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Comparison of Methods for Proving the Existence of Periodic Solutions for Continuous–Time Systems

Keywords: nonlinear system, periodic point, interval arithmetic, interval Newton method

ABSTRACT

In this paper we compare two different methods, which can be used for proving the existence of periodic orbits in continuous-time dynamical system. The first method (interval Newton method) allows also to prove the uniqueness of a periodic orbit in a certain set. The second method is based on topological conjugacy of the dynamics of the system around the periodic point with a linear system.

1. INTRODUCTION

Proving the existence of a periodic solutions in a nonlinear system is usually not a trivial task. The problem of existence and stability of periodic orbits is very important in analysis of nonlinear systems and also in many applications. One of the most sophisticated methods for controlling chaotic systems is to stabilize one of the periodic orbits embedded in the chaotic attractor [1].

Usually for identification of periodic solutions form a time series one uses the method close returns developed in [2]. In this method one scans a trajectory looking for parts which are almost periodic (the trajectory returns close to the initial point). We believe that in the neighborhood of such fragment there exist a real periodic orbit. However the existence of a real periodic trajectory is not ensured.

In this paper we study two different methods for proving the existence of periodic orbits. In both methods some parts of the proof are checked with the help of computer. The first method — the interval Newton method — belongs entirely to the interval arithmetic tools. The second method is based on the Brouwer's theorem and computer is used to check the assumptions of the topological theorem.

In the first part of the paper the methods are outlined briefly. Then we compare the feasibility of these two methods using a simple chaotic circuit as an example.

2. PROVING THE EXISTENCE OF PERIODIC ORBITS

2.1. Interval Newton Method

The interval Newton method allows to find zeros of a function

$$\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n. \tag{1}$$

In order to investigate the existence of zeros of $\mathbf{f}$ in an n-dimensional interval $\mathbf{X}$ one has to evaluate the *interval Newton operator*

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - (\mathbf{Df}(\mathbf{X}))^{-1}\mathbf{f}(\mathbf{x}_0), \tag{2}$$

where $\mathbf{x}_0$ is an arbitrary point belonging to the interval $\mathbf{X}$. In order to prove the existence and uniqueness of periodic orbits we use the following property of interval Newton operator [3]:

Proposition 1 *If* $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ *then there exist exactly one* $\mathbf{x} \in \mathbf{X}$ *such that* $f(\mathbf{x}) = 0$.

2.2. Topological Method

The second method, based on the Brouwer's theorem, was developed in [4]. Although the method is formulated for arbitrary dimension n of the map but we will describe it for the case $n = 2$. In order to prove the existence of a fixed point we have to find a parallelogram P such that its image under the map is enclosed in the stripe S defined by the "vertical" edges. The second condition is that the images of horizontal edges lie above and below the considered parallelogram (compare Fig. 1).

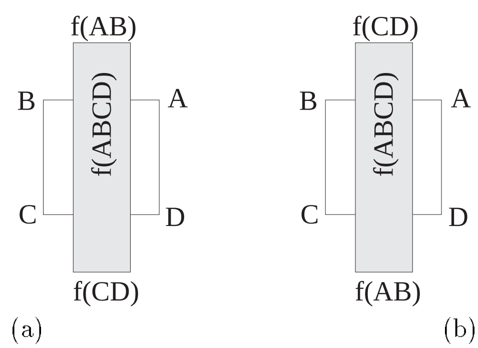

Fig. 1: Parallelogram $P = \overline{ABCD}$ and its image under f.

3. ELECTRONIC CIRCUIT

As an example let us consider the Chua's circuit — a simple third–order system — defined by:

$$C_1\dot{x} \quad = \quad G(y - x) - g(x),$$

$$
\begin{aligned}
C_2\dot{y} &= G(x-y)+z, \\
L\dot{z} &= -y-R_0 z,
\end{aligned} \tag{3}
$$

where $g(\cdot)$ is a three-segment piecewise-linear function

$$
g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|). \tag{4}
$$

For parameters: $C_1 = 1$, $C_2 = 9.3515$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$ the system (3,4) has a well-known "double–scroll" chaotic attractor. The state space $\mathbb{R}^3$ can be divided into three open regions $U_\pm = \{\, \mathbf{x} = (x,y,z)^T \in \mathbb{R}^3 : \pm x > 1 \,\}$, $U_0 = \{\, \mathbf{x} = (x,y,z)^T \in \mathbb{R}^3 : |x| < 1 \,\}$ separated by planes $V_\pm = \{\, \mathbf{x} \in \mathbb{R}^3 : x = \pm 1 \,\}$. In the regions U_0, $U_\pm$ the solution has the form $\mathbf{x}(t) = e^{\mathbf{A}_0 t}\mathbf{x}$ and $\mathbf{x}(t) = e^{\mathbf{A}_\pm t}(\mathbf{x} \mp \mathbf{p}) \pm \mathbf{p}$ respectively, where $\mathbf{A}_0$, $\mathbf{A}_+ = \mathbf{A}_-$ are matrices with real coefficients.

