Modern mathematical tools for analysing nonlinear effects in digital filters

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ABSTRACT — **A number of mathematical tools are used for analysing nonlinear overflow effects in digital filters. In our investigations we found if useful to use the probabilistic approach and the circle-map theory in the analysis of complex behaviour in second-order digital filter. In the first part we introduce some definition (rotation number, nonwandering and chain recurrent points, mixing, exactness) and present several theorems. In the second part we describe how to apply the theories described to the dynamical systems defined by digital filters.**

1 Mathematical tools

1.1 Circle-map theory

Let us recall some important definitions and theorems from the circle-map theory.

Let $f : S^1 \longmapsto S^1$ be a continuous map of S^1 to itself. The map $\Pi : I\!R \longmapsto S^1$ defined by $\Pi(t) = (cos(2\pi t), sin(2\pi t))$ is continuous and onto. Let $F : I\!R \longmapsto: I\!R$ be a *lift* of f, i.e. F is continuous, $\Pi \circ F = f \circ \Pi$, and for each $x \in I\!R$, $F(x+1) = F(x) + k$, where k is an integer constant. The integer k is unique for a given continuous map f and is called *the degree of* f. In this paper we are interested in degree-one maps only. An important concept in the study of degree-one maps is the rotation number. Let f be a degree-one map, and let F be a lift of f. If $x \in I\!R$, then the *rotation number* of x under F is defined by:

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}. \quad (1)$$

We say that a map of a circle is *non-decreasing* if its lift is non-decreasing.

Proposition 1 *([6]) If f is a non-decreasing degree-one map, F is a lift of f, then $\rho_F(x)$ exists for every $x \in I\!R$ and does not depend on initial point x. $\rho_F(x)$ is rational iff f has a periodic point.*

Thus for non-decreasing degree-one maps we can define the rotation number in the following way:

$$\rho_f = \rho_F(x) \ (mod\ 1),$$

where x is an arbitrary real value and F is an arbitrary lift of f.

We say that a point x is *nonwandering* if $\forall U$-neighbourhood of x $\exists n > 0 : f^n(U) \cap U \neq \emptyset$. We say that a point x is *chain recurrent* if $\forall \varepsilon > 0\ \exists n > 0\ \exists x_0, \ldots, x_n : d(f(x_i), x_{i+1}) < \varepsilon$ and $x = x_0 = x_n$. Using the concept of nonwandering and chain-recurrent points we were able to prove the following two lemmas.

Lemma 1 *If f is a non-decreasing degree-one map and f is not injective, then the rotation number of f is rational.*

Lemma 2 *If f is a non-decreasing degree-one circle map and the rotation number of f is rational then for every $x \in S^1$ the limit set $\omega(x)$ is periodic.*

1.2 Measure-theoretic theory

In this section we shall consider the probabilistic (measure-theoretic) approach in the analysis of chaotic system.
An important concept in studying measure preserving dynamical systems is mixing property. Mixing means that a set of initial conditions of nonzero measure will eventually spread over the whole phase space as the system evolves.

Definition 1 *Let (M, Ω, μ) be a normalized measure space, and $G : M \to M$ a measure-preserving transformation ($\mu(G^{-1}(A)) = \mu(A)$ for all $A \in \Omega$). G is called mixing if*

$$\lim_{n\to\infty} \mu(A \cap G^{-n}(B)) = \mu(A)\mu(B) \quad \forall A, B \in \Omega$$

Another concept is exactness. For exact systems a set of initial conditions of nonzero measure will eventually fill the whole phase space.

