On rigorous study of Poincaré maps defined by piecewise linear systems

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***Abstract*— In this work we study possibilities of rigorous analysis of piecewise linear systems using Poincaré map technique and interval methods. We show that successful analysis can be carried out provided that the Poincaré map is continuous in the region containing the attractor. In particular, complete classification of short periodic orbits embedded within the attractor is possible.**

I. Introduction

The method of Poincaré map is commonly used in analysis of continuous systems. In this work we describe methods for rigorous investigations of Poincaré maps defined by piecewise linear systems. For piecewise linear systems the planes separating linear regions are the most natural choice for the hyperplanes defining the Poincaré map. For this choice,however, it may happen that the Poincaré map is not continuous or even is not defined for the region of interest.

In [1] we have shown that for the Chua's circuit with parameter values for which the double-scroll attractor is observed the Poincaré map is not continuous. In this paper, we consider the Chua's circuit displaying the Roessler-type attractor. In this case the Poincaré map is well defined and continuous in the region containing the numerically observed attractor and a more complete analysis of the system is possible.

Let us consider a simple third order piecewise linear electronic circuit [2] described by the following state equation

$$\begin{aligned} C_1\dot{x}_1 &= (x_2 - x_1)/R - g(x_1), \\ C_2\dot{x}_2 &= (x_1 - x_2)/R + x_3, \\ L\dot{x}_3 &= -x_2 - R_0 x_3, \end{aligned} \qquad (1)$$

where $g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)$ is a three segment piecewise linear characteristics.

The circuit is studied with the following parameter values (after appropriate parameter rescaling)

$$\begin{aligned} &C_1 = 1, C_2 = 7.65, G_a = -3.4429, G_b = -2.1849, \\ &L = 0.06913, R = 0.33065, R_0 = 0.00036, \end{aligned} \qquad (2)$$

for which the Roessler-type attractor is observed in computer simulations (compare Fig. 1(a)). The double-scroll attractor (see Fig. 1(b)) exists for $C_2 = 9.3515$, all other parameters being the same as above. For the double-scroll attractor discontinuity of the Poincaré map can be seen in the projection onto the plane (x_1, x_2). Some trajectories turn close to the plane $x_1 = 1$, which means that intersections with this plane are not always transversal (compare Fig. 1(b)). On the other hand for the Roessler-type attractor in Fig. 1(a) the intersections are transversal and consequently the Poincaré map is continuous on the attractor.

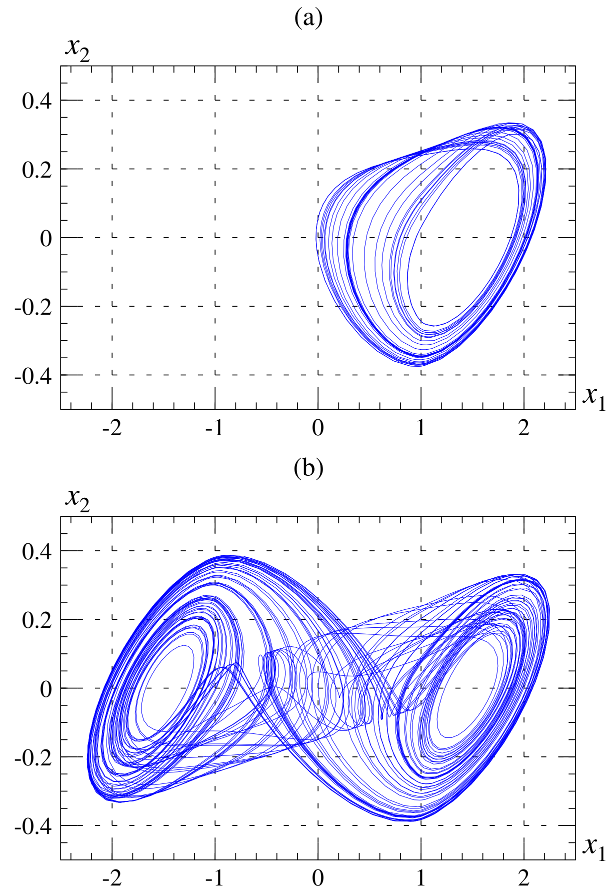

Fig. 1. Computer simulations of Chua's circuit, (a) Roessler-type attractor for $C_2 = 7.65$, (b) double-scroll attractor for $C_2 = 9.3515$

Another problem with the double-scroll attractor is that the stable manifold of the origin intersects the attractor. For points belonging to the stable manifold the Poincaré map is not defined (trajectories starting from the stable manifold never come back to the planes separating the linear regions).

