KRAWCZYK METHOD FOR PROVING THE EXISTENCE OF PERIODIC ORBITS OF INFINITE DIMENSIONAL DISCRETE DYNAMICAL SYSTEMS

Z. GALIAS

Department of Electrical Engineering,
AGH — University of Science and Technology,
al. Mickiewicza 30, 30-059 Kraków, POLAND
E-mail: galias@agh.edu.pl

In this work we describe how to prove with computer assistance the existence of fixed points and periodic orbits for infinite dimensional discrete dynamical systems. The method is based on Krawczyk operator. As an example we prove the existence of three fixed points, one period–2 and one period–4 orbit for the Kot-Schaffer growth-dispersal model.

1. Introduction

Interval methods[6,5] provide very powerful methods for proving the existence of periodic orbits and finding all short cycles of finite dimensional discrete dynamical systems[2,3]. In this work we use a modified version of Krawczyk operator to prove the existence of periodic orbits for an infinite dimensional dynamical system.

We consider the Kot-Schaffer growth-dispersal model $\Phi : L^2([-\pi,\pi]) \to L^2([-\pi,\pi])$

$$\Phi(a)(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} b(x,y)\mu a(x)\left(1 - \frac{a(x)}{c(x)}\right)dx, \qquad (1)$$

where $\mu > 0$ and $b(x,y) = b(x-y)$.

Elements of $L^2([-\pi,\pi])$ are represented using the basis $e_k = e^{ikx}$. Let a_k, b_k and c_k be the coefficients of Fourier expansion of functions a, b and $1/c$. Since the functions a, b and c are real valued we have $a_{-k} = a_k$, $b_{-k} = b_k$ and $c_{-k} = c_k$. In the basis (e_k) Eq. (1) is equivalent to the following set of maps

$$f_k(a) = \mu b_k \left(a_k - \sum_{j+l+n=k} c_j a_j a_l \right), \qquad k \in \mathbb{Z}. \tag{2}$$

Our study is devoted to the system (2). It was studied in Ref. 1 with the aim of proving the existence of complicated dynamics using Conley index theory. Here we develop tools, which allow to prove the existence of fixed points and periodic orbits for infinite dimensional maps.

In this work we consider the following parameter values: $\mu = 3.5$, $b_k = 2^{-k}$, $c_0 = 0.8$, $c_1 = -0.2$, $c_k = 0$ for $k > 1$. For these parameter values in simulations of Galerkin projections of (2) one observes chaotic behavior. The size of the attractor in the kth variable decreases very fast with k.

We use bold letters (e.g. $\mathbf{x}$, $\mathbf{A}$) to denote intervals, interval vectors and matrices and usual math italic to denote real quantities.

2. Analysis of finite dimensional system by means of Krawczyk operator

In this section we describe the Krawczyk operator in finite dimension and present analysis of fixed points of Galerkin projections of the system (2) using this operator.

Let us assume that $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1–function. Let $\mathbf{x} \subset \mathbb{R}^n$ be an interval vector (i.e. a product of intervals), let us choose $\hat{x} \in \mathbf{x}$ and an invertible matrix $C \in \mathbb{R}^{n \times n}$.

The Krawczyk operator is defined as

$$K(\mathbf{x}) = \hat{x} - CF(\hat{x}) + (I - CF'(\mathbf{x}))(\mathbf{x} - \hat{x}). \tag{3}$$

Krawczyk operator is simply a mean value form evaluation of the modified Newton operator $N(x) = x - CF(x)$. The preconditioning matrix C is usually chosen as the inverse of $F'(\hat{x})$.

The most important properties of the Krawczyk operator are[5]

1. if $K(\mathbf{x}) \cap \mathbf{x} = \emptyset$ then $F(x) \neq 0$ for all $x \in \mathbf{x}$,
2. if $K(\mathbf{x}) \subset \mathrm{int}\mathbf{x}$ then there exists exactly one zero of F in $\mathbf{x}$.