Definition 2 *Let (M,Ω,μ) be a normalized measure space, and $G : M \to M$ a measure-preserving transformation such that for all $A \in \Omega$, $G(A) \in \Omega$. G is called exact if*

$$\lim_{n\to\infty} \mu(G^n(A)) = 1 \quad \text{for every } A \in \Omega\ ,\mu(A) > 0$$

2 Results

Mathematical results presented in the previous section can be applied to analyse the dynamic behaviour of a digital filter due to the overflow nonlinearity. So far studies concentrated on stable behaviour – normal operation of the filter. New mathematical tools enable analysis for a wide range of filter parameters, allowing prediction of dynamic behaviour and possibly new unconventional applications (eq. filter-based noise generators). The simplest configuration for realizing the second-order filter is the direct form realisation, shown in Fig. 1.

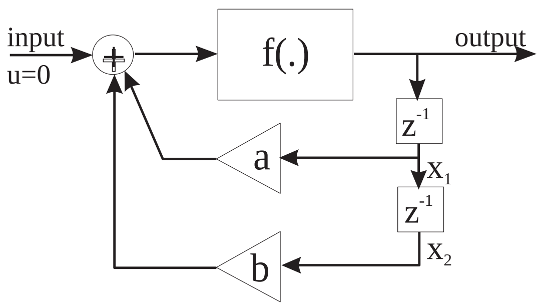


Figure 1: The direct form realization of the second-order digital filter

To simplify analysis, we neglect the quantization error which occurs in the finite wordlength representation. Under zero input conditions, the filter can be modeled by a two-dimensional discrete-time dynamical system with the following state equations [1]:

$$\begin{aligned} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} &= \begin{pmatrix} x_2(k) \\ f[bx_1(k)+ax_2(k)] \end{pmatrix} \\ &= \boldsymbol{F}\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \end{aligned} \tag{2}$$

where $f(x)$ is the overflow rule. The state space is the invariant set $I^2 = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$.

2.1 Saturation nonlinearity

When we use the saturation function for the overflow rule, $f(x)$ has the form:

$$f(x) = \frac{1}{2}(|x+1| - |x-1|) \tag{3}$$

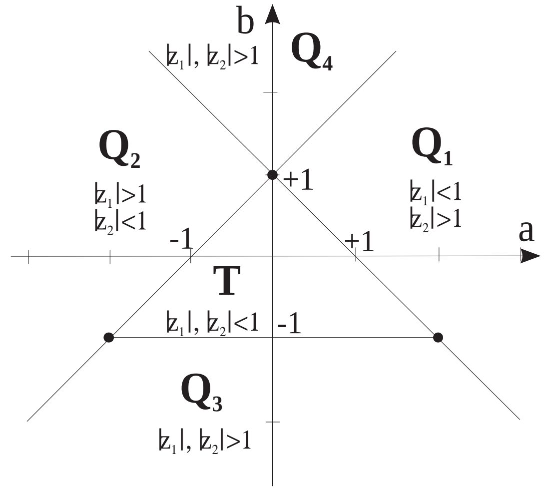


Figure 2: Partition of the parameter plane into sets T, Q_1, Q_2, Q_3, Q_4, for $(a,b) \in Q_1 \cup Q_2$ one eigenvalue lies inside and one outside the unit circle, for $(a,b) \in Q_3 \cup Q_4$ both eigenvalues lie outside the unit circle, for $(a,b) \in T$ both eigenvalues lie inside the unit circle and the linear system is asymptotically stable.

