PROVING THE EXISTENCE OF PERIODIC SOLUTIONS USING GLOBAL INTERVAL NEWTON METHOD

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ABSTRACT

In this paper we introduce the global interval Newton method and investigate the possibility of using this method for proving the existence of periodic orbits in continuous–time dynamical systems. We compare this method with a standard version of the interval Newton method. As an example we consider a simple third order electronic circuit for which we prove the existence of several unstable periodic orbits.

1. INTRODUCTION

The detection of periodic orbits in nonlinear systems is a problem of continuing interest in a variety of fields. Usually periodic orbits are found in numerical studies but there is no guarantee that there exists a true periodic trajectory that stays near a computer generated one. This problem is especially important for chaotic systems, as chaotic trajectories exhibit sensitive dependence on initial conditions.

In the present work we develop the technique for proving the existence of periodic orbits based on the *interval Newton's method* [1, 5]. An introduction to the interval arithmetic underlying this method is given in [2]. In interval analysis we are sure that the result obtained encloses the true solution (together with the rounding error). In this paper we use boldface lowercase letters to denote intervals and usual math italic lowercase letters to denote point quantities.

The interval Newton method uses set theoretic fixed point theorem and belongs to the class of *self validating algorithms* [1]. In this method in order to investigate the existence of zeros of a function $\mathbb{R}^m \ni x \mapsto f(x) \in \mathbb{R}^m$ in an m-dimensional interval $\mathbf{x}$ one computes the interval Newton operator:

$$\mathbf{N}(\mathbf{x}) = x_0 - (\mathbf{D}f(\mathbf{x}))^{-1} f(x_0), \qquad (1)$$

where $(\mathbf{D}f(\mathbf{x}))^{-1}$ is the interval matrix containing all Jacobian matrices of f of the form $(\mathbf{D}f(x))^{-1}$ for $x \in \mathbf{x}$ and x_0 is an arbitrary point belonging $\mathbf{x}$. If $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ then there exists exactly one point $x \in \mathbf{x}$ such that $f(x) = 0$. Hence the interval Newton's method can be used to prove the existence and uniqueness of zeros. By iterating the method one can easily sharpen the bounds of solutions.

In order to apply this method for proving the existence of periodic solutions for continuous–time systems one considers a Poincaré map P associated with the continuous–time flow and applies the interval Newton method to the map $\mathrm{id} - P$. In our previous studies [3] we have used this technique for proving the existence of low–period orbits in a simple electronic circuit. For longer orbits the method does not work due to the wrapping effect that causes a quick growth of the initial rectangle — the assumptions of the existence theorem cannot be checked with computer assistance in a reasonable time.

2. GLOBAL INTERVAL NEWTON METHOD

In this paper we present a modification of this method called the *global interval Newton method* which may be also used for proving the existence of longer periodic orbits.

Let us denote by $\varphi_t(x)$ the trajectory of the system starting at x. Let us consider an orbit $\{\varphi_t(\bar{x})\}_{t\in[0,T]}$ and let us choose p planes $\Sigma_1, \ldots, \Sigma_p$ which are transversal to this orbit. Let us denote by Σ the union of the planes Σ_i. We assume that $\bar{x} \in \Sigma$ and that the orbit does not intersect any of the sets $\Sigma_i \cap \Sigma_j$ for $i \neq j$. Let n be the number of points at which the trajectory intersects Σ.

Let us define a *generalized Poincaré map* $H : \Sigma \mapsto \Sigma$:

$$H(x) = \varphi_{\tau(x)}(x), \qquad (2)$$

where $\tau(x)$ is the time needed for the trajectory $\varphi_t(x)$ to reach Σ.

In order to prove that the trajectory $\{\varphi_t(\bar{x})\}_{t\in[0,T]}$ is periodic it is sufficient to prove that $H^n(\bar{x}) = \bar{x}$. One possible solution to this problem is to apply the interval Newton method to the map $\mathrm{id} - H^n$. This method

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works fine for short orbits. For longer orbits, especially when the matrix $\mathbf{D}H^n(\bar{x})$ is ill-conditioned, one cannot check the existence condition and the method fails (compare [3]).