Let us define a *Poincaré map* $\mathbf{P} : V_+ \ni \mathbf{x} \mapsto \phi_{\tau(\mathbf{x})}(\mathbf{x}) \in V_+$, where $\phi_t(\mathbf{x})$ is the trajectory of the system (3,4) based at $\mathbf{x}$ and $\tau(\mathbf{x})$ is the time needed for the trajectory $\phi_t(\mathbf{x})$ to return to V_+.

Similarly we define a *halfmap* $\mathbf{H} : V_- \cup V_+ \ni \mathbf{x} \mapsto \phi_{\tau(\mathbf{x})}(\mathbf{x}) \in V_- \cup V_+$, where $\tau(\mathbf{x})$ is the time needed for the trajectory $\phi_t(\mathbf{x})$ to reach one of the planes V_- or V_+.

Assuming that the trajectory based at $\mathbf{x} \in V_+$ visits k linear regions before returning to V_+ the Poincaré map can be decomposed as: $\mathbf{P}(\mathbf{x}) = \mathbf{H}^k(\mathbf{x})$. Hence the Jacobian of $\mathbf{P}$ can be computed in terms of Jacobians of the halfmap. For the computation of the Jacobian matrix of $\mathbf{H}$ we will use the following lemma [5]:

Lemma 1 *Let $\mathbf{x}_0 \in V_- \cup V_+$. Let us assume that the solution of the system for $t \in [0, t_0]$ is given by $\mathbf{x}(t) = e^{\mathbf{A}t}(\mathbf{x}_0 - \mathbf{p}) + \mathbf{p}$. Let $\mathbf{y}_0 = \mathbf{H}(\mathbf{x}_0) = \mathbf{x}(t_0)$. Let us also assume that the intersections of V_- and V_+ with the trajectory $\mathbf{x}(t)$ at $\mathbf{x}_0$ and $\mathbf{y}_0$ are transversal. Then the Jacobian of the halfmap $\mathbf{H}$ at $\mathbf{x}_0$ is the principal minor (created by removing the first row and the first column) of the matrix*

$$
\left[\mathbf{I} - \frac{\mathbf{A}(\mathbf{y}_0 - \mathbf{p})\mathbf{e}_1^T}{\mathbf{e}_1^T \mathbf{A}(\mathbf{y}_0 - \mathbf{p})} \right] e^{\mathbf{A}t_0}, \tag{5}
$$

where $\mathbf{e}_1 = (1,0,0)^T$ and $\mathbf{I}$ is the 3×3 identity matrix.

4. STUDY OF EXISTENCE OF PERIODIC ORBITS

For the extraction of periodic orbits we have used the combination of the method of close returns [2] and standard Newton method. We have found several periodic points of the Poincaré map $\mathbf{P}$ associated with the continuous flow. Some of them are shown in Fig. 2. Their approximate position and other parameters are collected in Table 1.

4.1. Interval Newton method

Now we describe how to prove the existence and uniqueness of a periodic orbit by means of the interval Newton method. Let n be the period of the orbit. First one