The above properties provide a powerful method for proving the existence of a unique zero in a given interval vector and for construction of a simple nonexistence criterion. The existence of fixed points of F in $\mathbf{x}$ can be studied by applying the Krawczyk operator to the map $G = F - \mathrm{id}$

$$K(\mathbf{x}) = \hat{x} - C(f(\hat{x}) - \hat{x}) + (I - C(f'(\mathbf{x}) - I))(\mathbf{x} - \hat{x}). \tag{4}$$

Table 1. Intervals containing first coordinates of fixed points for the Galerkin projection with m modes.

m	First coordinates		
1	$0.89285714285714^{33}_{25}$		
2	$0.171687825826289^{3}_{0}$	$0.51320252792568^{49}_{39}$	1.01153821767659^{9}_{7}
3	$0.253007602568588^{30}_{26}$	$0.58416707858910^{94}_{80}$	1.01684712604206^{8}_{6}
4	$0.265171408097915^{7}_{2}$	$0.61734072396246^{75}_{59}$	1.01700860077032^{7}_{5}
5	$0.26558688733343^{301}_{295}$	$0.62292700798956^{82}_{65}$	1.01701216480644^{4}_{2}
6	$0.26558676875900^{22}_{16}$	$0.62369567991200^{44}_{27}$	1.01701222393299^{8}_{6}
7	$0.265586935874199^{7}_{0}$	$0.62375796437171^{49}_{30}$	1.01701222469126^{2}_{0}
8	$0.26558694635922^{23}_{16}$	$0.62376168569651^{81}_{62}$	1.01701222469891^{4}_{2}
9	$0.26558694649282^{43}_{36}$	$0.62376185144285^{78}_{57}$	1.01701222469897^{6}_{4}
10	$0.26558694649310^{45}_{37}$	$0.62376185730101^{94}_{74}$	1.01701222469897^{6}_{4}

Let m be the number of modes of the Galerkin projection. Let us start with investigation of fixed points in the interval vector $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{m-1})$, where $\mathbf{a}_k = [-2^{-k}, 2^{-k}]$ for $k \geq 0$. Using Krawczyk method and generalized bisection algorithm we have found all fixed points in the interval vector $\mathbf{a}$ for Galerkin projections with $m = 1, 2, \ldots, 10$ modes.

Apart from the trivial fixed point $a = (0, 0, \ldots, 0)$ we found one fixed point for the Galerkin projection with one mode and three fixed points for Galerkin projections with $m = 2, 3, \ldots, 10$ modes. The intervals containing first coordinates of these fixed points are given in Table 1. One can see that positions of the fixed points do not change much when adding higher modes. This indicated that it may be possible to prove the existence of fixed points and periodic orbits of the infinite dimensional system by investigating its Galerkin projection with a certain number of modes and keeping into account errors caused by neglecting the higher modes.

3. The Krawczyk method in infinite dimension

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$ be an infinite dimensional interval vector, $\hat{x} \in \mathbf{x}$, and $C = (C_{ij})_{i,j=0}^{\infty}$ be an infinite dimensional matrix. We define the *infinite dimensional Krawczyk operator* as

$$K(\mathbf{x}) = \hat{x} - CF(\hat{x}) + (I - CF'(\mathbf{x}))(\mathbf{x} - \hat{x}). \tag{5}$$

The kth element of the vector $K(\mathbf{x})$ can be computed as

$$K_k(\mathbf{x}) = \hat{x}_k - \sum_{j=0}^{\infty} C_{kj} F_j(\hat{x}) + \sum_{j=0}^{\infty} \left(\delta_{kj} - \sum_{i=0}^{\infty} C_{ki} \frac{\partial F_i}{\partial x_j}(\mathbf{x}) \right) (\mathbf{x}_j - \hat{x}_j). \tag{6}$$

We make several assumptions on $\mathbf{x}$ and F to make the above definition valid. We assume that $\sum_k x_k^2 < \infty$, for all $x_k \in \mathbf{x}_k$. We assume that the function $x \mapsto F(x)$ is continuous on $\mathbf{x}$ and that the derivative $\frac{\partial F_i}{\partial x_j} : \mathbf{x} \mapsto \mathbb{R}$ is continuous for every i, j. Further, we assume that $\sum_{j=1}^{\infty} d_{ij} \sup_{x,y\in\mathbf{x}} |x_j - y_j| < \infty$ for every i, where $d_{ij} = \max_{x\in\mathbf{x}} \left|\frac{\partial F_i}{\partial x_j}(x)\right|$.

