INFLUENCE OF SYSTEM NON-UNIFORMITY ON DYNAMIC PHENOMENA IN ARRAYS OF COUPLED NONLINEAR NETWORKS

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In this paper we investigate the influence of system non-uniformity on the existence and stability of synchronous motion in an array of bi–directionally coupled electronic circuits. In computer simulations we find the level of non-uniformity for which synchronous behavior is sustained. We also present several examples of attractors, which appear when the synchronous motions is no longer stable.

1   Introduction

Dynamics of generalized cellular neural networks (CNN) with higher-order cells is one of still vividly studied areas of research [1,2,3,4]. Such networks provide a versatile model for a variety of phenomena observed in real systems in such areas as physics, biology or medicine [5].

Depending on dynamics of individual cells in the network and the type and strength of coupling between them a variety of interesting behaviors can be observed, including hyper–switching and clustering [1], attractor crowding and various kinds of spatial, temporal or spatio–temporal ordered structures referred to as self–organization [6].

Among various types of dynamical behaviors occurring in coupled systems is the synchronization behavior when some or all cells behave in the same manner (the notions of weak and strong synchronization are used to distinguish these two cases [7,8]).

Stability of the synchronous motion becomes a very important problem. In this paper we investigate how synchronization phenomena depend on disturbance of coupling parameters (non-uniformity of connections).

2   Dynamics of the Network

We consider a one–dimensional array composed of simple third–order electronic oscillators (Chua's circuits).

A single circuit is described by the following set of ordinary differential equations:

$$\begin{aligned}
C_2\dot{x} &= -y + G(z - x), \\
L\dot{y} &= x, \\
C_1\dot{z} &= G(x - z) - f(z),
\end{aligned} \tag{1}$$

where x and z denotes the voltages across the capacitances C_2 and C_1 respectively, and y is the current through the inductance L. f is a five–segment piecewise linear
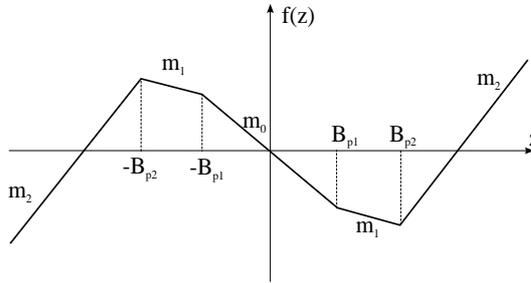
Figure 1. A five–segment piecewise linear function.

function (Fig. 1):

$$f(z) = m_2 z + \frac{1}{2}(m_1 - m_2)(|z + B_{p_2}| - |z - B_{p_2}|)$$

$$+ \frac{1}{2}(m_0 - m_1)(|z + B_{p_1}| - |z - B_{p_1}|). \quad (2)$$
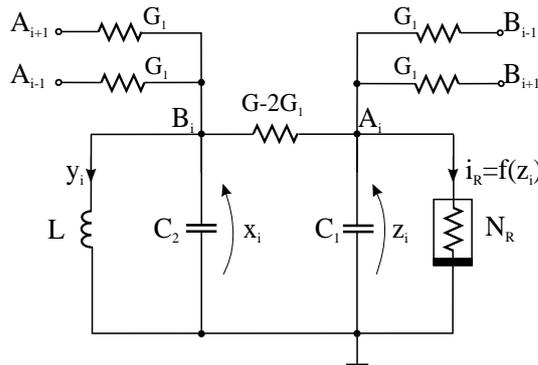
Figure 2. Nonlinear oscillator and its connection to the neighbors.