Here we propose to use the interval Newton method to the map $F : (\mathbb{R}^m)^n \mapsto (\mathbb{R}^m)^n$ defined by

$$[F(z)]_k = x_{(k+1) \bmod n} - H(x_k) \quad \text{for } 0 \leq k < n, \quad (3)$$

where $z = (x_0, \ldots, x_{n-1})$. As a starting point for the interval Newton method we choose an interval centered at $z = (\bar{x}, H(\bar{x}), \ldots, H^{n-1}(\bar{x}))$. See that $F(z) = 0$ if and only if $\bar{x}$ is a fixed point of H^n. In the global interval Newton method the problem of existence of periodic orbits is translated to the problem of existence of zeros of a higher–dimensional function.

Once the sequence of the planes Σ_i is defined we may introduce a local coordinate system on each of the planes Σ_i and consider H as a map from $\mathbb{R}^{m-1}$ to $\mathbb{R}^{m-1}$ reducing the dimension of the map F from mn to $(m-1)n$.

3. ELECTRONIC CIRCUIT

As an example we consider the Chua's circuit, a simple third–order system defined by the following set of ordinary differential equations:

$$\begin{aligned} C_1\dot{x} &= G(y-x) - g(x), \\ C_2\dot{y} &= G(x-y) + z, \\ L\dot{z} &= -y - R_0 z, \end{aligned} \quad (4a)$$

where $g(\cdot)$ is a three-segment piecewise-linear function

$$g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|). \quad (4b)$$

For parameters: $C_1 = 1$, $C_2 = 9.3515$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$ the system (4) has a "double–scroll" chaotic attractor. The state space $\mathbb{R}^3$ can be divided into three regions (where the system is linear) separated by planes $V_\pm = \{\mathbf{x} \in \mathbb{R}^3 : x = \pm 1\}$. For our circuit we choose this planes as the planes defining the generalized Poincaré map ($\Sigma_1 = V_+$, $\Sigma_2 = V_-$).

The Jacobian matrix of the map F at the point $z = (x_0, \ldots, x_{n-1})$ can be computed as

$$\mathbf{D}F(z) = \begin{pmatrix} -\mathbf{J}_0 & \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{J}_1 & \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{J}_{n-1} \end{pmatrix} \quad (5)$$

where $\mathbf{I}_2$ is the identity matrix of dimension 2 and $\mathbf{J}_i = \mathbf{D}H(x_i)$ is the Jacobian of H at x_i (see [3] for the formula of $\mathbf{D}H$).

4. PERIODIC ORBITS

In this section we apply the global interval Newton method for proving the existence of periodic orbits for Chua's circuit.

In order to identify periodic orbits and prove their existence we propose to use the following procedure. First we extract periodic orbits using the method of close returns [4]. We monitor a trajectory and look for parts of the trajectory coming closely to the initial point. Then using the standard Newton method we sharpen the approximation obtaining a trajectory of length n of the generalized Poincaré map. We create an interval $\mathbf{x}$ centered at the approximate position of the orbit with the same diameter at all points along the orbit. Finally we apply the interval Newton operator to the map F at $\mathbf{x}$ and check whether $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$. If this condition is fulfilled the existence of periodic orbit is proven. In the opposite case we modify $\mathbf{x}$ and repeat the computations.

Using the method of close returns we have found several quasi–periodic trajectories. We have applied the above procedure to some of the periodic orbits found. For most of the orbits we have succeeded in proving their existence. For few of them the method failed. We have observed that in all unsuccessful cases the orbit spent a long time in one linear region before returning to Σ. If such cases introducing greater number of planes Σ_i should be helpful.

Some of the orbits for which the existence was proven are shown in Fig. 1. Their parameters are collected in Table 1. In the first column we give the position of the orbit in Fig. 1. Periodic orbits are sorted according to their periods which are printed in the second column. In Table 1 we use the following notation: n is the number of intersections of the orbit with the planes $V_\pm$ (it is the period of the orbit on the generalized Poincaré map H), n_+ (n_-) is the period of the orbit on the Poincaré map defined by the section plane V_+ (V_-). The position of the orbit on the plane V_+ is given in the sixth column and the uncertainty of the position of the orbit on the generalized Poincaré map is given in the last column. For all the orbits by iterating the interval Newton method we have obtained the uncertainty smaller than 10^{-7}.

For every of the orbit shown in Fig. 1 there exist an orbit symmetric to it with respect to the origin. This is true also for orbits (j) and (k) which seem to be symmetric. Close to the orbit (j) we have found orbits of approximately two (n,o), three (r), four (t) and five (v) times longer period.

