Cluster formation in arrays of interconnected chaotic circuits*

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ABSTRACT

We present the results of computer simulations of an array composed of locally interconnected chaotic circuits. We investigate the phenomena of synchronization and cluster formation in such networks.

Keywords: coupled nonlinear networks, synchronization, cluster formation,

1 INTRODUCTION

Networks of locally coupled oscillators has become an extensively studied subject in the last decade [3, 4, 1]. They provide a model for a variety of phenomena observed in real systems.

Depending on the connection type and strength of coupling a variety of interesting phenomena can be observed. This includes synchronization behavior, when all cells behave in the same manner and clustering, when some cells in the network are fully synchronized [5, 3, 2].

In this work we study the behavior of a ring of coupled chaotic oscillator. We find examples of full synchronization, clustering and weak synchronization in this network. We discuss conditions under which such phenomena can be observed.

2 DYNAMICS OF THE NETWORK

Let us consider a one–dimensional array composed of simple third–order electronic oscillators (Chua's circuits). The circuits are coupled by means of conductances G_1. Every circuit is connected with its two nearest neighbors. The dynamics of the one–dimensional lattice

*This research has been supported by the European Community research program "COSYC of SENS", no. HPRN-CT-2000-00158.

composed of n circuits can be described by the following set of equations:

$$\begin{aligned} C_2\dot{x}_i =& G(z_i - x_i) - y_i + G_1(x_{i-1} - x_i) + G_1(x_{i+1} - x_i), \\ L\dot{y}_i =& x_i, \\ C_1\dot{z}_i =& G(x_i - z_i) - f(z_i), \end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, n$ and where x_i and z_i denotes the voltages across the capacitances C_2 and C_1 respectively, and y_i is the current through the inductance L in the ith circuit. f is a five–segment piecewise linear function:

$$f(z) = m_2 z + \frac{m_1 - m_2}{2}(|z+B_{p_2}| - |z-B_{p_2}|) + \frac{m_0 - m_1}{2}(|z+B_{p_1}| - |z-B_{p_1}|). \tag{2}$$

The lattice forms a ring, i.e. $x_{n+1} = x_1$, $z_{n+1} = z_1$, $x_0 = x_n$, $z_0 = z_n$. In our study we use typical parameter values for which an isolated circuit generates chaotic oscillations — the "double scroll" attractor ($C_1 = 1/9$, $C_2 = 1$, $L = 1/7$, $G = 0.7$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$). In this work we consider the network composed of $n = 15$ cells.

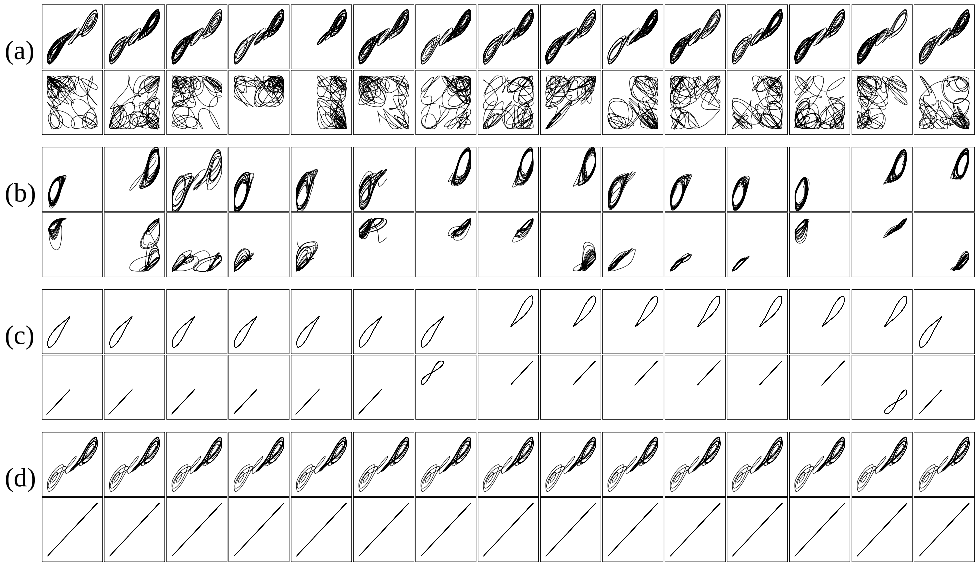

Figure 1: "Steady–state" behavior for (a) $G_1 = 0$, (b) $G_1 = 10$, (c) $G_1 = 50$, (d) $G_1 = 100$