The circuits are coupled bi-directionally (see Fig. 2) by means of two conductances G_1 cross-connected between the capacitors C_1 and C_2 of the neighboring circuits. Every circuit is connected with its two nearest neighbors. In our simulations we use balanced chaotic circuits, where the value of the resistor connecting capacitors C_1 and C_2 in a single circuit is decreased by $2G_1$. This ensures the existence of a synchronized chaotic solution. If we apply identical initial conditions to every oscillator in the ring $(x_i(0) = x(0), y_i(0) = y(0), z_i(0) = z(0)$ for $i = 1, \ldots, n)$ then all the circuits oscillate synchronously and the equations describing the array have the form (1) with $x_i = x$, $y_i = y$ and $z_i = z$ for $i = 1, \ldots, n$. In the case of equal initial conditions every cell oscillates in the same way as a single uncoupled chaotic cell. The dynamics of the one–dimensional lattice composed of n circuits

can be described by the following set of equations:

$$C_2 \dot{x}_i = -y_i + (G - 2G_1)(z_i - x_i) + G_1(z_{i-1} - x_i) + G_1(z_{i+1} - x_i),$$
$$L \dot{y}_i = x_i, \tag{3}$$
$$C_1 \dot{z}_i = (G - 2G_1)(x_i - z_i) - f(z_i) + G_1(x_{i-1} - z_i) + G_1(x_{i+1} - z_i),$$

where $i = 1, 2, \ldots, n$ and the lattice forms a ring ($x_{n+1} = x_1$, $z_{n+1} = z_1$, $x_0 = x_n$, $z_0 = z_n$).

In our study we use typical parameter values for which an isolated circuit generates chaotic oscillations — the "double scroll" attractor ($C_1 = 1/9F$, $C_2 = 1F$, $L = 1/7H$, $G = 0.7S$, $m_0 = -0.8$, $m_1 = -0.5$, $m_2 = 0.8$, $B_{p_1} = 1$, $B_{p_2} = 2$).

3 Stability of the Synchronous Motion

In our previous work [9] we have found the theoretical conditions for the stability of the synchronous motion using the concept of Lyapunov exponents and the master stability function [10].

We have found the range of G_1, for which synchronization is possible:

$$G_1 \in \left(\frac{-\beta}{2}, \frac{-\alpha}{2} \right), \qquad\qquad \text{for } n = 2,$$

$$G_1 \in \left(\frac{-\beta}{4 \sin^2 \frac{\pi}{n}}, \frac{-\alpha}{4} \right), \qquad\qquad \text{for even } n \geq 4, \tag{4}$$

$$G_1 \in \left(\frac{-\beta}{4 \sin^2 \frac{\pi}{n}}, \frac{-\alpha}{4 \sin^2 \frac{(n-1)\pi}{2n}} \right), \qquad\qquad \text{for odd } n \geq 3,$$

where $\alpha \approx -1.171$ and $\beta \approx -0.229$.

For a given network size n there exists the values of G_1 for which the synchronous state is stable if

$$n < \frac{\pi}{\arcsin \sqrt{\beta/\alpha}} \approx 6.86, \qquad\qquad \text{for even } n \geq 4, \tag{5}$$

$$n < \frac{\pi}{2 \arcsin 0.5 \sqrt{\beta/\alpha}} \approx 7.05, \qquad\qquad \text{for odd } n \geq 3. \tag{6}$$

It follows that for $n = 2, 3, \ldots, 7$ the synchronized state can be stable. In Table 1 we collect the values of the coupling strength G_1 for which the stability condition holds. Theses results are also plotted in Fig. 3.

In our previous work we have also shown that the theoretical predictions presented above agree very well with the results of computer experiments. We have studied the case of a uniform network, where the circuit parameters and the connections strengths were identical. This is not a very realistic assumption, when we try to investigate the real systems. In real systems we never have two identical systems, nor the connections are equal. In this paper we address the problem of the influence of non-uniformity on the system behavior, and in particular on the stability of the synchronous motion.

n	G_1
2	$(0.1145, 0.5855)$
3	$(0.0763, 0.3903)$
4	$(0.1145, 0.2928)$
5	$(0.1657, 0.3237)$
6	$(0.2290, 0.2928)$
7	$(0.3041, 0.3080)$