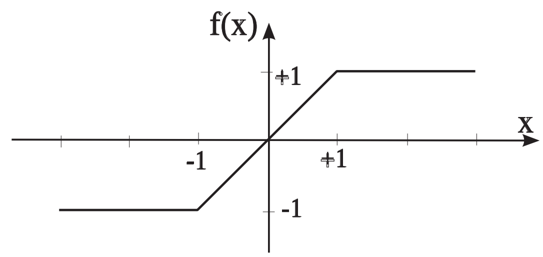


Figure 3: The saturation characterictic

We consider the case $(a,b) \in Q_3 = \{(a,b) : b < -1, b < a+1, b < -a+1\}$. For other cases the behaviour of the filter is well-known, the only possible limit sets are period-1 and period-2 orbits [4, 7]. In [7] it was shown that for $(a,b) \in Q_3$ there exists an absolutely convex polygon with invariant boundary $\mathbf{W}^\infty$. It was proved that every non-trivial trajectory in finite time enters the set $\mathbf{W}^\infty$ and remains in it. Thus we can reduce our study to the analysis of one-dimensional map of $\mathbf{W}^\infty$ into itself. As $\mathbf{W}^\infty$ is homeomorphic to a circle we can define the rotation number of $\Phi := \boldsymbol{F}|\mathbf{W}^\infty : \mathbf{W}^\infty \longmapsto \mathbf{W}^\infty$. The map Φ is weakly monotone which implies (Proposition 1) the existence of a unique rotation number for each pair (a,b), denoted by $\rho_{\boldsymbol{F}}$.

Theorem 1 *If $(a,b) \in Q_3$, $\boldsymbol{x} \neq O$, ρ is the rotation number of Φ, then*

1. *If Φ is not a homeomorphism then ρ is rational.*

Figure 4: The rotation number as a function of parameter a.

2. *If ρ is rational ($\rho = p/q$) then the limit set of $\boldsymbol{x}$ is a period-q orbit contained in $\mathbf{W}^\infty$.*

3. *If ρ is irrational then the limit set of $\boldsymbol{x}$ is dense in $\mathbf{W}^\infty$.*

From the above theorem it follows that no chaotic behaviour is possible when we use the saturation nonlinearity for the overflow rule. In the region Q_3 the behaviour of the filter strongly depends on the circuit parameter values. A very small change of filter parameters can cause the qualitative change of circuit behaviour (change of the period of the periodic orbit or change of the type of the orbit, from periodic to the unperiodic one).

Now we will discuss the structure of the Arnold tongues.

Definition 3 *We say that point (a, b) belongs to the ω-Arnold tongue denoted by A_ω if there exists an orbit with rotation number $\omega \in I\!R$ for $\boldsymbol{F}$ with parameters (a, b):*

$$A_\omega = \{(a, b) : \exists \boldsymbol{x} \neq O : \rho_{\boldsymbol{F}}(\boldsymbol{x}) = \omega\}. \quad (4)$$

From Proposition 1 it follows that $(a, b) \in A_{p/q}$ implies the existence of a periodic orbit with period q and rotation number p/q. It is also clear that Arnold tongues with different ω are disjoint. For any real ω the ω-Arnold tongue is closed and pathwise connected [8].

In Fig. 4 we present the rotation number as a function of parameter a. One can easily see the

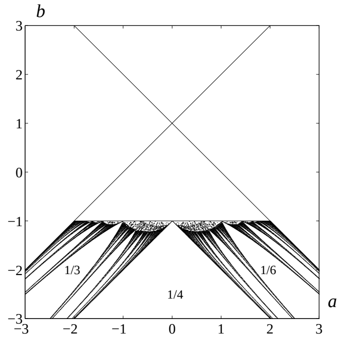


Figure 5: The ranges of parameters (a, b) with a given rotation number, points (a, b) lying inside the half-circular regions correspond to homeomorphic Φ (possible quasi-periodic limit sets), below these regions the map Φ is not homeomorphic and the rotation number is rational (periodic limit sets).

devil's staircase structure. It can be proved that if we change the parameter a for a given value of b then the rotation number changes in a weakly monotonic way. It can also be proved that if ω is irrational then the interior of A_ω is empty. It is clear that all Arnold tongues have nonempty intersection with the interval $T_3 = \{(a, b) : b = -1, a \in [-2, 2]\}$ - the bottom side of the triangle T. Namely $A_\omega \cap T_3 = \{(2\cos(2\pi\omega), -1)\}$. In Fig. 5 the structure of Arnold tongues on parameter plane is shown.

It is clear that for such parameter choices the structure does not have the filtering properties but can generate oscillations of any chosen period.