3 Stability of the synchronous state

Observe that for the connection type considered there exists a synchronized chaotic solution. If we apply identical initial conditions to every oscillator in the ring then all the circuits oscillate synchronously. In the first experiment we study the stability of synchronous motion. We start the network from a point close to the synchronization space and observe its behavior for different values of coupling strength. We consider four cases, namely $G_1 = 0, 10, 50$ and 100. The long term behavior (a trajectory for $t \in [1300, 1400]$) is shown in Fig. 1. For each simulation we show two lines of plots. The upper line shows individual trajectories of circuits, i.e., projection of the trajectory of the system onto the plane z_i, y_i. In the second line in the ith plot we show the z_i variable versus the variable z_{i+1} from the next cell. By inspecting this plot we can easily check whether neighboring cells are synchronized. In case of perfect synchronization between cells i and $i+1$ the plot is on the diagonal line.

For $G_1 = 0$ the circuits are not coupled and hence they are oscillating independently. Each circuit forms the double–scroll attractor but their trajectories are uncorrelated. For $G_1 = 10$ there is some correlation between the circuit trajectories. Every cell oscillates chaotically and forms the double–scroll attractor, but the switchings between the scrolls are less frequent than for the uncoupled case. Observe that due to short observation time trajectories of most of the cells belong to one of the scrolls only.

For $G_1 = 50$ the steady state of the system is a periodic trajectory. The network is divided into two clusters of cells with trajectories in the upper or lower part of the state space. In each cluster the cells are fully synchronized (diagonal lines in the second row of plots). There is a phase offset between the clusters corresponding to the "eight"-type trajectory. For $G_1 = 100$ the trajectory does not escape from the synchronization subspace. In each cell the double–scroll attractor is formed, and all cells oscillate in a full synchrony.

From this experiment one can conclude that for $G_1 = 100$ the synchronous chaotic state is stable while for other values of coupling strength it is not. The steady state for $G_1 = 100$ is the synchronous behavior. For smaller values of G_1 the system displays various steady states including two clusters of cells oscillating synchronously ($G_1 = 50$), and non-synchronized behavior for $G_1 = 10$.

4 CLUSTER FORMATION

Above we have seen an example of existence of clusters in the network, where cells oscillate synchronously, although the network as a whole is not synchronized. In the second part of the paper we study properties of the system in this state and the process of cluster formation.

We have run a number of simulations, where the system was started from a perturbed synchronized state, i.e. the synchronous state was modified by adding a small random number to each system variable. Initially the cells were strongly connected ($G_1 = 100$). In each simulation we apply a series of changes to the coupling strength (coupling changes in time).

In the first experiment we have changed the coupling conductance to $G_1 = 50$ at $t = 30$. It follows from the previous simulations that for $G_1 = 50$ the synchronous state is not stable. The system trajectory leaves the synchronization subspace somewhere around $t = 100$. Initially only two cells switch the scroll, and a two cluster structure with 2 and 13 cells is formed. This structure is unstable and after a short time some cells lying on the border of the larger cluster switch the scroll and leave the cluster. In Fig. 2(b) one can see two clusters with sizes 5 and 10. After some more time the larger cluster decreases to have 8 cells and this state is stable in the sense that in quite a long integration time no cell changes the scroll and