In Ref. 4 we have proved that for the infinite dimensional Krawczyk operator (5) there is a similar property on existence of a unique zero as for the standard Krawczyk operator. Namely, we have shown that if

$$K_k(\mathbf{x}) = (\hat{x} - CF(\hat{x}) + (I - CF'(\mathbf{x}))(\mathbf{x} - \hat{x}))_k \subset \mathrm{int}\mathbf{x}_k, \tag{7}$$

for all $k \geq 0$ then there exists a unique $x \in \mathbf{x}$, such that $F(x) = 0$.

Proof of this fact is similar to the proof of properties of Krawczyk operator in finite dimension. There is a number of technical assumptions necessary to carry out the proof. We assume that the entries in the matrix C are bounded and they are zero far from the diagonal (i.e. there exists d such that $C_{ij} = 0$ if $|i-j| > d$). We also assume that the matrix C has the property $Cx = 0 \Rightarrow x = 0$. Observe that all the assumptions on matrix C are automatically fulfilled by the choice of C given in the next section. We also assume that the inclusion (7) is satisfied uniformly in the sense that there exists $\varepsilon > 0$ such that for all $k \geq 0$

$$\frac{\mathrm{diam}\,(K_k(\mathbf{x}))}{\mathrm{diam}\,(\mathbf{x}_k)} \leq 1 - \varepsilon < 1. \tag{8}$$

For the precise formulation, the proof and other details see Ref. 4.

4. Proving the existence of fixed points

To prove the existence of a unique fixed point of f in $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots)$ we use the infinite dimensional Krawczyk operator for the map $f - \mathrm{id}$.

$$K(\mathbf{a}) = \hat{a} - C(f(\hat{a}) - \hat{a}) + (I - C(f'(\mathbf{a}) - I))(\mathbf{a} - \hat{a}). \tag{9}$$

An infinite dimensional interval vector $\mathbf{a}$ is represented as the finite dimensional interval vector $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{M-1})$ of length M and the tail. The tail is stored as two positive real numbers s and A such that $-As^{-k} \leq a_k \leq As^{-k}$ for all $a_k \in \mathbf{a}_k$ and $k \geq M$.

Now let us describe how we choose C and $\hat{a}$. Let $m \leq M$ be a positive integer. m is the number of modes in the Galerkin projection based on which we choose the preconditioning matrix C. If m is sufficiently large then the other variables are not very important.

Let $C^{(m)}$ be the preconditioning matrix obtained in the Krawczyk method for the mth projection. C is defined as

$$C_{i,j} = \begin{cases} C^{(m)}_{i,j} & \text{for } i,j < m \\ -1 & \text{for } i = j \geq m \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Let us assume that $0 \in \mathbf{a}_k$ for $k \geq M$. The point $\hat{a}$ is chosen as $\hat{a}_k = \mathrm{mid}\,(\mathbf{a}_k)$ for $k < M$ and $\hat{a}_k = 0$ for $k \geq M$.

In order to carry out the existence proof we have to check the condition $K_k(\mathbf{a}) \subset \mathbf{a}_k$ for each $k \geq 0$. Each of the first M conditions is checked independently. To obtain an enclosure for $K_k(\mathbf{a})$, we evaluate Eq. (6). Elements containing only terms of order smaller than M are computed in interval arithmetic. It is necessary to derive analytically estimates for infinite sums containing higher order terms. These estimates are inserted into formulas for K_k, allowing us to obtain enclosures for the $K_k(\mathbf{a})$.

Conditions for $k \geq M$ are checked all at once. We construct numbers A' and $s' = s$ representing the tail of $K(\mathbf{a})$.

Once this is done checking the assumption of the existence theorem is equivalent to checking the following conditions: $K_k(\mathbf{a}) \subset \mathbf{a}_k$ for $k < M$ and $A' < A$.

5. The existence of fixed point and periodic orbits

As a first example we have shown the existence of a fixed point. Namely with computer assistance we have proved the following theorem.

Theorem 5.1. *There exists a single fixed point in* $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots)$, *where* $\mathbf{a}_0 = 1.01^{75}_{65}$, $\mathbf{a}_1 = 0.194^{7}_{0}$, $\mathbf{a}_2 = 0.030^{8}_{2}$, $\mathbf{a}_3 = 0.00^{42}_{35}$, $\mathbf{a}_k = [-As^{-k}, As^{-k}]$ *for* $k \geq 4$, $A = 0.1$, $s = 2$.