Table 1. Coupling strength G_1 for which the synchronized state is stable.
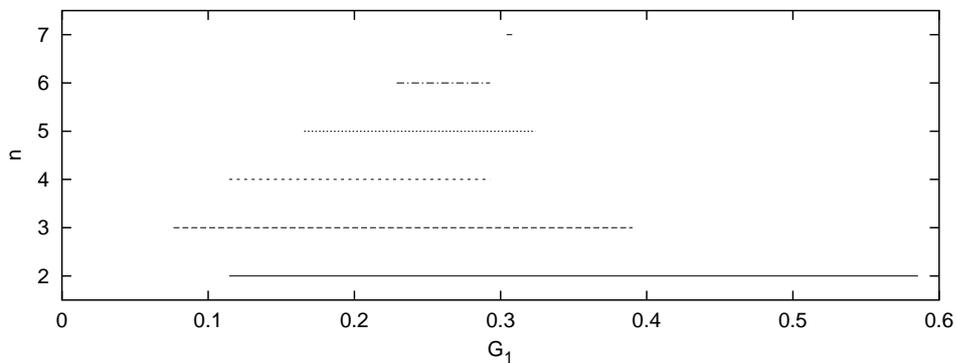
Figure 3. Ranges for the coupling strength ensuring stability of the synchronous motion calculated using the master stability function method.

4   Nonuniform connections

In our simulations we consider the case when the couplings are not equal. We assume each of these conductances being a random variable from the interval $G_1[1-e, 1+e]$ with a uniform distribution. When the connections are not uniform the synchronization manifold is no longer invariant. However if the deviation from the nominal value of G_1 is small, then there should exist a stable invariant manifold, close to the synchronization subspace ($x_i = x_j$, $y_i = y_j$, $z_i = z_j$).

In order to test the stability of a particular solution one can perturb this solution by a random additive signal with a small amplitude and observe the steady–state behavior of the system. If the system converges to the solution under consideration one claims that the solution is stable. We follow this approach and integrate the system for $t \in [1, T]$, where $T = 1000$, starting from initial conditions close to the synchronization subspace. We monitor the trajectory and check whether the solution is always close to the synchronization subspace ($|x_i - x_j| < \varepsilon$, $|y_i - y_j| < \varepsilon$, $|z_i - z_j| < \varepsilon$). If this condition is true we say that the parameter deviation does not destroy the synchronous behavior. In order to get better statistics we repeat this computations several times. For each value of the maximum parameter deviation e we generate several ensembles of the network (in each case the values of coupling

conductances are generated according to he uniform distribution with the mean value G_1 and the maximum deviation eG_1) and we integrate the system starting from different initial conditions close to the synchronization subspace.
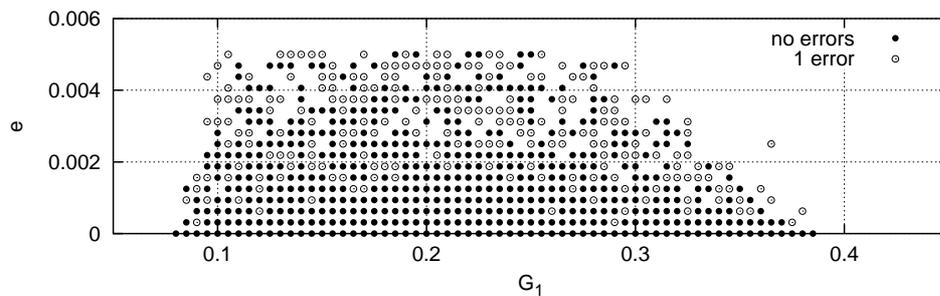
Figure 4. Synchronization in a non–uniformly coupled network composed of $n = 3$ cells

In Fig. 4 we plot the results of these simulations for the network size $n = 3$. A filled circle at the position (G_1, e) means that for all chosen examples of the system with coupling strengths from the interval $[G_1(1 - e), G_1(1 + e)]$, the solution of the system did not leave close neighborhood of the synchronization subspace. An empty circle means that this was true in all but one case. Other points are not plotted. We have performed these computations for $G_1 \in [0.05, 0.40]$ and $e \in [0, 0.005]$. In case of no deviation ($e = 0$) the results agree very well with the theoretical predictions based on the master stability function method (compare Fig. 3).

