent is $l_{\max} \approx -0.135$. This value agrees very well with the convergence rate observed in Fig. 3:

$$\frac{\log(e_i(t_1)) - \log(e_i(t_0))}{t_1 - t_0} \approx \log \frac{0.5 \cdot 10^{-6}}{100} \approx -0.145.$$

If the systems does not synchronize the maximum Lyapunov exponent cannot be computed directly by observing a time plot. As the chaotic attractors of the systems are bounded the state variables of the difference system cannot grow infinitely and hence we cannot obtain positive Lyapunov exponent in this way.

The synchronization process shown in Fig. 4 agrees with the maximum conditional Lyapunov exponent $l_{\max}$ plotted in Fig. 5. For the first three cases ($k_3 = 0, 0.3, 0.4$) the maximum conditional Lyapunov exponent is positive and the synchronization does not occur, while in the last two cases ($k_3 = 0.6, 1.0$) l_1 is negative and we observe perfect synchronization. For $k_3 = 0.6$ the maximum conditional Lyapunov exponent is $l_{\max} \approx -0.1$ and the convergence rate is slow. For $k_3 = 1.0$ the value of $l_{\max}$ is approximately -0.135 and the state v_1' converges to v_1 at a higher rate.

We would like to stress that (in opposite to the method for proving synchronization presented in Section 2) the method based on conditional Lyapunov

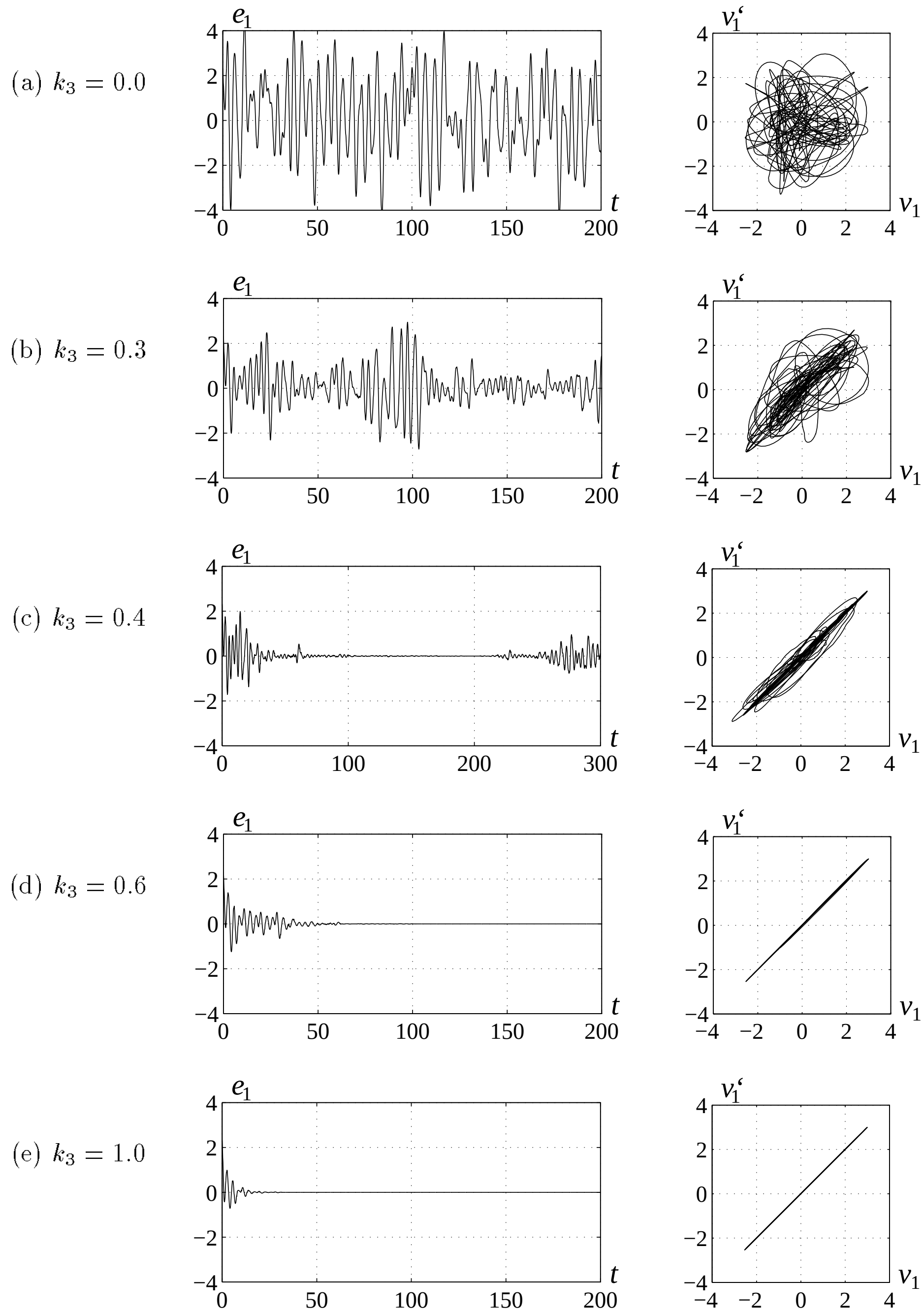

Figure 4: Synchronization of the four-dimensional circuit for different values of the coupling coefficient k_3.

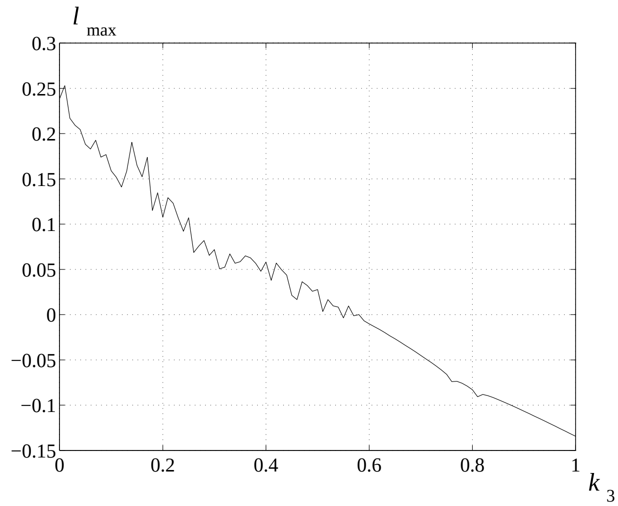

Figure 5: The maximum conditional Lyapunov exponent of the four-dimensional synchronized circuit for different values of coupling coefficient k_3

exponents does not allow us to obtain any global characteristics of the coupled systems. Conditional Lyapunov exponents are computed locally along the trajectory. Hence one has no information about the size of the set of initial conditions for which the synchronization take place. Another drawback is that CLE's are usually obtained from computer simulations of the system, so one cannot be sure what their exact values are.

5 Conclusions

In this paper we investigated the possibility of synchronization of simple hyperchaotic circuits by linear coupling of the systems. Using the global Lyapunov function method we have proved that for certain values of coupling coefficients the synchronization is ensured. In spite of existence of two positive Lyapunov exponents for the system we managed to synchronize them in simulations using only one variable from the autonomous system.

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