In the next section we show how to rigorously analyse the system in case the Poincaré map is continuous.

II. Rigorous analysis of the system

The state space $\mathbb{R}^3$ can be divided into three open regions $U_1 = \{x \in \mathbb{R}^3 \colon x_1 < -1\}$, $U_2 = \{x \colon |x_1| < 1\}$ and $U_3 = \{x \colon x_1 > 1\}$ separated by planes $\Sigma_1 = \{x \colon x_1 = -1\}$ and $\Sigma_2 = \{x \colon x_1 = 1\}$. In the regions U_i the system is linear, the state equation can be written as: $\dot{x} = A_i(x - p_i)$, where A_i are matrices with real coefficients, p_i are vectors, and the solution has the form

$$\varphi(t, x) = \mathrm{e}^{A_i t}(x - p_i) + p_i.$$

A. Poincaré map

Let $\Sigma = \Sigma_1 \cup \Sigma_2$. The *Poincaré map* $P : \Sigma \mapsto \Sigma$ is defined as

$$P(x) = \varphi(\tau(x), x), \tag{3}$$

where $\varphi(t, x)$ is the trajectory of the system based at x, and $\tau(x)$ is the time needed for the trajectory $\varphi(t, x)$ to reach Σ. Usually a Poincaré map is defined by a single hyperplane. If the set defining the Poincaré map is composed of two or more planes the map is sometimes called a *generalized Poincaré map*.

B. Evaluation of the Poincaré map

For rigorous evaluation of P we use analytical formulas for solutions of linear systems. Let us assume that $\mathbf{x}$ is the rectangle enclosed in one of the planes Σ_1, Σ_2. We assume that trajectories based at $x \in \mathbf{x}$ enter the linear region where the state equation has the form $\dot{x} = A(x - p)$. First, we find t_1 such that $\varphi(s, x) \notin \Sigma$ for all $x \in \mathbf{x}$ and $s \in (0, t_1]$. Then we find $t_2 > t_1$ such that for all $x \in \mathbf{x}$ the point $\varphi(t_2, x)$ belongs to another linear region. It follows that the interval $\mathbf{t} = [t_1, t_2]$ is the enclosure of the return time for all points in $\mathbf{x}$, i.e. $\mathbf{t} \supset \{\tau(x) \colon x \in \mathbf{x}\}$. We find the box

$$\mathbf{R} = \mathrm{e}^{A\mathbf{t}}(\mathbf{x} - p) + p. \tag{4}$$

In order to obtain a narrow enclosure we use the mean value form for the evaluation of the above formula. For the details see [3]. Finally, the enclosure $\mathbf{y}$ of $\{P(x) \colon x \in \mathbf{x}\}$ is computed as the intersection of $\mathbf{R}$ with Σ.

The Jacobian of P at $x \in \Sigma$ can be computed using the following formula (compare [4]):

$$P'(x) = \left(I - \frac{A(y - p)e_1^T}{e_1^T A(y - p)} \right) e^{At}, \tag{5}$$

where $t = \tau(x)$ is the return time, $y = P(x)$, $e_1 = (1, 0, 0)^T$ and I is the 3×3 identity matrix. The above formula holds if the trajectory $\varphi(t, x)$ intersects Σ transversally at points x and y.

The enclosure of $\{P'(x) \colon x \in \mathbf{x}\}$ is computed using the formula (5) with the interval quantities $\mathbf{t}$ and $\mathbf{y}$ found in the computation of $P(\mathbf{x})$.

