# Coexistence of attractors in a one-dimensional CNN array

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**ABSTRACT:** *In this paper we investigate long-term (steady-state) behavior of a one-dimensional array of chaotic circuits for different connection strength. Using computer experiments we have confirmed the existence of a very large number of stable steady–states depending only on the initial conditions applied in the individual cells and on the connection strength. Of special interest is the coexistence of large amplitude periodic oscillations in some cells and chaotic oscillations in the others, that forms very complex spatial patterns.*

## 1 Introduction

In the recent years there has been a growing interest in studies of systems composed of coupled nonlinear oscillators (bi-stable, oscillatory or chaotic cells). Usually one considers a one of two–dimensional lattice of nonlinear oscillators.

Lattice models naturally arise when using neural networks, arrays of electronic oscillators whose dynamical behavior can be very complex both in time and space [4, 5, 6, 7, 8]. Such models, exhibiting required collective behavior can be also used for simulations of a variety of phenomena observed in real systems and for studies in physics, solid state electronics, chemical reactions, biology and medicine [1, 2, 3, 9].

The dynamics of individual cells and the coupling between them predefines the overall system behavior. Among other types of collective dynamics one can observe various kinds of spatial, temporal or spatio-temporal ordered structures referred to as self-organization [1] or "pattern formation". "Organized" behavior is usually linked with coherent (synchronized) behavior of a number of cells in the network. Organized spatio-temporal behavior includes also propagation of waves including solitons and autowaves, target waves, spiral waves and traveling wavefronts [8].

In this paper we study long-term (steady-state) behavior observed in a ring (one–dimensional array with connected edges) of coupled Chua's chaotic circuits. Using computer experiments we have confirmed the existence of a very large number of stable final states depending only on the initial conditions applied in the individual cells.

## 2 Experimental setup

Let us consider a one–dimensional lattice of simple third–order electronic oscillators (Chua's circuits). The oscillators are coupled bi-directionally by means of two resistors cross-connected between the capacitors $C_1$ and $C_2$ of the neighboring cells. Every cell is connected with two nearest neighbors. The dynamics of the lattice composed of $n$ cells can be described by the following set of ordinary differential equations [4], [5]:

$$\left\{ \begin{array}{rcl} C_2\dot{x}_i &=& -Gx_i - y_i + Gz_i + G_1(z_{i-1} - x_i) + G_1(z_{i+1} - x_i) \\ L\dot{y}_i &=& x_i \\ C_1\dot{z}_i &=& Gx_i - Gz_i - f(z_i) + G_1(x_{i-1} - z_i) + G_1(x_{i+1} - z_i) \end{array} \right. \quad \text{for i=1,2, ... ,n} \qquad (1)$$

where $f$ is a five–segment piecewise linear function:

$$f(z) = m_2 z + \frac{1}{2}(m_1 - m_2)(|z + B_{p_2}| - |z - B_{p_2}|) + \frac{1}{2}(m_0 - m_1)(|z + B_{p_1}| - |z - B_{p_1}|) \qquad (2)$$

We consider the array of size 31. The first and the last cells are also connected and the lattice forms a ring. In order to achieve this efect we use the following boundary conditions: $x_0 = x_n$, $z_0 := z_n$, $x_{n+1} := x_1$ and $z_{n+1} := z_1$.

In simulation experiments we used typical parameter values for which an isolated Chua's circuit generates chaotic oscillations — the double scroll attractor ($C_1 = 1/9F$, $C_2 = 1F$, $L = 1/7H$, $G = 0.7S$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$). For these parameter values together with a chaotic attractor there exist periodic orbit with a large amplitude. In the experiments we have considered the uniform coupling $G_1 \in [0.01, 0.1]$. For the integration of the system the fourth-order Runge-Kutta method was used with the time step $\tau = 0.1$.

## 3 Steady–states of the array for different types of initial conditions

The steady–state of the network depends on the type of initial conditions. Two types of natural initial conditions are possible. The first one corresponds to a cell oscillating in a chaotic regime with a trajectory forming the double–scroll attractor. The second one corresponds to a cell sustaining periodic oscillations with large amplitude. In this section we study the influence of initial conditions on the steady–state of the array. We consider three examples. In the first example all the cells are started with initial conditions on the chaotic attractor. In the second case some cells have initial conditions on the chaotic attractor while other cells have initial conditions on the large amplitude periodic orbit. In the last case initial conditions of all the cells are picked on the periodic orbit. For each case we couple the circuits using different $G_1$.