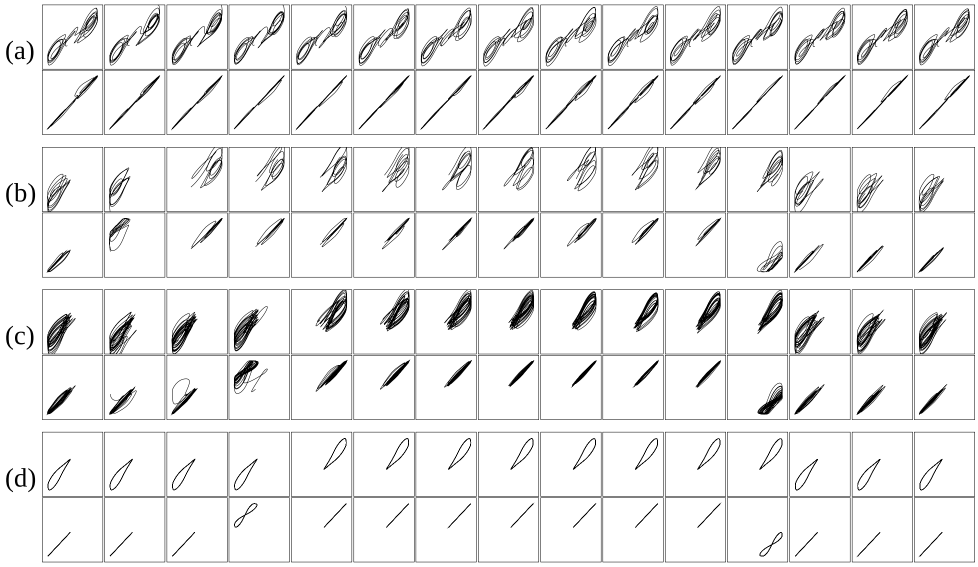

Figure 2: Coupling strength: $G_1 = 100$ for $t \in [0, 30]$ — full synchronization, $G_1 = 50$ for $t \in [30, 650]$ — process of cluster formation (clusters of large size are not stable), $G_1 = 100$ for $t \in [650, 800]$ — cluster structure is sustained, trajectory becomes periodic, (a) $t \in [130, 170]$, (b) $t \in [180, 200]$, (c) $t \in [200, 250]$, (d) $t \in [750, 850]$

the cluster structure persists (compare Fig. 2(c)). In this steady state each circuit oscillates chaotically and the synchronization within a cluster is not full (phase synchronization). After the steady state is achieved the connection strengths were increased to the initial value of $G_1 = 100$. See the system does not return to the synchronization subspace. Instead, the cluster structure is sustained. The trajectory in the steady state is periodic (see Fig. 2(d)).

In Fig. 3 one can see the results of similar simulations. These time the cluster structure is quite different. The number of clusters is much larger. Clusters have sizes 2,3 and 4 and they are separated by single cells operating in a different region of the state space. For $G_1 = 50$ in the steady state circuits display quasi-periodic trajectory. After increasing the connection strength to $G_1 = 100$ the cluster structure is unaltered. The steady state however changes from quasiperiodic one to the period-2. The cells in the clusters become fully synchronized.

In Fig. 4 we show the results of simulation when we did not wait until steady state develops for the smaller connection strength. Initially $G_1 = 100$, at $t \in [50, 70]$ connection strengths were decreased to $G_1 = 50$ and at $t = 70$ it was changed to the initial value $G_1 = 100$. In consequence for $G_1 = 50$ the steady state was not obtained and the large cluster with 12 cells survived. From the observation of the steady state for $G_1 = 100$ it follows that such a cluster is stable for this connection strength. In the steady state all cells within the cluster are fully synchronized, but on contrary to the previous cases the steady state is chaotic. There is no generalized synchronization between the clusters, i.e. there is no one to one relation between the states (see the z_i versus z_{i+1} plot on the border of the cluster).

(a)

(b)

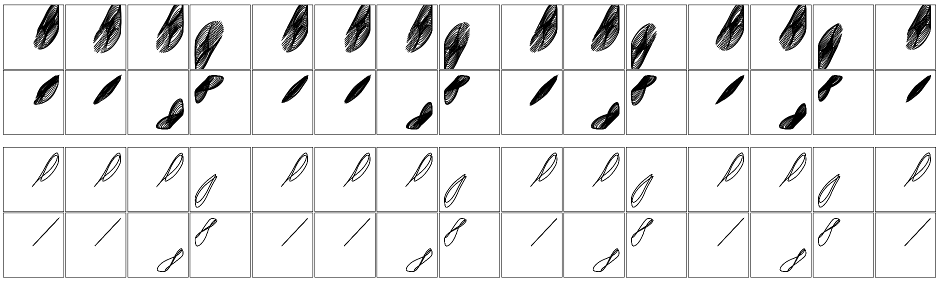

Figure 3: Coupling strength, $G_1 = 100$ for $t \in [0, 200]$ — synchronous state, $G_1 = 50$ for $t \in [200, 700]$ — synchronization is lost, steady state with many clusters of cells oscillating in a quasiperiodic way develops, $G_1 = 100$ for $t \in [700, 1300]$ — cluster structure is sustained, period–2 orbit, (a) $t \in [600, 700]$, (b) $t \in [1200, 1300]$

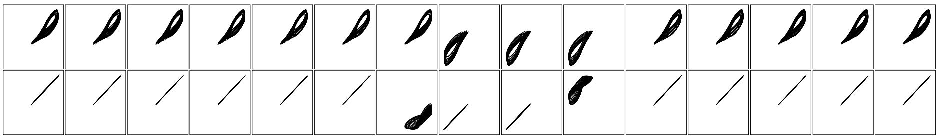

Figure 4: Coupling strength, $G_1 = 100$ for $t \in [0, 50]$ — synchronous state, $G_1 = 50$ for $t \in [50, 70]$ — the trajectory leaves the synchronization subspace, the steady state is not achieved, $G_1 = 100$ for $t \in [70, 1000]$ — steady state with two fully synchronized clusters oscillating chaotically, trajectory for $t \in [1200, 1300]$ shown