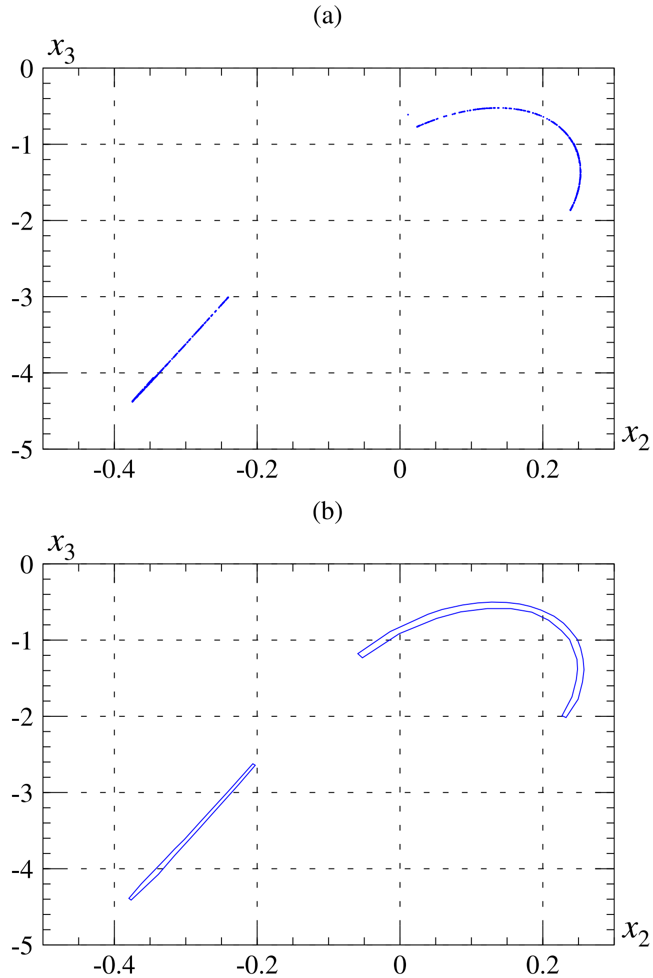

Fig. 2. (a) computer generated trajectory of the Poincaré map P, (b) the trapping region composed of two polygons

C. Trapping region

In the first step of the rigorous analysis we locate the trapping region containing the numerically observed attractor. We say that a set A is a *trapping region*, if it is *positively invariant*, i.e. $P(x) \in A$ for all $x \in A$. As was mentioned before if the Poincaré map is not continuous on the attractor it is not possible to prove the existence of such a region using techniques described here.

A computer generated trajectory of the Poincaré map is shown in Fig. 2(a). The trapping region is found by constructing a set of polygons enclosing the trajectory and modifying corners' positions by hand to satisfy the condition for the trapping region. The polygons found have 16 and 36 edges, respectively. In order to prove that the region A is positively invariant it is sufficient to check that the image of its border is enclosed in A and that the Poincaré map is well defined on A. This can be done by covering the region by a number of boxes, computing image of each box and checking whether a proper condition is satisfied. We have proved that the image of the first polygon is enclosed in the second one and that the image of the second polygon is enclosed in the first one, thus showing that the set composed of the two polygons is a trapping region for the Poincaré map.

D. Graph representation

In the second step, we find the graph representation of the dynamics of the system in the trapping region (see [5] for details). The trapping region is covered by ε–boxes, i.e. sets of the form

$$\mathbf{v} = [k_1\varepsilon_1, (k_1+1)\varepsilon_1] \times [k_2\varepsilon_2, (k_2+1)\varepsilon_2], \qquad (6)$$

where k_i are integer numbers, ε_i are fixed positive real numbers, and $\varepsilon = (\varepsilon_1, \varepsilon_2)$. ε–boxes define vertices and admissible connections between boxes define edges of the graph.

The computation results are shown in Table I. The trapping region is covered by 6067 ε–boxes, with $\varepsilon = (0.001, 0.0025)$. An enclosure of the image of each box is found and the set E composed of 36090 nonforbidden transitions between boxes is constructed:

$$E = \{(i,j)\colon P(\mathbf{v}_i) \cap \mathbf{v}_j \neq \emptyset\}.$$

Next, we reduce the graph by removing vertices corresponding to boxes having empty intersection with the invariant part of the trapping region (see also [6], [7], [5]). This results in the covering composed of 1722 boxes with 10676 admissible connections.

Computation of the invariant part is combined with the subdivision technique, where invariant part is found successively for finer divisions (see Table I).

TABLE I

THE NUMBER OF ε–BOXES AND CONNECTIONS FOR DIFFERENT ε.

ε	covering $\#V$	covering $\#E$	invariant part $\#V$	invariant part $\#E$	t [s]
$(0.001, 0.0025)$	6067	36090	1722	10676	1199.51
$(0.001, 0.0025)/2$	6888	39721	2882	16463	2138.36
$(0.001, 0.0025)/2^2$	11528	65476	5752	32691	3811.76
$(0.001, 0.0025)/2^3$	23008	130331	11891	67485	7662.16
$(0.001, 0.0025)/2^4$	47564	269741	24482	139553	15480.82

