EXISTENCE AND UNIQUENESS OF LOW–PERIOD CYCLES AND ESTIMATION OF TOPOLOGICAL ENTROPY FOR THE HÉNON MAP

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Abstract— In this paper we investigate periodic orbits for the Hénon map. We find all low period ($n \leq 15$) cycles belonging to a set containing the trapping region and the attractor observed numerically. Using this result we estimate the topological entropy of the Hénon map.

I. INTRODUCTION

In this paper we study rigorously the existence of periodic orbits for the Hénon map [1]:

$$h(x, y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$.

Analysis of chaotic systems in terms of periodic orbits has many advantages. One of the main features of chaotic attractors is the existence of infinitely many periodic orbits. Periodic orbits determine the spatial layout of the chaotic attractor. Short periodic orbits give good approximation of the attractor while by recovering more unstable cycles one obtains better approximations [2].

Hence the methods for extraction of periodic orbits and proving their existence are of very high importance. One of the well–known methods for finding periodic orbits is the method of close returns [2]. Its main advantage is that it allows to find periodic orbits from experimental data. It is however not possible to use this method to prove the existence of periodic orbit (one cannot be sure whether a real periodic orbit actually exists in a neighborhood of the ε–pseudo periodic orbit).

As a main tool in our study we use the interval Newton method [3], which allows to prove with computer assistance the existence and uniqueness of periodic orbits within a given interval. In order to investigate the existence of zeros of a function $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ in an n-dimensional interval $\mathbf{X}$ one has to evaluate the *interval Newton operator*

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - (\mathbf{Df}(\mathbf{X}))^{-1}\mathbf{f}(\mathbf{x}_0), \tag{2}$$

where $(\mathbf{Df}(\mathbf{X}))^{-1}$ is the interval matrix containing all matrices of the form $(\mathbf{Df}(\mathbf{x}))^{-1}$ for $\mathbf{x} \in \mathbf{X}$ and $\mathbf{x}_0$ is an arbitrary point belonging to the interval $\mathbf{X}$. One usually chooses $\mathbf{x}_0$ to be the center of $\mathbf{X}$.

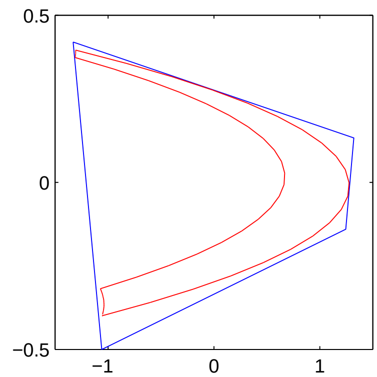

Figure 1: Invariant quadrangle Ω and its image under the Hénon map.

The following theorem [3] states the relation between the zeros of $\mathbf{f}$ in $\mathbf{X}$ and the position of $\mathbf{N}(\mathbf{X})$ with respect to $\mathbf{X}$.

Theorem 1 *If* $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ *then there exist exactly one point* $\mathbf{x} \in \mathbf{X}$ *such that* $f(\mathbf{x}) = 0$*. If* $\mathbf{N}(\mathbf{X}) \cap \mathbf{X} = \emptyset$ *then there are no zeros of* $\mathbf{f}$ *in* $\mathbf{X}$*.*

The above theorem can be used to prove both the existence and uniqueness of zeros. By iterating the method one can easily sharpen the bounds of solutions.

During the computer assisted proof we have used the procedures for interval computations form BIAS and PROFIL packages. Programs were compiled using gnu C++ compiler (gcc version 2.7.2.1) and run on Sun Ultra 1 computer. Program code is available at the following www location: `http://fractal.zet.agh.edu.pl/~galias/int.html`.