### 3.1 Initial conditions on chaotic attractor

In the first experiment we have coupled 31 chaotic Chua's circuits with initial conditions on the chaotic double–scroll attractor. We have checked what is the steady–state behavior for different values of coupling conductance $G_1 \in [0.01, 0.1]$. The results are plotted in Fig. 1. In each line we show the behavior of 12 adjacent cells. For each cell we plot projection of the cell state on the $y_i, z_i$ plane. In Fig. 1a–d the range for $y_i$ and $z_i$ variables is $[-2, 2]$ while in Fig. 1e–h the range is increased to $y_i \in [-12, 12]$ and $z_i \in [-5, 5]$.

For very weak coupling ($G_1 = 0.01$) all the cells behaves as they are not coupled at all (Fig. 1a). For every cells the plot is very similar to the double–scroll attractor. The only difference is that switching between two scrolls are less frequent. If we increase the coupling slightly ($G_1 = 0.02$) we observe that all the cells behave periodically (see Fig. 1b). Fourteen of the cells have trajectories belonging to the upper plane and seventeen to the lower plane. For the coupling conductance $G_1 = 0.03$ the trajectories are again of double–scroll type but switchings between two scrolls are now very seldom (see Fig. 1c). For $t = 200$ trajectory of only one cell visit both half-planes. See also that the attractor is smaller than for $G_1 = 0.01$ (compare Fig. 1a). If we again increase coupling strength ($G_1 = 0.04$) we observe that the steady–state is chaotic. This time every cell displays a Roessler–type attractor (single scroll). After a transient we observe no switchings between upper and lower half-planes.

For $G_1 = 0.045$ the behavior of the array is very interesting. Two cells ($i = 3, 4$) display oscillations of large amplitude (see Fig. 1e). These two cells are almost perfectly synchronized. Two adjacent cells ($i = 2, 5$) behave chaotically (medium amplitude) while other cells behave chaotically (small amplitude) with trajectories forming Roessler–type attractors. In this case all the cells behave chaotically. Chaos is not visible in the behavior of two cells with large oscillations (small chaotic signal is added to large periodic oscillations). For stronger coupling $G_1 = 0.05$ ten cells are in a large amplitude "periodic" mode, four cells behave unperiodically (medium size oscillations), other cells show very small oscillations (amplitude of oscillation depends on the distance from cells sustaining large amplitude oscillations. For $G_1 = 0.06$ all the cells behave periodically. For two cells ($i = 15, 16$) the amplitude of oscillations is small while for the other cells the amplitude of oscillations is large (Fig. 1g). If we further increase the coupling strength ($G_1 = 0.07$) we observe that all the cells are perfectly synchronized and display periodic oscillations of large amplitude (compare Fig. 1h). Similar results are observed for stronger coupling $G_1 = 0.08$, $G_1 = 0.1$.

### 3.2 Mixed initial conditions

In the second experiment initial conditions of 3 cells were picked on the chaotic attractor ($i = 9, 15, 25$) while other cells were initially behaving periodically (large amplitude oscillations). For these initial