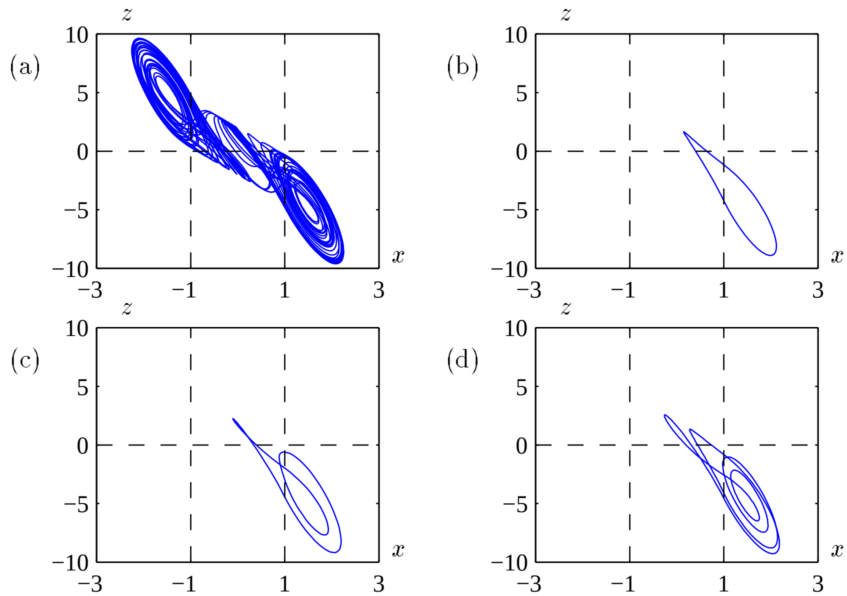

Fig. 2: Chaotic trajectory (a) and periodic orbits of the Chua's circuit, period-1 orbit (b), period-2 orbits (c)–(e), period-3 orbit (f)

n	$n_{\mathbf{H}}$	length	approx. position	λ_1	λ_2
1	2	7.38	$(1, -0.333, -4.240)$	-3.18	-0.00412
2	4	14.38	$(1, -0.352, -4.439)$	-9.09	$-5.39 \cdot 10^{-6}$
3	6	28.62	$(1, -0.185, -2.441)$	45.57	$3.69 \cdot 10^{-12}$

Table 1: Periodic orbits of $\mathbf{P}$. n is the period of the orbit. $n_{\mathbf{H}}$ is the number of the regions U_0 and $U_{\pm}$ visited by the orbit (with multiplicities), $\lambda_{1,2}$ are the eigenvalues of $\mathbf{P}^n$

chooses a rectangle $\mathbf{X}$ on the Poincaré map which encloses the periodic point found numerically. Then one evaluates the image of the center $\mathbf{x}_0$ of $\mathbf{X}$ under the n^{th} iteration of the Poincaré map. We also need to compute the Jacobian matrix of $\mathbf{P}^n$ at the interval $\mathbf{X}$. Finally the interval Newton operator for the map $\mathrm{id} - \mathbf{P}^n$ is computed:

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - (\mathbf{I} - \mathbf{D}\mathbf{P}^n(\mathbf{X}))^{-1} (\mathbf{x}_0 - \mathbf{P}^n(\mathbf{x}_0)) . \tag{6}$$

If $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ then there exists exactly one periodic point of $\mathbf{P}$ with period n belonging to $\mathbf{X}$. In the opposite case one has to modify the initial rectangle $\mathbf{X}$ and repeat the computations.

First let us consider a fixed point of $\mathbf{P}$ (compare Fig. 2b). Using the interval Newton method we were able to prove the existence and uniqueness of the fixed point of $\mathbf{P}$ within the rectangle $(y, z) = (-0.333^{2181}_{0109}, -4.239^{9987}_{7915})$. Its diameter is greater than 0.0002 in both y and z. We would like to stress that for proving the existence and uniqueness of a periodic orbit we need to evaluate the interval Newton operator only once.

By applying the interval Newton operator iteratively (3 iterations) we were able to prove the existence of the fixed point within the interval $(y, z) = (-0.333114482^{207}_{009}, -4.23989511^{63}_{47})$. In this way we have sharpened the bounds of the result to an uncertainty under $1.98 \cdot 10^{-10}$ in y and $1.6 \cdot 10^{-9}$ in z. Using this result we have computed the Jacobian matrix of the Poincaré map at the fixed point and the eigenvalues of this Jacobian. The eigenvalues belong to the intervals: -3.1798^{239}_{308} and -0.0041^{23}_{30}. The uncertainty is below $7 \cdot 10^{-6}$. Hence the fixed point is of a saddle type.

Let us now consider the period–2 orbit shown in Fig. 2c. This orbit has four intersections with the planes $V_{\pm}$. Due to the "wrapping effect" which causes quick growth of initial rectangle when we compute its trajectory using interval arithmetic we are not able to prove the existence of this periodic orbit using directly Proposition 1 — the diameter of $\mathbf{N}(\mathbf{X})$ is greater than the diameter of $\mathbf{X}$ for any choice of $\mathbf{X}$.