Finally we consider the case when large connection strengths are applied to freely running uncoupled cells. Initially the cells oscillate independently and in consequence the cluster structure (i.e. the position of the trajectory with respect to scrolls) is random. At $t = 50$ we apply strong connection of $G_1 = 100$. Such a strong connection preserves the cluster structure. In the steady state we observe two clusters of size 3 and 8 separated by single cells. Cluster are fully synchronized and are oscillating chaotically (see Fig. 5(b)). In the steady state at $t = 700$ the connection strength is decreased to $G_1 = 50$. The cluster structure remains unmodified but the cells within the cluster are not fully synchronized any more (see Fig. 5(c)). After increasing the connection strength to $G_1 = 100$ the system achieves the same steady state as before decreasing G_1.

5 CONCLUSIONS

We have performed a series of simulations of a ring of locally connected chaotic oscillators. From these experiments one can draw several conclusions about the cluster formation and stability of particular cluster structures. Full synchronization is only possible for large coupling ($G_1 = 100$ in our experiments). For smaller values of coupling strength (for example $G_1 = 50$) clusters are formed. Large clusters however are not stable and they loose border

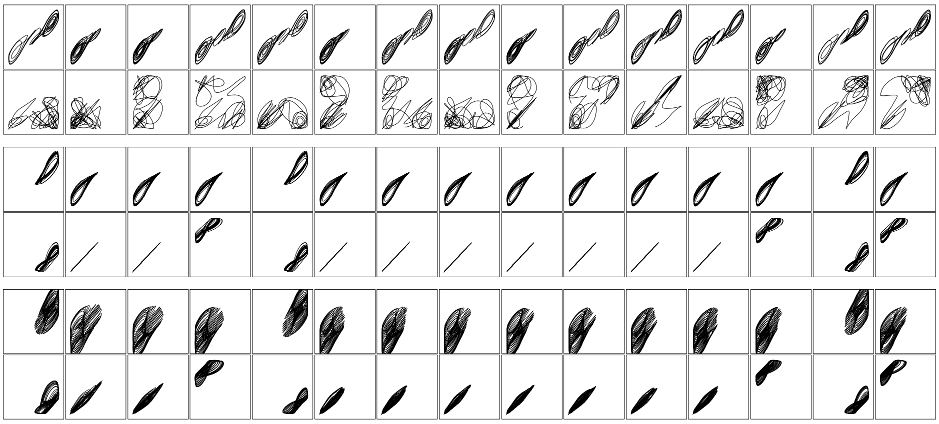

Figure 5: Coupling strength: $G_1 = 0$ for $t \in [0, 50]$ — independent oscillations, $G_1 = 100$ for $t \in [50, 700]$ — steady state with two fully synchronized clusters oscillating chaotically, $G_1 = 50$ for $t \in [700, 1100]$, $G_1 = 100$ for $t \in [1100, 1500]$, (a) $t \in [0, 50]$, (b) $t \in [650, 700]$, (c) $t \in [1000, 1100]$

cells until maximum stable size is achieved. In this case cells within a cluster may or may be not fully synchronized. Trajectories of individual cells may form periodic orbits (Fig. 2(c)), quasiperiodic orbits (Fig. 3(a)) or chaotic orbits (Fig. 5(c)). Increasing the coupling strength back to the original value does not cause the system to return to the state of full synchronization between all cells. Instead the cluster structure is sustained, and the cells within clusters become fully synchronized if this was not the case before. Individual cells may oscillate periodically (period–1, see Fig. 2(d), period–2, see Fig. 3(b)) or chaotically (see Fig. 5(d)). The maximum size of the stable cluster increases with the connection strength.

REFERENCES

[1] Z. Galias and M. Ogorzałek. Study of synchronized motions in a one–dimensional array of coupled chaotic circuits. *Int. J. Bifurcation and Chaos*, 9(11):2219–2224, 1999.

[2] H. Haken. Synergetics; from pattern formation to pattern analysis and pattern recognition. *Int. J. Bifurcation and Chaos*, 4:1069–1083, 1994.

[3] K. Kaneko. Clustering, coding, switching, hierarchical ordering and control in a network of chaotic elements. *Physica D*, 41:137–172, 1990.

[4] A. Perez-Muñuzuri, V. Perez-Muñuzuri, M. Gomez-Gesteira, L. Chua, and V. Perez-Villar. Spatio–temporal structures in discretely–coupled arrays of nonlinear circuits: A review. *Int. J. Bifurcation and Chaos*, 5:17–50, 1995.

[5] K. Pyragas. Weak and strong synchronization of chaos. *Phys. Rev. E*, 54:4508–4511, 1996.