We stress that the method is very powerful in a sense that in order to prove the existence of the orbit one has to evaluate the interval Newton operator only at one

orbit (Fig. 1)	period	n	n_+	n_-	position	uncertainty
a	7.38058439^{9}_{7}	2	1	0	$(-0.3331144821^{0}_{2}, -4.2398951154^{4}_{6})$	1.1×10^{-10}
b	14.38443804^{5}_{1}	4	2	0	$(-0.3515008453^{4}_{7}, -4.439012186^{2}_{5})$	4.9×10^{-10}
c	21.330218^{4}_{2}	4	2	0	$(-0.3587931237^{3}_{8}, -4.527224417^{2}_{7})$	1.4×10^{-9}
d	24.70392^{9}_{7}	4	2	0	$(-0.36693626^{2}_{4}, -4.6261202^{41}_{5})$	1.9×10^{-8}
e	28.68369^{8}_{6}	4	2	0	$(-0.3633332^{6}_{8}, -4.5823077^{6}_{8})$	6.1×10^{-9}
f	29.52941^{4}_{2}	4	2	0	$(-0.36503367^{1}_{3}, -4.60297200^{0}_{5})$	5.1×10^{-9}
g	21.6768157^{2}_{0}	6	3	0	$(-0.32688953^{78}_{8}, -4.14219691^{85}_{9})$	6.3×10^{-10}
h	28.6270866^{6}_{2}	6	3	0	$(-0.1853416631^{6}_{8}, -2.44125998^{0}_{3})$	2.0×10^{-9}
i	31.062129^{9}_{4}	6	3	0	$(-0.31976123^{1}_{3}, -4.0564093^{0}_{2})$	7.3×10^{-9}
j	32.99894^{7}_{5}	8	2	2	$(-0.3729433^{08}_{1}, -4.6994901^{0}_{3})$	2.2×10^{-8}
k	44.1795015^{6}_{1}	8	2	2	$(-0.380808052^{1}_{3}, -4.79680872^{72}_{8})$	4.2×10^{-9}
l	59.064059^{2}_{0}	12	2	4	$(-0.377142740^{2}_{4}, -4.75110728^{3}_{6})$	4.0×10^{-8}
m	63.74541^{9}_{7}	12	4	2	$(-0.365322212^{2}_{6}, -4.6065197^{0}_{2})$	1.7×10^{-8}
n	66.42711^{8}_{5}	16	4	4	$(-0.3748804^{5}_{7}, -4.7232727^{5}_{7})$	4.8×10^{-8}
o	66.58^{2001}_{1997}	16	4	4	$(-0.37217029^{5}_{7}, -4.69002^{199}_{202})$	4.9×10^{-8}
p	74.95806^{4}_{1}	18	5	4	$(-0.37426646^{3}_{5}, -4.727655^{28}_{30})$	3.6×10^{-8}
q	79.12980^{4}_{1}	18	4	5	$(-0.379759199^{2}_{4}, -4.78391023^{7}_{9})$	2.0×10^{-8}
r	100.31057^{9}_{4}	24	6	6	$(-0.37212670^{4}_{6}, -4.689488^{3}_{5})$	5.6×10^{-8}
s	118.258476^{5}_{3}	26	9	5	$(-0.3820839497^{1}_{7}, -4.81724127^{6}_{8})$	4.6×10^{-9}
t	132.1431^{4}_{2}	32	8	8	$(-0.2631955^{46}_{5}, -3.382313^{6}_{8})$	4.2×10^{-8}
u	140.9263^{9}_{7}	34	9	8	$(-0.26419576^{1}_{3}, -3.3954757^{3}_{6})$	6.0×10^{-8}
v	165.4853^{8}_{6}	40	10	10	$(-0.37472862^{01}_{1}, -4.7214127^{7}_{9})$	5.0×10^{-8}
w	187.79237^{8}_{6}	38	7	12	$(-0.382607950^{2}_{4}, -4.82284234^{0}_{2})$	1.6×10^{-8}
x	214.52700^{9}_{4}	40	10	10	$(-0.371070036^{1}_{6}, -4.6766596^{1}_{3})$	8.0×10^{-8}

Table 1. Periodic orbits for the Chua circuit, compare Fig. 1, n is the number of intersections with the planes $V_{\pm}$

interval. We would like to point out that this method (unlike the standard version of the interval Newton method) is not limited to short periodic orbits. The longest periodic orbit shown in Fig. 1 has period 29 times longer than the shortest one. Using standard version of interval Newton method we were able to prove the existence of the shortest orbit only (Fig. 1a). By means of the method of intermediate sections for computation of $\mathbf{N}(\mathbf{x})$ we proved the existence of the orbit (b) with the second shortest period. For longer orbits even using the method of intermediate sections we were not able to complete the proof in a reasonable time.