E. Return time

Once the graph is generated, we can find the bounds for the return time for all points belonging to the attractor. Let $\boldsymbol{\tau}_i$ be the interval containing return times for points in the box $\mathbf{v}_i$ ($\boldsymbol{\tau}_i$ is found in the process of finding the image of $\mathbf{v}_i$ under Poincaré map). The return time $\boldsymbol{\tau}$ for the whole set $\bigcup \mathbf{v}_i$ is computed as a hull of intervals $\boldsymbol{\tau}_i$, i.e. $\boldsymbol{\tau} = \bigcup \boldsymbol{\tau}_i$. Starting with the covering composed of 24482 boxes of size $(0.001, 0.0025)/2^4$, we have shown that the return time for all points belongs to $\mathbf{T}_1 = \boldsymbol{\tau} = [1.1986, 4.3658]$. Using the information on admissible connections between boxes we can obtain bounds for the return time of the n–th iteration of the Poincaré map. The results are collected in Table II.

It is clear that the average return time between two crossings of the set Σ belongs to the interval $\mathbf{T}_n/n$ for each n. For example for $n = 1000$ we obtain:

$$\tau_{\text{aver}} \in [3.2704, 3.3141]. \qquad (7)$$

TABLE II

RIGOROUS ESTIMATES $\mathbf{T}_n$ FOR TIME NEEDED TO COMPLETE n ITERATIONS OF THE POINCARÉ MAP.

n	$\mathbf{T}_n$	$\mathbf{T}_n/n$
1	[1.1986,4.3658]	[1.1986,4.3658]
2	[5.1574,7.6279]	[2.5787,3.8140]
3	[8.3655,11.5561]	[2.7885,3.8521]
4	[12.1315,13.9024]	[3.0329,3.4756]
5	[14.8024,17.6637]	[2.9605,3.5328]
6	[18.6087,20.9189]	[3.1014,3.4865]
7	[21.5266,24.8744]	[3.0752,3.5535]
8	[25.3095,27.2978]	[3.1636,3.4123]
9	[27.6289,31.0721]	[3.0698,3.4525]
10	[31.4984,34.2038]	[3.1498,3.4204]
11	[34.5306,38.1560]	[3.1391,3.4691]
12	[38.4023,40.8458]	[3.2001,3.4039]
13	[40.6944,44.7794]	[3.1303,3.4446]
14	[44.6086,47.4692]	[3.1863,3.3907]
15	[47.2592,51.4259]	[3.1506,3.4284]
16	[51.2181,54.1062]	[3.2011,3.3817]
100	[326.08,332.40]	[3.2608,3.3240]
1000	[3270.4,3314.1]	[3.2704,3.3141]

It follows that the period of the orbit having n intersections with Σ belongs to the interval $[3.2704 \cdot n, 3.3141 \cdot n]$.

F. Periodic orbits

Using the graph representation we can also find all low-period cycles of the Poincaré map. We start by finding all period–n cycles in the graph. Each cycle may correspond to a periodic orbit of the dynamical system. In order to prove the existence of a periodic orbit or to prove that there are no periodic orbits corresponding to this cycle we use the Hansen–Sengupta operator H, which is a standard interval tool for proving the existence of zeros of nonlinear maps (compare [8], [9]). We evaluate the interval operator H on the interval vector $\mathbf{z}$, corresponding to the cycle under study. If $\mathbf{z} \cap \mathrm{H}(\mathbf{z}) = \emptyset$, then there is no period–n orbits in $\mathbf{z}$. If $\mathrm{H}(\mathbf{z}) \subset \mathbf{z}$, then there is exactly one period–n orbit inside $\mathbf{z}$.

In none the two above conditions is fulfilled we try to proceed by increasing the size of $\mathbf{z}$. Namely, we set the new value of $\mathbf{z}$ to be the convex hull of $\mathbf{z}$ and $(1+\varepsilon)\,\mathrm{H}(\mathbf{z}) - \varepsilon\,\mathrm{H}(\mathbf{z})$. If the diameter of $\mathrm{H}(\mathbf{z})$ is larger than the diameter of $\mathbf{z}$, we stop the computations and try again with a finer grid.

The computation time depends on the number of period–n cycles in the graph. In order to reduce this number we find the cycles of the graph for a finer division, and then construct the graph for boxes two (or more) times larger in each direction.

We start with the covering by boxes of size $(0.001, 0.0025)/2^3$. After finding all cycles of length n we increase the box size 8 times in each direction obtaining a smaller number of cycles. In Table III we show the results of the search for short periodic orbits. Q_n is the number of period–n cycles, P_n is the number of fixed points of P^n, C_n is the number of period–n cycles in the graph for the initial box size, and D_n is the number of period–n cycles for which the existence was verified. We have found all periodic orbits having at most 16 intersections with the set Σ (see Fig. 3).