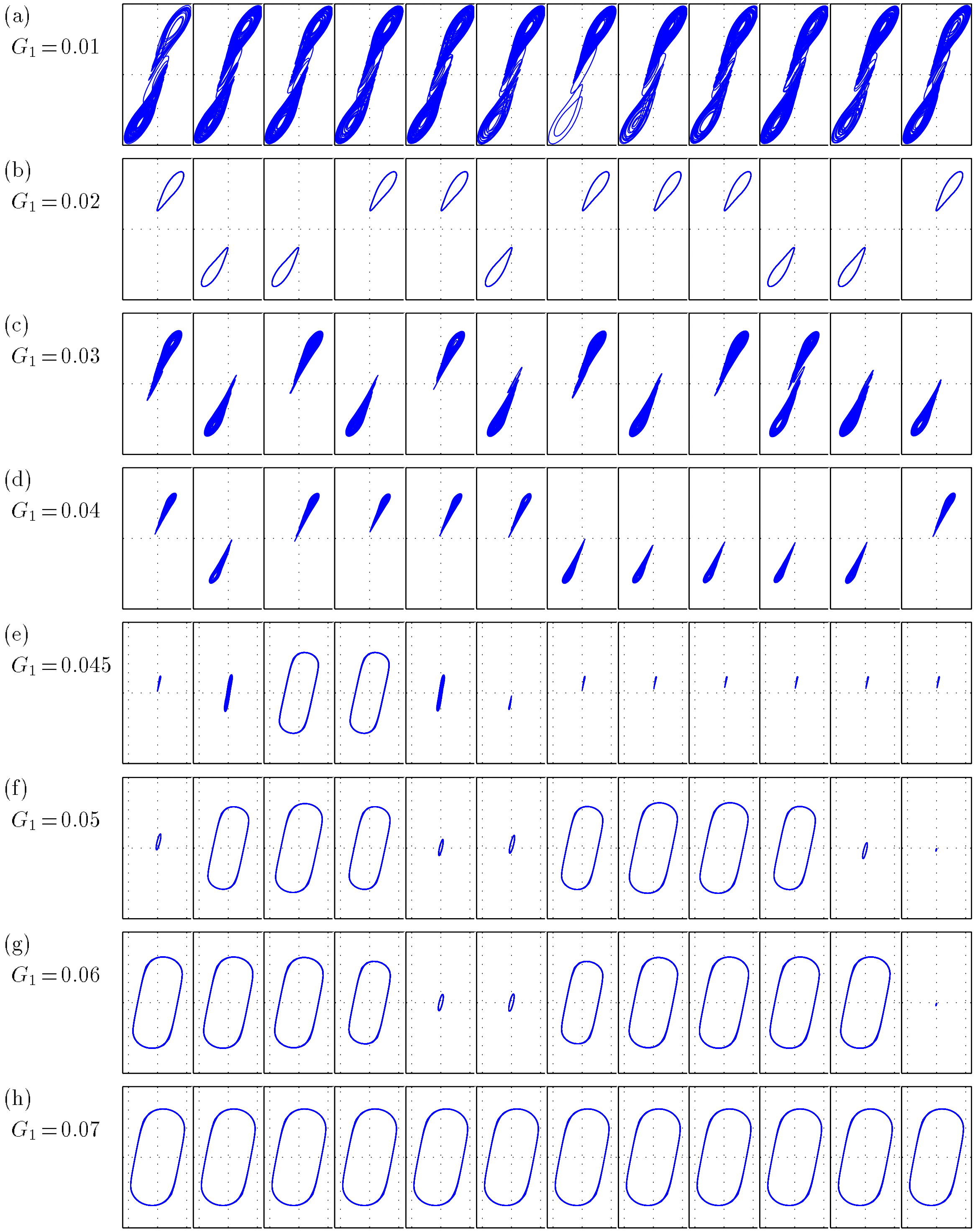


*Figure 1: Steady state behavior of the lattice of chaotic cells with initial conditions lying on the chaotic attractor for different coupling conductance* $G_1$*. The projection of attractor onto the* $y_i, z_i$ *plane for cells* $i = 1, \ldots, 12$ *(*$i = 11, \ldots, 22$ *for case* (g)*) is shown.* (a)–(d) $y_i, z_i \in [-2, 2]$, (e)–(h) $y_i \in [-12, 12]$ *and* $z_i \in [-5, 5]$

conditions there is a smaller number of different steady–state behavior types. For $G_1 = 0.01$ and $G_1 = 0.02$ the three cells with initial conditions on chaotic attractor remains chaotic while the other cells display large "periodic" oscillations. Due to connection between cells small chaotic signal is observed also in cells behaving "periodically". For $G_1 = 0.02$ one can see long periods of almost periodic behavior. For $G_1 = 0.05$ two cells ($i = 9, 25$) remains in a chaotic regime of small amplitude while the third cell starting from the chaotic attractor escapes to "periodic" mode. For $G_1 = 0.06$, $G_1 = 0.07$ we observe long transient but finally all the cells are synchronized (large amplitude periodic oscillations). For $G_1 = 0.08$ two cells ($i = 15, 25$) escape to "periodic" mode while the cell $i = 9$ remains in a chaotic regime. The neighbors of the $9^{\text{th}}$ cell oscillate with the opposite phase forming a "sea-saw" profile of the wave. For $G_1 = 0.085$ all the cells finally are in a periodic mode (large oscillations) but this time cells are not synchronized. One can observe traveling wave front moving in the left direction. For $G_1 = 0.09$ all the cells are synchronized, and for $G_1 = 0.1$ we again observe wave front moving in the left direction.

### 3.3 Initial conditions on the large amplitude periodic orbit

In this experiment we have coupled 31 cells with initial conditions on the large amplitude periodic orbit. For this type of initial conditions we have observed either full synchronization or a traveling wave. For weak coupling $G_1 = 0.01, 0.02, 0.03$ the steady–state of the network was a traveling wave moving in the right direction. For strong coupling $G_1 = 0.04, 0.05, 0.07, 0.08, 0.1$ all the cells in the steady–state were synchronized.