It is interesting to note that a relatively small level of non-uniformity in coupling strength makes the synchronous state unstable. If $e > 0.5\%$ then in most cases the trajectory of the system starting close to the synchronization subspace leaves its neighborhood — the synchronized behavior is unstable or does not exist.
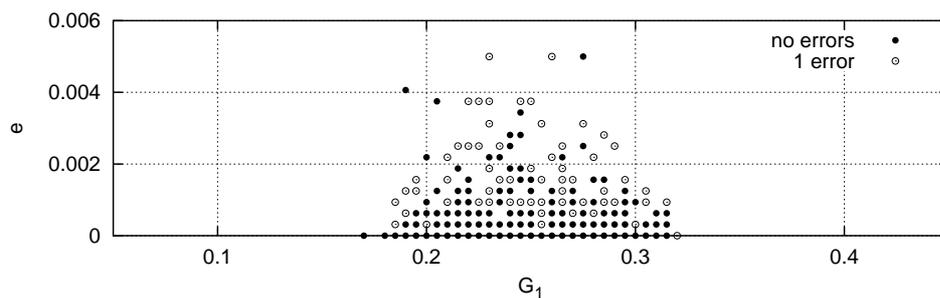
Figure 5. Synchronization in a non–uniformly coupled network composed of $n = 5$ cells

Similar results for $n = 5$ are shown in Fig. 5. One can see that the range of e for which the synchronous behavior is not destroyed is much smaller than for $n = 3$.

A interesting question which arises is what is the limit set of the trajectory, which leaves the close neighborhood the synchronization subspace. Does the trajectory of each sub-circuit form the double–scroll attractor, or maybe the influence of neighboring circuits is so important that the trajectories of the individual circuits change qualitatively.

To answer these questions we have run a series of simulations observing the steady-state behavior of the system. Some examples are shown below. All of them were obtained in the system composed of $n = 5$ circuits for the nominal value of the coupling coefficient $G_1 = 0.2$.
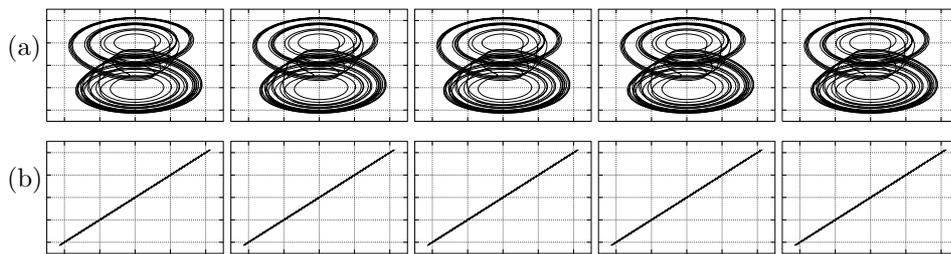
Figure 6. Uniform coupling $G_1 = 0.2$, complete synchronization, (a) y_i versus x_i for $i = 1, \ldots, 5$, x_i range is $[-0.5, 0.5]$, y_i range is $[-2.5, 2.5]$ (b) y_{i+1} versus y_i,

In the Figures 6-12 the first line of plots shows trajectories of individual circuits (x_i versus y_i). In the second line we plot the second state variables from neighboring cells ($y_{i \bmod n+1}$ versus y_i). This kind of presentation allows us to say whether the neighboring oscillators are synchronized.