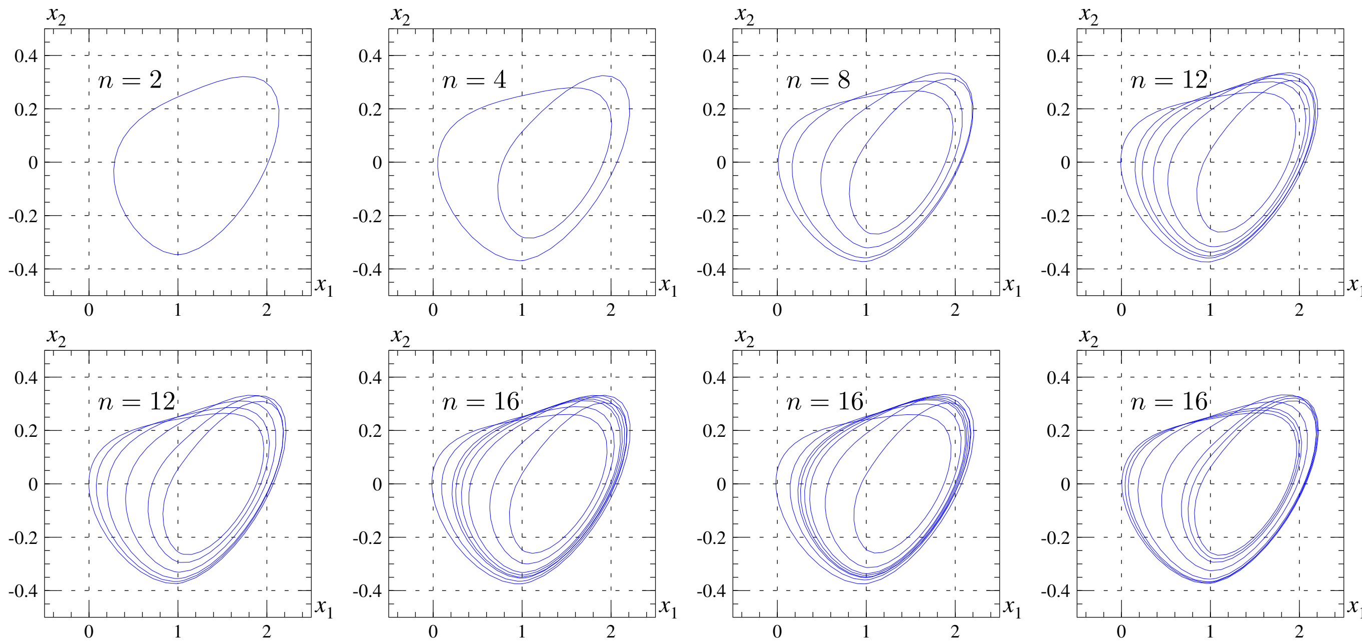

Fig. 3. Short periodic orbits of the Chua's circuit, n is the number of intersections with Σ

TABLE III
PERIODIC ORBITS OF THE POINCARÉ MAP

n	Q_n	P_n	C_n	D_n	t [s]
2	1	2	5	1	0.67
4	1	6	123	7	4.07
6	0	2	221	2	2.19
8	1	14	14780	62	63.04
10	0	2	9031	3	4.47
12	2	30	1968941	421	650.99
14	0	2	371209	5	14.08
16	3	62	232528160	30388	38479.85

It is clear that there are no periodic orbits with odd number of intersection with Σ. One should note that there are no periodic orbits with 6, 10 and 14 intersections with the set Σ.

Since the average return time belongs to the interval $[3.2704, 3.3141]$ we can claim that all periodic orbits with period shorter than 58 are found ($3.2704 \cdot 18 > 58$).

III. Conclusions

In this paper we have studied a possibility of rigorous investigation of piecewise linear system by means of the combination of the Poincaré map technique and interval methods. We have shown that if the Poincaré map is continuous in the region containing the numerically observed attractor it is possible to find a trapping region and all low-period cycles for the system.

As an example, we have considered the third-order electronic circuit with parameter values for which the Poincaré map is continuous in the region containing the numerically observed attractor. We have located a positively invariant region in the domain of the Poincaré map. For a very fine division of the trapping region into boxes we have found the graph representation of the dynamics of the system. Using this information we have found a very good rigorous approximation for the average return time. We have also found all periodic orbits with period $n \leq 16$ of the generalized Poincaré map enclosed in the trapping region.

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