### 3.4 Influence of initial conditions — summary

If the cell before coupling was behaving chaotically two different types of steady state behavior are observed. For small coupling the cell displays chaotic oscillations or periodic oscillations of small amplitude. For larger coupling the cell may enter the large amplitude periodic regime. Its trajectory may also remain chaotic or converge to a fixed point (possibly modulated with chaotic motion of a very small amplitude).

If the behavior of the cell before coupling was periodic (large amplitude oscillations) only one type of behavior is observed. The cell in the steady–state displays oscillations of large amplitude. These oscillations are periodic. These large amplitude oscillations may be modulated with a chaotic oscillations of a small amplitude if there are some cells in the array oscillating chaotically.

## 4 Steady–states for different coupling strength

In this section we investigate the influence of coupling strength on the number and types of steady states existing in lattice.

### 4.1 Strong coupling

First we consider strong coupling $G_1 = 0.1$. We simulated the array of Chua's circuits starting from random initial conditions. We have observed that eventually all the cells were displaying periodic oscillations of large amplitude. The network as a whole converged to one of the five steady–state behaviors corresponding to different wave profiles. Four of them are shown in Fig. 2. The first type is a synchronized behavior (see Fig. 2a), where all cells follow the same trajectory and there is no difference in phase between the cells. In the second type all the cells also follow the same trajectory but there is a slight difference in the phase between adjacent cells. The total difference is $2\pi$ and one can see a single–hump wave moving in the left direction (compare Fig. 2b). The next type of steady–state behavior is very similar to the second one — the only difference is the direction of the wave (compare Fig. 2c). The last two types correspond to double–hump waves moving in the left and right directions (compare Fig. 2d). In spite of extensive simulations we have not found initial conditions leading to a different steady–state behavior.

One could expect that for a longer lattice there exist stable waves with many humps developing in the system. The development of a single-hump wave is very interesting. Although initially the slopes of the hump are steep and the length of the wave is small eventually the length of the wave becomes exactly the length of the lattice. It seems that the "natural" length of this wave is smaller than the length of the considered lattice. We think that even for $n = 31$ one could observe a three–hump wave but the chance to get one starting from random initial conditions is very small.

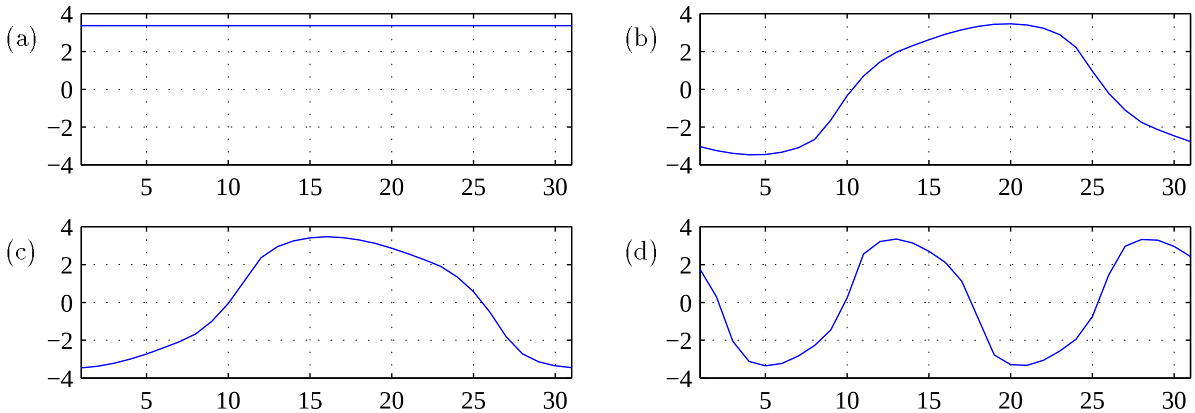


*Figure 2: Possible profiles of traveling waves observed in the lattice for strong coupling $G_1 = 0.1$, (a) synchronized behavior, (b) single–hump wave moving in the right direction, (c) single–hump wave moving in the left direction, (d) double–hump wave moving in the left direction*

### 4.2 Medium coupling

Now we consider the circuits coupled by the conductances $G_1 = 0.05$. As before we initiate the network using random initial conditions. For small amplitude of initial conditions the steady–state is zero (all the cells settle in a fixed point $x = y = z = 0$) (compare also fixed point attractor for some cells in Fig. 1f). One can easily prove that the origin is stable for $G_1 \in [0.05, 0.0586]$. If we increase the amplitude of initial conditions then some of the cells enters oscillations of large amplitude and some of them settle in a fixed point. It is also possible that some cells behave chaotically. Four of the possible wave profiles are shown in Fig. 3.