In order to overcome this problem we can use the method of intermediate sections [6]. When we use this method for $\mathbf{X} = (-0.351500^{8}_{9}, -4.43901^{119}_{245})$ with division into 4 rectangles at each intermediate section for the evaluation of both $\mathbf{DP}^n(\mathbf{X})$ and $\mathbf{P}^n(\mathbf{x}_0)$ we obtain $\mathbf{N}(\mathbf{X}) = (-0.3515008^{12}_{79}, -4.43901^{19}_{25}) \subset \mathbf{X}$, with diameter $6.7 \cdot 10^{-8}$ in y and $6 \cdot 10^{-7}$ in z. The existence and uniqueness of period two orbit follow from Proposition 1.

We have also tried to use the interval Newton method for proving the existence of the third periodic orbit from Table 1 (compare also Fig. 2d). We have estimated that in order to prove the existence of this orbit it would be necessary to use 4^7 rectangles at each intermediate section.

4.2. Topological method

Let us now consider the second method. In order to implement this method we have to find a parallelogram with the desired properties. The most natural choice of a parallelogram is based on the approximate position of the periodic orbit and the eigenvectors of the Jacobian matrix at this point. We have found the approximate positions of periodic orbits using the standard Newton method and the Jacobian matrix at these points using equation (5). Then one has to check whether the images of edges lie appropriately with respect to the parallelogram. This is done by covering the edges by rectangles and computing their images using interval arithmetic tools.

In order to check the assumptions of the topological theorem for the period-1 orbit we had to evaluate the Poincaré map at 2605 rectangles. The number of rectangles is almost independent on the parallelogram size. For the diameter 0.02 we need 2605 rectangles, for the diameter 2×10^{-7} we need 2801 rectangles while for the diameter 2×10^{-8} we need 3801 rectangles. For the diameter 2×10^{-9} we were not able to complete the proof. Even if we use the smallest representable rectangles their images under the second iteration of the Poincaré map are greater than the parallelogram.

We have also estimated the amount of computations necessary to complete the proof for the period-2 orbit. In order to check the assumptions of the topological theorem we have to evaluate the second iteration of $\mathbf{P}$ at more than 4×10^6 rectangles.

There are several possible modifications of this method. The first possibility is to use the concept of intermediate section. Another option is to construct parallelograms for each plane $V_\pm$ separately.

5. CONCLUSIONS

In this paper we have compared two methods for proving the existence of periodic orbits in continuous–time systems. The main advantage of the interval Newton method is that it allows to prove also the uniqueness of the orbit. The implementation of this method is easier. For short orbits one has to evaluate the interval Newton operator only once, while in the topological method the number of rectangles at which we have to evaluate the map is rather large even for short orbits. By iterating the interval Newton method one can easily sharpen the bounds of solutions.

It seems that the interval Newton method is superior to the topological method in the investigations of periodic orbits.

ACKNOWLEDGMENTS

This research has been supported by KBN, grant no. 0449/P3/94/06.

REFERENCES

[1] E. Ott, C. Grebogi, and J.A. Yorke, "Controlling chaotic dynamical systems," in *Chaos — Soviet-American Perspectives on Nonlinear Science*, D.K. Campbell, Ed., pp. 153–172. American Institute of Physics, New York, 1990.

[2] D.P. Lathrop and E.J. Kostelich, "Characterisation of an experimental strange attractor by periodic orbits," *Phys. Rev. A*, vol. 40, no. 7, pp. 4028–4031, 1989.

[3] Götz Alefeld, "Inclusion Methods for Systems of Nonlinear Equations – The Interval Newton Method and Modifications," in *Topics in Validated Computations*, Jürgen Herzberger, Ed. IMACS–GAMM, 1994, pp. 7–26.

[4] C. Miranda, "Un'osservazione su un teorema di Brouwer," *Boll. Un. Mat. Ital.*, vol. 2, no. 3, pp. 5–7, 1940.

[5] Z. Galias, J.A. Nossek, and M.J. Ogorzałek, "Negative results on control of chaotic circuits," in *Proc. Int. Workshop, Nonlinear Dynamics of Electronics Systems, NDES'94*, Cracow, 1994, pp. 174–178.

[6] Z. Galias, "Investigations of periodic orbits in electronic circuits with interval Newton's method," accepted for presentation at ISCAS'98, 1998.