# A FEEDBACK CHAOS CONTROLLER: THEORY AND IMPLEMENTATION

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## ABSTRACT

The occasional proportional feedback (OPF) controller introduced by Hunt [1] for stabilizing periodic solutions in an autonomous chaotic system has the disadvantage that it requires an external synchronizing signal. In this paper we describe a modified version of Hunt's controller which obviates the need for an external synchronizing signal. We show how the controller has been used successfully to remove chaos in the chaotic Colpitts oscillator. We also describe some theoretical results on the possibilities for controlling a given chaotic system using the OPF method. We prove that the possibility of stabilization depends on the behavior of the system in the neighborhood of the periodic orbit and not on the dimensionality of the attractor on the Poincaré surface.

## 1. INTRODUCTION

Let us start with the description of two methods of control, namely OGY [2] and OPF [1] control. Many other control schemes exist [3] but these two are of interest because they use the special properties of chaotic systems and very small control signals are required.

One of the special properties of chaotic attractors is that they contain an infinite number of unstable periodic orbits. The OGY control method was introduced by Ott, Grebogi and Yorke [2] in 1990. This method allows one to stabilize any unstable periodic orbit by perturbing one of the system parameters over a small range about some nominal value.

We assume that we have a three-dimensional continuous time system of first-order autonomous ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, p), \tag{1}$$

where $\mathbf{x} \in I\!R^3$ and $p \in I\!R$ is a system parameter which we can change. We also assume that parameter $p$ can be modified within a small interval around its nominal value $p_0$ ($p \in [p_0 - \delta p_{\max}, p_0 + \delta p_{\max}]$, where $\delta p_{\max}$ is the maximum permissible change of the parameter $p$). We choose a two-dimensional Poincaré surface $\Sigma$ which defines a Poincaré map $\mathbf{P}$ (for $\xi \in \Sigma$, we define by $\mathbf{P}(\xi)$ the point at which the trajectory starting from $\xi$ intersects $\Sigma$ for the first time). Since the vector field $\mathbf{F}$ depends on $p$, the Poincaré map $\mathbf{P}$ also depends on this parameter $p$. Thus, we have

$$\mathbf{P}\colon I\!R^2 \times I\!R \ni (\xi, p) \longrightarrow \mathbf{P}(\xi, p) \in I\!R^2. \tag{2}$$

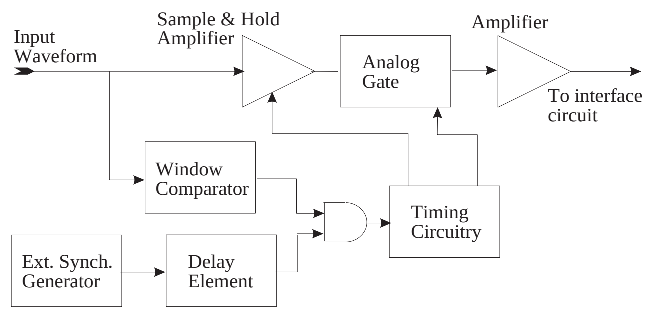


**Figure 1. Hunt's implementation of the OPF method**

Let us assume that $\mathbf{P}$ is differentiable. Say we have selected one of the unstable periodic orbits embedded in the system's attractor as the goal of our control because, for example, it offers an improvement in system performance over the original chaotic behavior. For simplicity, we assume that this is a period-1 orbit (a fixed point of the map $\mathbf{P}$).

Let us denote by $\xi_F$ an unstable fixed point of $\mathbf{P}$ for $p = p_0$ ($\mathbf{P}(\xi_F, p_0) = \xi_F$). Let the first-order approximation of $\mathbf{P}$ in the neighborhood of $(\xi_F, p_0)$ be of the form

$$\mathbf{P}(\xi, p) \approx \mathbf{P}(\xi_F, p_0) + \mathbf{A}\cdot(\xi - \xi_F) + \mathbf{w}\cdot(p - p_0), \tag{3}$$

where $\mathbf{A}$ is a Jacobian matrix of $\mathbf{P}(\cdot, p_0)$ at $\xi_F$, and $\mathbf{w} = \frac{\partial \mathbf{P}}{\partial p}(\xi_F, p_0)$ is the derivative of $\mathbf{P}$ with respect to the parameter $p$.

Stabilization of the fixed point is achieved by realizing feedback of the form

$$p(\xi) = p_0 + \mathbf{c}^T(\xi - \xi_F). \tag{4}$$

In the original description of the OGY method [2], the vector $\mathbf{c}$ is computed using the expression

$$\mathbf{c} = -\frac{\lambda_u}{\mathbf{f}_u^T \mathbf{w}} \mathbf{f}_u^T, \tag{5}$$

where $\lambda_u$ is the unstable eigenvalue and $\mathbf{f}_u$ is the corresponding left eigenvector of $\mathbf{A}$.