Let us look closer into the first example (see Fig. 3a). The cells $i = 13$ displays periodic oscillations of large amplitude. Its neighbors ($i = 12, 14$) show periodic oscillations of small amplitude (symmetric to the origin). Their neighbors $i = 13, 15$ also oscillate periodically but the amplitude is much smaller. This is an example of coexistence of large amplitude oscillations ($i = 13$) and a fixed point behavior ($i = 12, 14$), modulated by periodic motion of the neighbor. Due to the non–zero coupling this modulation propagates through the array. The amplitude of periodic oscillations depends on the distance from the cell sustaining large periodic oscillations. In this figure we have also groups of cells with large periodic oscillations. One of the groups consists of two cells $i = 5, 6$. The second one is composed of 7 cells ($i = 28, \ldots 3$). The cells in each group are quite well synchronized. Such a group of cells oscillating in a synchronized way is called a cluster. Of particular interest is the cell $i = 4$. Its neighbors belong to different clusters. Hence its fixed point attractor is modulated by non-synchronized oscillations from two cells. It displays very complex (possibly chaotic) behavior.

If we further increase the amplitude of initial conditions all the cells enter the large amplitude oscillations and we may see waves similar to the ones observed for strong coupling (synchronized behavior, single and double–hump waves)(comapre Fig. 2).

All the steady–states observed seem to be stable. We have performed several experiments adding some noise to the steady–state. If the amplitude of the noise was small enough the steady–state of the network was not changed.

## 5 Conclusions

In this paper we have studied the influence of initial conditions and coupling strength on long-term behavior of the array of locally coupled chaotic circuits. We have found an extremely rich variety of patterns and wave profiles existing as a steady–state of the considered system. When all the cells operate in the large amplitude periodic mode the array as a whole displays a wave with one of the several profiles (synchronization, single–hump and double–hump waves). In the opposite case one can observe the coexistence of large amplitude oscillations with a fixed point behavior or the coexistence of large amplitude oscillations with chaotic behavior. For particular coupling it is also possible that all the cells sustain periodic oscillations with small amplitude.

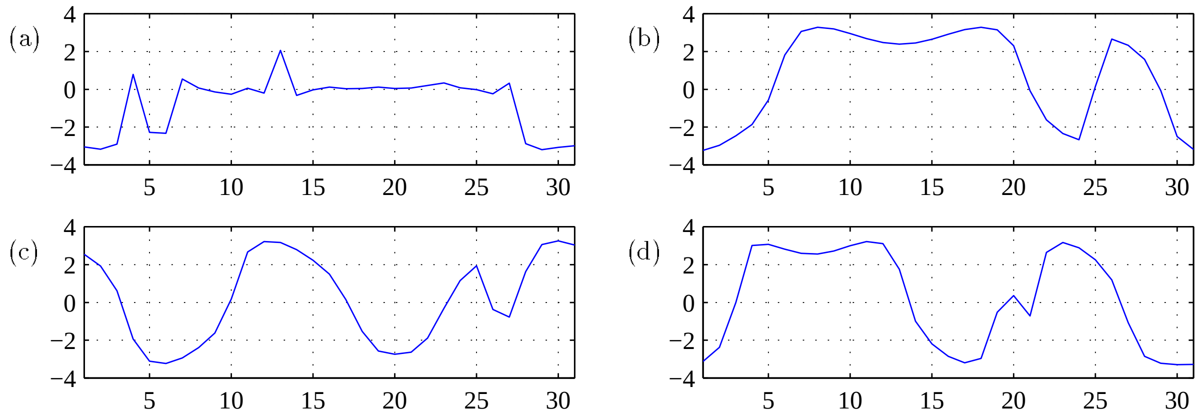


Figure 3: Examples of waves observed in the lattice for medium coupling $G_1 = 0.05$

In our future work we would like to investigate the possibility of controlling the patterns observed in the array by imposing some boundary conditions (for examples constant or periodic) instead of connecting the edges of the lattice. We would like also to study the possibility of existence of such waveforms in a two–dimensional array of chaotic cells. In our previous work [4, 5] we have observed the synchronized behavior and various patterns composed of spiral waves but we did not observe wavefront traveling from right to left or from bottom to top.

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