II. PERIODIC ORBITS

Let us denote by Ω the quadrangle $ABCD$, where $A = (-1.33, 0.42)$, $B = (1.32, 0.133)$, $C = (1.245, -0.14)$ and $D = (-1.06, -0.5)$. One can easily show [1] that Ω is a trapping region: $h(\Omega) \subset \Omega$. The sets Ω and its

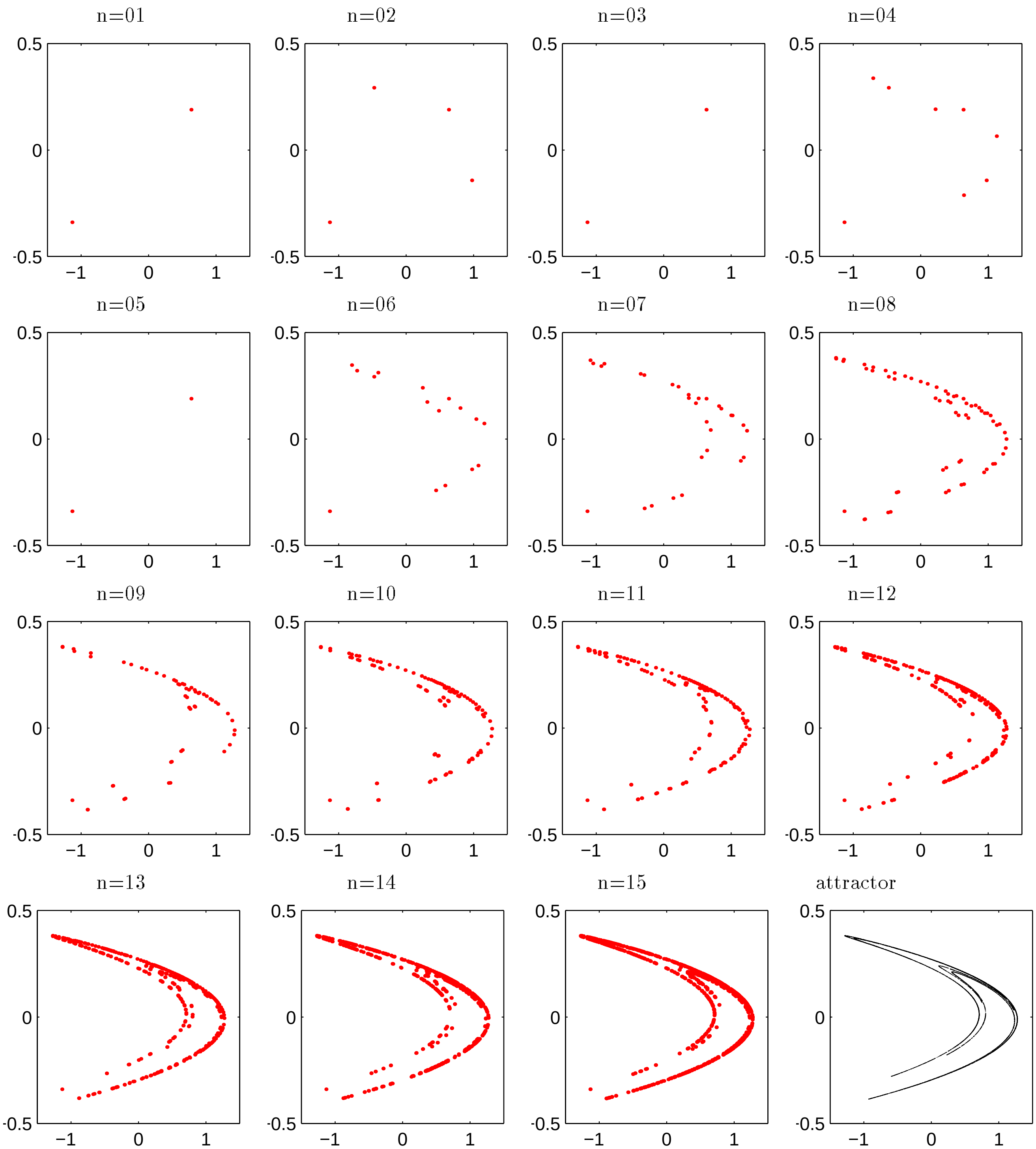

Figure 2: Fixed points of h^n for $n = 1, \ldots, 15$ within $M = [-5,5] \times [-5,5]$. See for example that for $n = 1$, $n = 3$ and $n = 5$ the number of fixed points of h^n is the same, which means that there are no period–3 and period–5 points of h within M. The last picture shows the trajectory of the Hénon map consisting of 10000 points.