5. CONCLUSIONS

In this paper we have introduced the global interval Newton method for proving the existence of periodic orbits in continuous time systems. We have shown that this method is much more powerful that the non-global version of the interval Newton method. The method is very efficient as in order to perform the proof we need to evaluate the map under investigation only at one interval. This technique may be automated for performing an exhaustive search of periodic orbits in the state space. By iterating the Newton operator one can easily reduce the uncertainty of the position of periodic orbits.

6. REFERENCES

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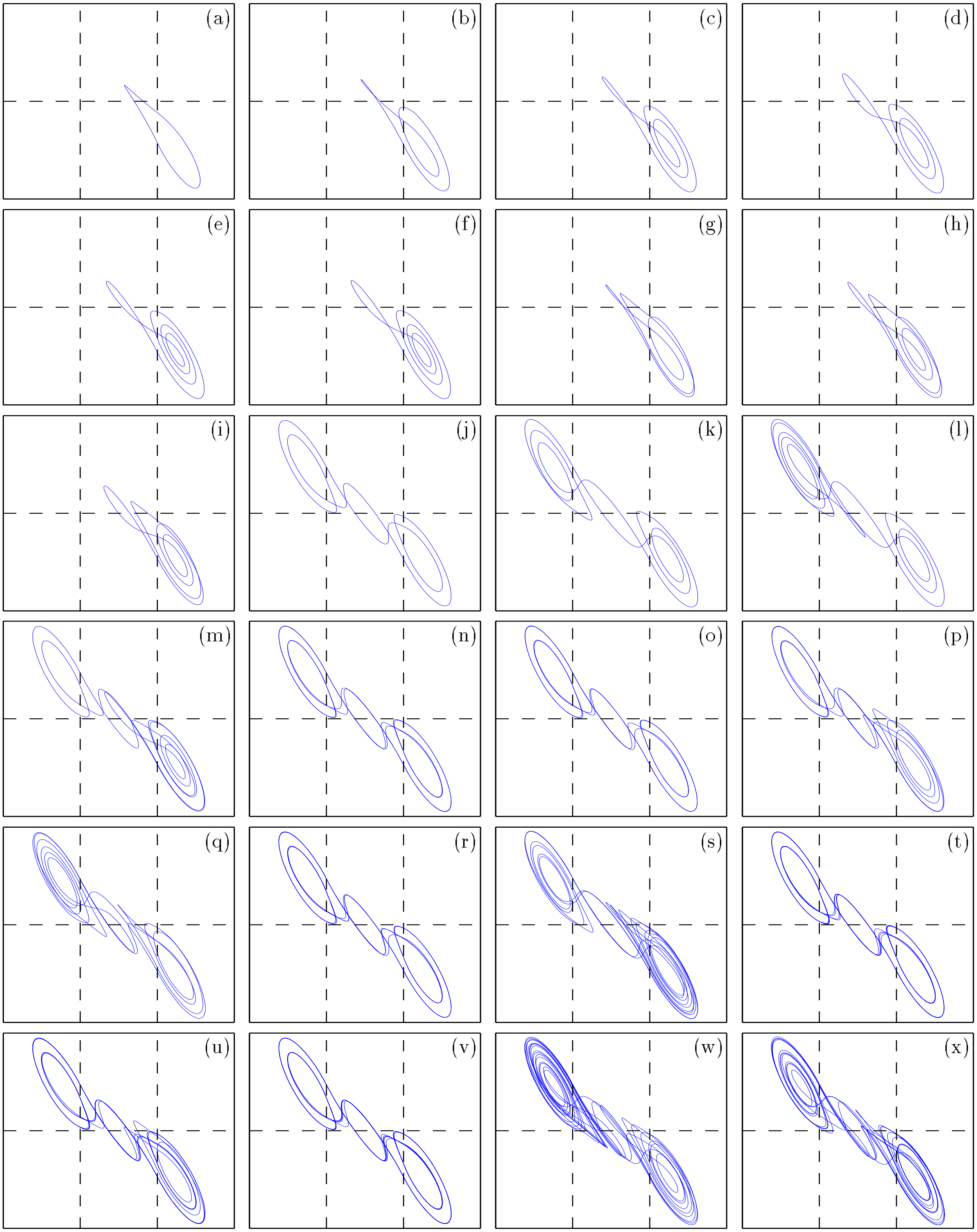

Figure 1. Periodic orbits of the Chua's circuit. For the explanation see text and Table 1.