Numerical studies of the Hénon map

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Abstract: In this paper we perform a rigorous study of the Hénon map. We prove with computer assistance the existence of symbolic dynamics for h^2 and h^7 and the existence of periodic orbits of all periods but 3 and 5.

Key Words: chaos, computer assisted proof, interval arithmetic.

1 Introduction

In this paper we consider the Hénon map given by (1) with the "classical" parameter values: $a = 1.4$ and $b = 0.3$.

In the first part of the paper we prove the existence of symbolic dynamics for h^2 and what follows the existence of periodic points of h with all even periods.

In [7] the dynamics of topological horseshoe was proved for h^7. From this follows the existence of symbolic dynamics for h^7 and the existence of periodic orbits of h of period $7n$ for all natural n.

In the second part of the paper we repeat the proof performed by Zgliczyński using interval arithmetic. We show that using interval arithmetic the number of points for which we must check certain conditions can be significantly reduced. Then checking some more conditions we prove the existence of periodic points with period 8 and all periods greater or equal to 10.

Finally by means of the interval Newton method we prove that within the region $[-5, 5] \times [-5, 5]$ there exists no periodic point with period 3 and 5 and we prove that within this region there exist periodic points with period 9.

The symbolic dynamics for h^2 and h^7 is proved for invariant sets embedded in the strange attractor observed numerically. Also all the periodic orbits the existence of which is proved (apart from one of the fixed points) lie in the region where the strange attractor is observed. This indicates that the dynamics of the system is very complicated. However the existence of a strange attractor for classical values of parameters still remains an open problem.

2 Main results

The Hénon map [5] is defined by the following equation:

$$h(x, y) = (1 + y - ax^2, bx). \tag{1}$$

The above equation is considered with the "classical" parameter values: $a = 1.4$ and $b = 0.3$. In this paper we show rigorously with computer assistance that

A. subshift on two symbols with the transition matrix (2) (this corresponds to the deformed topological horseshoe [2]) is embedded in h^2,
B. full shift on two symbols with the transition matrix (3) (the topological horseshoe) is embedded in h^7,
C. h has periodic points of all periods but 3 and 5,
D. h has no periodic points with periods 3 and 5 within the set $[-5,5] \times [-5,5]$.

2.1 Symbolic dynamics for h^2

Items A and B of the above list are proved by means of the technique of TS-maps (topological shifts) introduced by Zgliczyński in [7]. For the proof of part A we define the sets N_i as follows: N_0 is a quadrangle $\overline{A_1A_2A_3A_4}$ and N_1 is the quadrangle $\overline{A_5A_6A_7A_8}$, where $A_1 = (-0.82, 0.29)$, $A_2 = (-0.82, 0.39)$, $A_3 = (-0.26, 0.34)$, $A_4 = (-0.26, 0.24)$, $A_5 = (0, 0.19)$, $A_6 = (0.08, 0.29)$, $A_7 = (0.42, 0.2)$ and $A_8 = (0.34, 0.1)$. We also define sets E_0, E_1 and E_2 lying respectively to the left, between and to the right of the sets N_0 and N_1.

With the computer assistance we have proved that the image of the N_0 "covers horizontally" $N_0 \cup N_1$ and the image of the N_1 "covers horizontally" N_0. This is formally written in the following lemma.

Lemma 1. *Let the Hénon map h be defined by Eq.* (1)*. Let $a = 1.4$ and $b = 0.3$.*

1. $h^2(\overline{A_1A_2}) \subset E_2$ *and* $h^2(\overline{A_3A_4}) \subset E_0$,
2. $h^2(\overline{A_5A_6}) \subset E_0$ *and* $h^2(\overline{A_7A_8}) \subset E_1$,
3. $h^2(\overline{A_1A_4})$, $h^2(\overline{A_2A_3})$, $h^2(\overline{A_5A_8})$, $h^2(\overline{A_6A_7}) \subset W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$.

Proof. During the proof we use the procedures for interval computations form BIAS and PROFIL packages [6]. For the proof of 1 and 2 we have covered the vertical edges $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$ and $\overline{A_7A_8}$ by 1, 1, 1 and 3 rectangles respectively. Using interval arithmetic we have proved that their images under h^2 are enclosed in the appropriate sets E_i.

For the proof of 3 we have covered the horizontal edges $\overline{A_1A_4}$, $\overline{A_2A_3}$, $\overline{A_5A_8}$ and $\overline{A_6A_7}$ by 9, 11, 4 and 4 rectangles respectively. We have proved that each of the images is enclosed within the set W.