In Fig. 6 the behavior of the network with uniform coupling is shown. The circuits are perfectly synchronized.
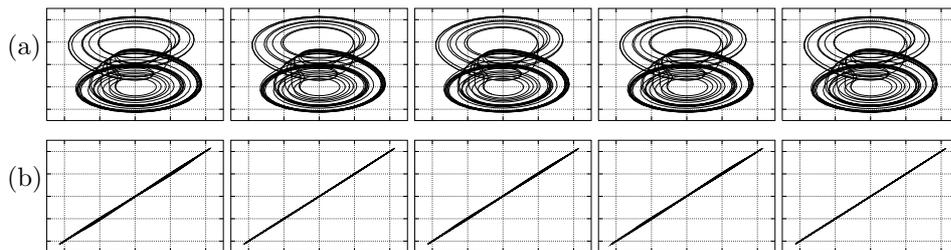
Figure 7. Non-uniform coupling, $G_1 \in 0.2[1-e, 1+e]$, $e = 0.005$, the oscillators are synchronized, trajectories of individual circuits form the double-scroll attractor

In Fig. 7-10 the coupling values are modified, with the maximum deviation of 0.5%. In the first case we observe synchronization between the circuits. It is not a perfect synchronization since due to the differences between coupling strengths

the system is not completely symmetrical and the synchronization subspace is not invariant. Nevertheless, from a practical point of view the systems are synchronized, since the synchronization error remains small all the time. In the three other examples the circuits are not synchronized. Trajectories of individual cells are considerably altered, when compared to the double-scroll attractor. One observes a thin or wide version of the Roessler-type attractor, or some periodic type behavior. It is interesting to note that in each case there exist a cluster of two neighbors oscillating in full synchrony. This is evident from the second line of plots, where the state variable y_{i+1} is plotted versus y_i.
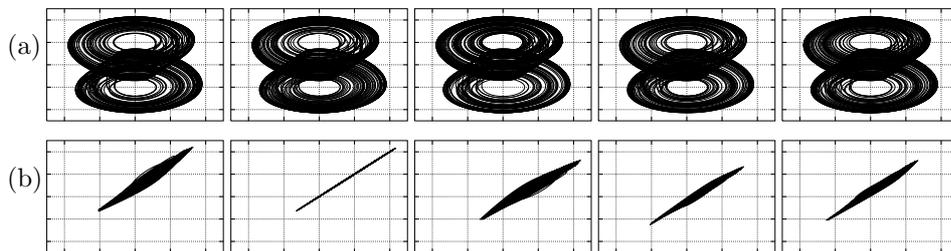
Figure 8. Non-uniform coupling, $G_1 \in 0.2[1-e, 1+e]$, $e = 0.005$, the oscillators are not synchronized, trajectories of individual circuits form the Roessler-type attractor
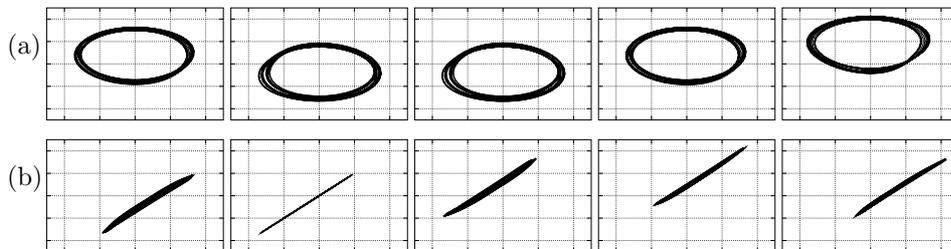
Figure 9. Non-uniform coupling, $G_1 \in 0.2[1-e, 1+e]$, $e = 0.005$, the oscillators are not synchronized