2.2 Modular nonlinearity

In this subsection we consider 2's complement characteristic for the overflow rule, i.e.:

$$f(x) = (x + 1) mod\ 2 - 1 \quad (5)$$

We apply the measure-theoretic theory to the

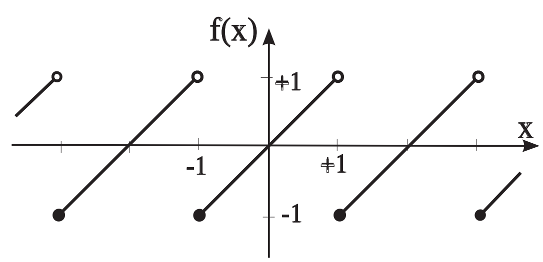
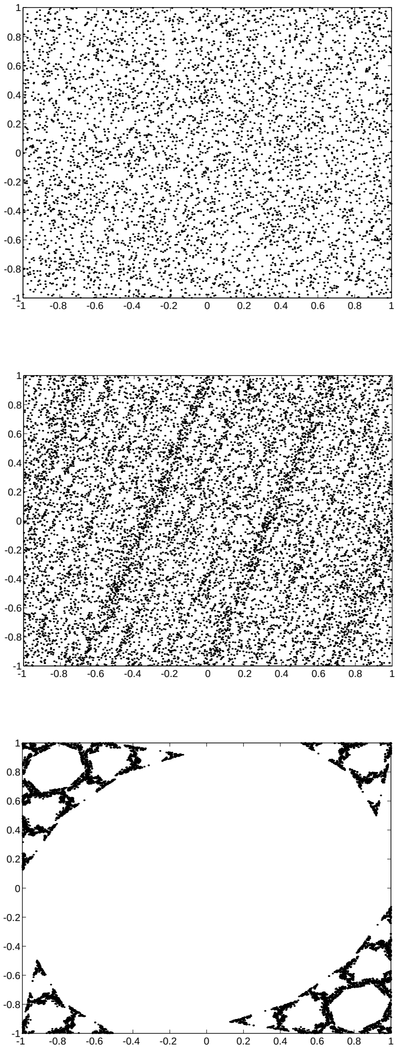


Figure 6: The 2's complement characterictic

considered digital filter. The measure we will use for the map $\boldsymbol{F}$ will be the Borel measure.



(a)

(b)

(c)

Figure 7: The trajectory of 10000 points for (a) $a = 4, b = -1$, (b) $a = 3, b = -1.1$, (c) $a = 0.4, b = 1.03$

Lemma 3 *Let $b \neq 0$ be an integer. Then the map $\boldsymbol{F}$ is measure-preserving.*

Theorem 2 *If $b = -1$, $a > 2$ and a is an integer, then $\boldsymbol{F}$ is mixing.*

Theorem 3 *If a,b are integers, such that $b \neq a+1$, $b \neq -a+1$ and $b \neq 0, \pm 1$, then the map $\boldsymbol{F}$ is exact.*

It can be proved that exactness implies mixing. The converse is not necessarily true; the mixing map $\boldsymbol{F}$ for $b = -1$ and a an integer larger than 2, is not an exact map.

Fig. 7 shows three examples of complex trajectories. Of particular interest is Fig. 7(a), which shows uniform distribution of points in the state space – such a filter structure could be possibly used as a noise generator. In Fig. 7(b) the uniform distribution is no longer valid. One could see stripes that are more frequently visited. Fig. 7(c) shows an example when the trajectory visits fragments of the state space. Thus one can see that changing parameters we can influence the statistical properties of the generated sequences.

3 Conclusions

We have described some mathematical tools useful in analysis of nonlinear effects in digital filters. In the case of saturation characteristic we used the circle-map theory. We proved in this case the existence of periodic and quasi-periodic limit sets only. In the case of modular characteristic we showed the strong chaotic filter's behaviour (mixing, exactness) in a wide range of parameters.

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