From Lemma 1 using the theorem on TS-maps [7] one can conclude that the subshift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

is embedded within the map $h^2|N_0 \cup N_1$.

For the whole proof of the existence of symbolic dynamics for h^2 we needed to compute images of 34 rectangles under h^2.

2.2 Symbolic dynamics for h^7

In [7] Zgliczyński introduced the quadrangles N_0, N_1 (different to the sets defined above), the sets E_0, E_1 and E_2 lying to the left, between and to the right of the sets N_0 and N_1 and the set $W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$. For these sets using the technique of TS-maps he proved the existence of the topological horseshoe. He proved that the full shift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3}$$

is embedded within the map h^7. This proof required the computation of h^7 for approximately 60000 points.

Using the same sets N_i and E_i we have repeated the proof. In order to prove the existence of symbolic dynamics associated to the full shift we have to prove that the images of N_0 and N_1 under h^7 cover horizontally the set $N_0 \cup N_1$. Namely, we have proved the following lemma.

Lemma 2. *The map h is defined by Eq. (1), $a = 1.4$ and $b = 0.3$.*

1. $h^7(\overline{A_1A_2}) \subset E_2$ *and* $h^7(\overline{A_3A_4}) \subset E_0$,
2. $h^7(\overline{A_5A_6}) \subset E_0$ *and* $h^7(\overline{A_7A_8}) \subset E_2$,
3. $h^7(\overline{A_1A_4})$, $h^7(\overline{A_2A_3})$, $h^7(\overline{A_5A_8})$, $h^7(\overline{A_6A_7}) \subset W$.

Proof. Our proof was performed using interval arithmetic. For the proof of the existence of topological horseshoe within the sets N_i we computed the images of 131 rectangles under h^7 proving that they lie within appropriate subsets.

Notice that the number of points at which the image is computed is significantly reduced in comparison with the original proof. Probably Zgliczyński overestimated the error (he did not use the interval arithmetic).

2.3 Periodic points with all natural periods but 3 and 5

In Lemma 2 we have proved that the images of sets N_i under h^7 covers horizontally the sets N_0 and N_1. Now we extend this result. We have checked the positions of horizontal and vertical edges of N_0 and N_1 under h^i, for $i = 1, \ldots, 6$. The results are shown in Table 1.

From these results one can easily prove the existence of periodic points for all periods greater or equal to 7 with the exception of period 9.

Lemma 3. *For every integer $n \geq 7$, $n \neq 9$ there exist periodic point of h with period n.*

So far we have shown that there exist periodic points with all periods but 1, 3, 5 and 9. The existence of a fixed point can be proved analytically. In fact there exist two such points (x_1, bx_1) and (x_2, bx_2) where $x_{1,2} = (b - 1 \pm \sqrt{(1-b)^2 + 4a})/(2a)$. One of the fixed points is embedded within the numerically observed strange attractor.

In order to decide the existence of periodic points with periods 3, 5 and 9 we have used the interval Newton method [1]. We have proved the following lemma.

i	$h^i(N_0)$	$h^i(N_1)$
1	N_1	—
2	N_0	—
3	N_0, N_1	—
4	N_0, N_1	—
5	N_0, N_1	—
6	N_0, N_1	—
7	N_0, N_1	$0, N_1$

Table 1: Images of N_0 and N_1 under h^i ($i = 1, \ldots, 7$). In the last two columns the sets which are covered horizontally by $h^i(N_0)$ and $h^i(N_1)$ are given

Lemma 4. *Let $M = [-5, 5] \times [-5, 5]$.*

1. *There exists no periodic point with period 3 within the set M.*
2. *There exists no periodic point with period 5 within the set M.*
3. *There exist 6 period-9 orbits within the set M.*

Proof. To prove part 1 we have covered the set M by 493 rectangles. Using the interval Newton method we have proved that there are no period-3 orbits within each of these rectangles. Similarly using 4241 rectangles to cover the set M we have proved that there are no period-5 orbits within the set M. For the proof of part 3 we have covered the set M by 2974053 rectangles. We have proved the existence of exactly 54 periodic points with period 9 within M which corresponds to 6 different period-9 orbits.

Acknowledgments

This work was sponsored by the University of Mining and Metallurgy under grant no. 10.120.132.

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