n	$Q(n)$	$P(n)$	$\mathrm{H}_n(h)$	$R(n)$
1	2	2	0.693	81
2	1	4	0.693	225
3	—	2	0.231	273
4	1	8	0.519	1905
5	—	2	0.138	3061
6	2	16	0.462	21657
7	4	30	0.485	67093
8	7	64	0.519	353945
9	6	56	0.447	1019625
10	10	104	0.464	3767613
11	14	156	0.459	11445321
12	19	248	0.459	44520813
13	32	418	0.464	140036237
14	44	648	0.462	533037209
15	72	1082	0.466	1742355589

Table 1: Periodic orbits for the Hénon map. $Q(n)$ is the number of cycles with principle period n, $P(n)$ is the number of fixed points of h^n, $\mathrm{H}_n(h) = \log(P(n))/n$ is the estimation of topological entropy of h based on $P(n)$, $R(n)$ is the number of rectangles at which the interval Newton operator was evaluated.

image under h are shown in Fig. 1. The invariant set Ω encloses the attractor observed numerically.

In our investigations we consider the region $M = [-5,5] \times [-5,5]$, which encloses the trapping region Ω and hence the Hénon attractor.

In order to find fixed points of h^n we use the following procedure. The set M is divided into several rectangles (the number of them increases with n).

For each rectangle $\mathbf{X}$ the interval Newton operator for the map $\mathrm{id} - h^n$ is computed:

$$\mathbf{N}(\mathbf{X}) = \mathbf{x}_0 - (\mathbf{I} - \mathbf{D}h^n(\mathbf{X}))^{-1} (\mathbf{x}_0 - h^n(\mathbf{x}_0)), \quad (3)$$

where $\mathbf{x}_0$ is the center of $\mathbf{X}$. If $\mathbf{N}(\mathbf{X}) \subset \mathbf{X}$ then there exists exactly one fixed point of h^n with period belonging to $\mathbf{X}$. If $\mathbf{N}(\mathbf{X}) \cap \mathbf{X} = \emptyset$ then there is no fixed points of h^n in $\mathbf{X}$. The remaining rectangles for which none of the above cases is true are covered by larger rectangles and we try to apply Theorem 1 again. If this action is not successful we divide the remaining rectangles into smaller parts and repeat the computations.

We have applied the above procedure for $n = 1,\ldots,15$. Positions of the fixed points of h^n for different n are shown in Fig. 2. For the reference in the lower right corner of Fig. 2 we show the trajectory of the Hénon map consisting of 10000 points. One can see that the collection of longer periodic orbits gives better approximation of the attractor.

The numbers $P(n)$ of fixed points of h^n and the numbers $Q(n)$ of cycles with principle period n are collected in Table 1. There are two fixed points (this

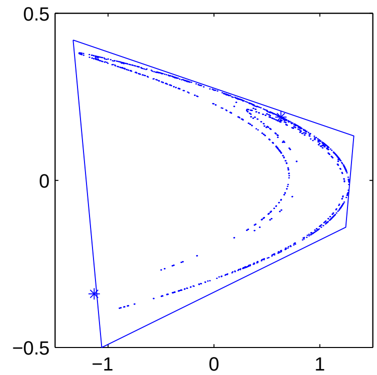

Figure 3: All periodic orbits with period $n \leq 15$. Positions of fixed points are denoted by stars. All periodic orbits, apart from one of the fixed points lie on the strange attractor observed numerically.

result can be also proven analytically):

$$\mathbf{x}_1 \approx (0.6314, 0.1894), \mathbf{x}_2 \approx (-1.1314, -0.3394). \quad (4)$$

First of the fixed points belongs to the trapping set, while the second one lies outside of it. There is one period–2 orbit. There are no period–3 and period–5 orbits within the region M.

For $n = 12,\ldots,15$ the results are not complete as we were not able to prove the uniqueness of the fixed point of h^n in a very small neighborhood of the fixed point $\mathbf{x}_2$ lying outside of the trapping set. Strictly speaking for $n = 1,\ldots,11$ we have found all fixed points of h^n in M, while for $n = 12,\ldots,15$ we have found all fixed points of h^n in $M \setminus \mathrm{B}(\mathbf{x}_2, 2 \cdot 10^{-8})$.

In Fig. 3 we plot all periodic orbits found. One can see that all of them (apart from one of the fixed points) lie on the strange attractor observed numerically. For all of these periodic orbits we have computed the Floquet multipliers proving that they are of a saddle type.