In the proof we have used $M = 13$ and $m = 4$. We have shown that $K_k(\mathbf{a}) \subset \mathrm{int}\mathbf{a}_k$ for $k < M$. We have also proved that the tail of $K(\mathbf{a})$ can be represented by $s = 2$ and $A' = 0.089158 < A$. By iterating the Krawczyk operator we obtained the sharp bound for the position of the fixed point. The details of the computer assisted proof are reported in Table 2.

In a similar way we have proved the existence of three other fixed points. In fact we have proved that there are exactly 4 fixed points in the interval vector $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$, where $\mathbf{a}_k = [-2, 2] \cdot 2^{-k}$.

We have also proved the existence of a period–2 orbit.

Table 2. Details of the computer assisted proof of existence of the fixed point.

k	$\mathbf{a}_k$	$K_k(\mathbf{a})$	sharp bound
0	$[1.0165, 1.0175]$	$[1.01660, 1.01742]$	1.01701222^{58}_{36}
1	$[0.194, 0.1947]$	$[0.19406, 0.19460]$	0.19433733^{73}_{61}
2	$[0.0302, 0.0308]$	$[0.030278, 0.030741]$	0.030518985^{55}_{08}
3	$[0.0035, 0.0042]$	$[0.0035517, 0.00418302]$	0.003884168^{22}_{01}
4	$0.00625[-1, 1]$	$[-0.0047240, 0.0056177]$	0.0004066^{902}_{896}
5	$0.003125[-1, 1]$	$[-0.0018224, 0.0018881]$	$3.5968^{74}_{59} \cdot 10^{-5}$
6	$0.0015625[-1, 1]$	$[-0.00051562, 0.00051946]$	$2.753^{16}_{09} \cdot 10^{-6}$
7	$0.00078125[-1, 1]$	$[-0.00013436, 0.00013453]$	$1.85^{82}_{75} \cdot 10^{-7}$
8	$0.000390625[-1, 1]$	$3.53435 \cdot 10^{-5}[-1, 1]$	$1.1^{28}_{13} \cdot 10^{-8}$
9	$0.0001953125[-1, 1]$	$9.35659 \cdot 10^{-6}[-1, 1]$	$[3.36, 8.86] \cdot 10^{-10}$
10	$9.765625 \cdot 10^{-5}[-1, 1]$	$2.46934 \cdot 10^{-6}[-1, 1]$	$[-0.988, 1.049] \cdot 10^{-9}$
11	$4.8828125 \cdot 10^{-5}[-1, 1]$	$6.49886 \cdot 10^{-7}[-1, 1]$	$[-3.233, 3.237] \cdot 10^{-9}$
12	$2.44140625 \cdot 10^{-5}[-1, 1]$	$1.70616 \cdot 10^{-7}[-1, 1]$	$[-8.621, 8.623] \cdot 10^{-9}$
M	$A = 0.1$	$A' = 0.089158$	$A' = 0.08908$

Theorem 5.2. *There exists a single period two orbit in* $\mathbf{a}^0, \mathbf{a}^1$, *where* $\mathbf{a}^0_0 = 0.6^{204}_{044}$, $\mathbf{a}^0_1 = 0.2^{435}_{375}$, $\mathbf{a}^0_2 = 0.04^{37}_{17}$, $\mathbf{a}^0_3 = 0.00^{46}_{26}$, $\mathbf{a}^1_0 = 1.^{2056}_{1956}$, $\mathbf{a}^1_1 = 0.1^{935}_{895}$, $\mathbf{a}^1_2 = 0.01^{92}_{80}$, $\mathbf{a}^1_3 = 0.00^{36}_{22}$, $\mathbf{a}^0_k = \mathbf{a}^1_k = [-0.1, 0.1] \cdot 2^{-k}$ *for* $k \geq 4$.

In the proof we used $m = 5$, $M = 14$. Iterating the Krawczyk operator we have found intervals with diameter below 10^{-7} containing the orbit.

Similarly we have proved the existence of one period–4 orbit.

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