In the second set of experiments we have allowed larger changes in the coupling conductances of 10% of the nominal value. The results are shown in Fig. 11-13. We observe existence of a very simple periodic attrractor (see Fig. 11), a very thin chaotic attractor located close to the period three orbit (Fig. 12) and a fully developed chaotic attractor. In the first two cases the sub-circuits are phase–synchronized, while in the last case they are not.
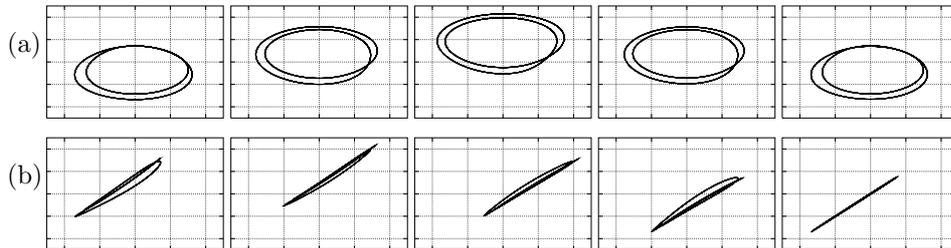
Figure 10. Non-uniform coupling, $G_1 \in 0.2[1 - e, 1 + e]$, $e = 0.005$, the oscillators are not synchronized, the system as a whole oscillates periodically
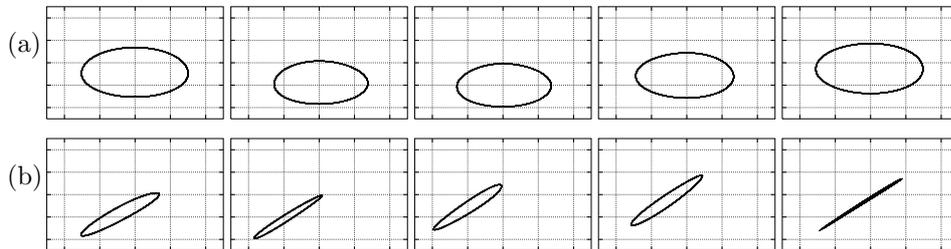
Figure 11. Non-uniform coupling, $G_1 \in 0.2[1 - e, 1 + e]$, $e = 0.1$, periodic orbit
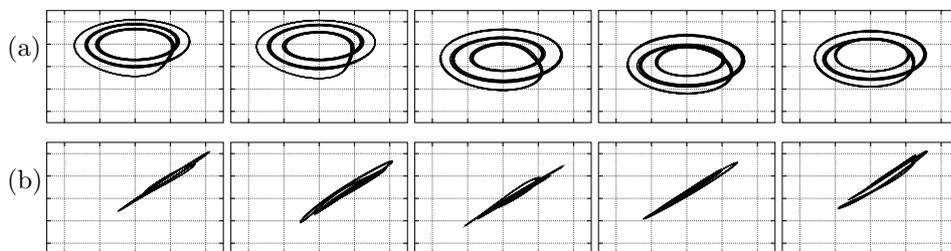
Figure 12. Non-uniform coupling, $G_1 \in 0.2[1 - e, 1 + e]$, $e = 0.1$, thin chaotic attractor

5 Conclusions

In this paper we have investigated the influence of parameter deviation on the stability of synchronous solution of a one–dimensional array of bi–directionally coupled chaotic circuits. We have found an abundance of attractors coexisting with the synchronous motion. We have shown that if the coupling strength is not uniform, then very often the trajectory of the system as a whole is attracted to one of the non-synchronous behaviors. This phenomenon occurs in spite of the fact that all coupling strengths belong to the region where for the uniform system the
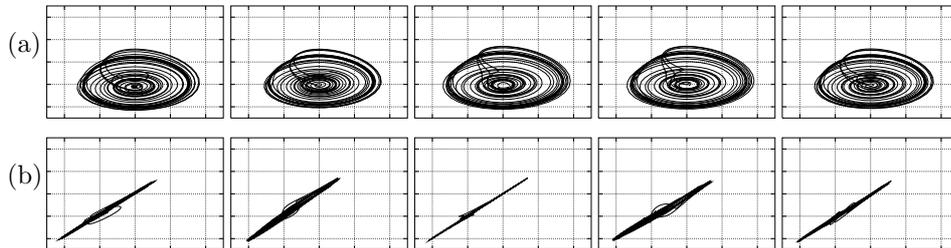
Figure 13. Non-uniform coupling, $G_1 \in 0.2[1-e, 1+e]$, $e = 0.1$, unsynchronized chaotic behavior

synchronous state is stable.

Acknowledgments

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