III. ESTIMATION OF TOPOLOGICAL ENTROPY

In this section we use the number of periodic orbits for the estimation of topological entropy of the Hénon map.

Topological entropy $\mathrm{H}(f)$ of a map f characterizes "mixing" of points by the map f. One of the equivalent definitions of topological entropy is based on the notion of (n,ε)–separated sets (see [4]).

Definition 1 *A set $E \subset X$ is called (n,ε)–*separated *if for every two different points $x, y \in E$, there exists $0 \leq j < n$ such that the distance between $f^j(x)$ and $f^j(y)$ is greater than ε. Let us define the number $s_n(\varepsilon)$ as the cardinality of a maximum (n,ε)–separated set:*

$$s_n(\varepsilon) = \max\{\operatorname{card} E : E \quad \textit{is} \quad (n,\varepsilon)\textit{-separated}\}$$

The number

$$\mathrm{H}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon), \qquad (5)$$

is called the topological entropy *of the map f.*

The number of periodic orbits is closely related to the topological entropy. For axiom A diffeomorphisms we have

$$\mathrm{H}(f) = \lim_{n \to \infty} \frac{\log C(f^n)}{n},$$

where $C(f^n)$ denotes the number of fixed points of f^n. It is also possible to use the number of periodic orbits for the estimation of topological entropy when there exists a symbolic dynamics for the map.

Using the existence of symbolic dynamics for h^7 one can prove that (compare [5, 6]):

$$\mathrm{H}(h) > \frac{1}{7} \log 2 > 0.099.$$

Similarly one can obtain the estimation of topological entropy based on the existence of symbolic dynamics for h^2 (compare [6]):

$$\mathrm{H}(h) > \frac{1}{2} \log \frac{\sqrt{5}+1}{2} > 0.24.$$

Here we use the formula $\mathrm{H}_n(h) = \log(P(n))/n$ as the aproximation of topological entropy. We believe that for the Hénon map in the limit we have $\mathrm{H}(h) = \lim_{n\to\infty} \mathrm{H}_n(h)$. The results are plotted in Fig. 4 (see also Table 1). One can see that $\mathrm{H}_n(h)$ is almost constant for $n \geq 10$. This let us state the hypothesis that the topological entropy of the Hénon map is close to 0.46.

In Fig. 4 we also plot the curves $\log(P(n) - n)/n$ and $\log(P(n)+n)/n$, which would be $\mathrm{H}_n(h)$ if there is one less or one more cycle of period n. They give the maximum accuracy one can obtain in computation of topological entropy for given n using $\mathrm{H}_n(h)$.

IV. CONCLUSIONS

In this paper we have studied periodic orbits for the Hénon map. We have found all cycles with period $n \leq 15$ in the trapping region enclosing the attractor observed numerically. Using the number of low–period cycles we have estimated that the topological entropy of the map is approximately 0.46, which shows that the dynamics of the Hénon map may be more complicated than the one reported in [6].

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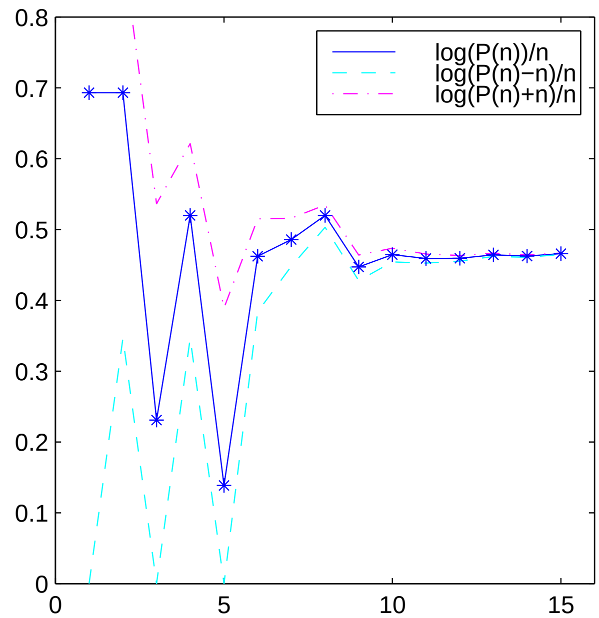

Figure 4: Estimation of topological entropy of the Hénon map based on the number of low–period cycles.

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