The occasional proportional feedback (OPF) method [4, 5] is a one-dimensional version of the OGY method. A schematic diagram, showing Hunt's implementation of the OPF control method, is presented in Fig. 1. The window comparator, taking the input waveform, gives a logical high when the input waveform is inside the window. This

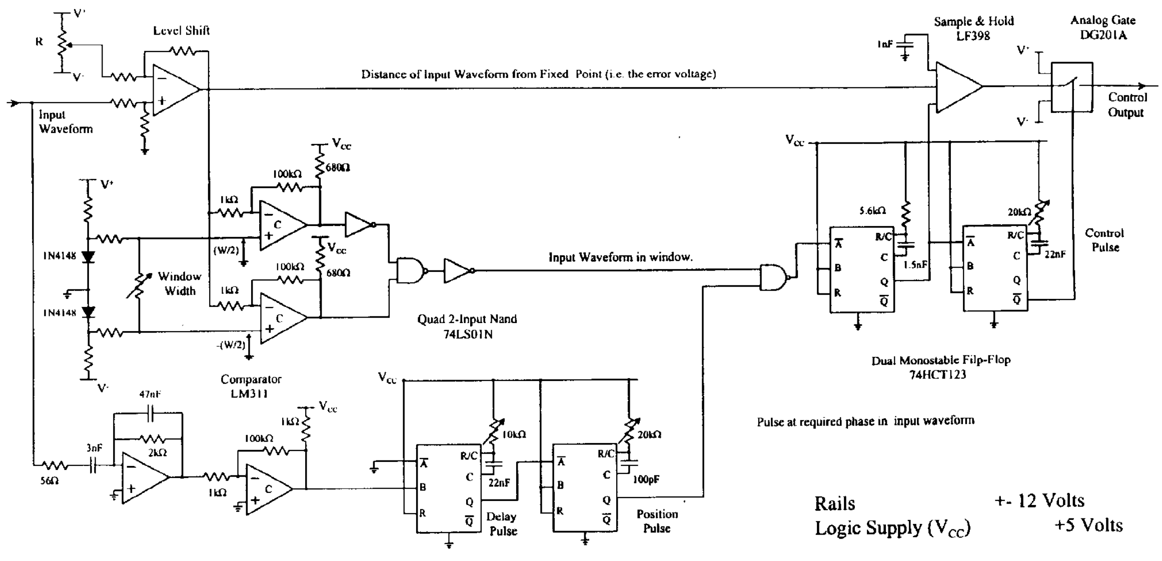


**Figure 2. Circuit diagram for the modified OPF controller**

is then ANDed with the delayed output from the external frequency generator. This logical signal drives the timing block which triggers the sample-and-hold and then the analog gate. The output from the gate, which represents the error signal at the sampling instant, is then amplified and applied to the interface circuit which transforms the control pulse into a perturbation of the system. The frequency, delay, control pulse width, window position, width and gain are all adjustable. The interface circuit used depends on the chaotic system under control.

One of the major advantages of Hunt's controller over OGY is that the control law depends on only one variable and does not require any complicated calculations in order to generate the required control signal. The disadvantage of the OPF method is that there is no systematic method for finding the embedded unstable orbits (unlike OGY).

## 2. CHAOS CONTROLLER FOR AUTONOMOUS CIRCUIT

In this section we introduce our modified controller which uses Hunt's method without the need for an external synchronising oscillator.

Hunt uses the peaks of one of the system variables to generate the $1^D$ map. He then uses a window around a fixed level to set the region where control is applied. This approach means that his controller needs just one of the system variables as input. In order to find the peaks, Hunt's scheme uses a synchronizing generator.

In our modified controller [6], we simply take the derivative of the input signal and generate a pulse when it passes through zero. We use this pulse instead of Hunt's external driving oscillator as the "synch" pulse for our Poincaré map. This obviates the need for the external generator and so makes the controller simpler and cheaper to build.

A circuit diagram for our modified controller is given in Fig. 2. The variable level window comparator is implemented using a window comparator around zero and a variable level shift. Two comparators and three logic gates form the window around zero. The synchronizing generator used in Hunt's controller is replaced by an inverting differentiator and a comparator. A rising edge in the comparator's output corresponds to a peak in the input waveform. We use the rising edge of the comparator's output to trigger a monostable flip-flop. The falling edge of this monostable's pulse triggers another monostable, giving a delay. We use the monostable's output pulse to indicate that the input waveform peaked a fixed time earlier. If this pulse arrives when the output from the window comparator is high then a monostable is triggered. The output of this monstable triggers a sample-and-hold on its rising edge which samples the error voltage; on its falling edge, it triggers another monostable. This final monostable generates a pulse which opens the analog gate for a specific time (the control pulse width). The control pulse is then applied to the interface circuit, which amplifies the control signal and converts it into a perturbation of one of the system parameters, as required.

## 3. THEORETICAL RESULTS

In this section, we address the problem of whether or not it is possible to stabilize a given periodic orbit using the OPF method. We describe our approach for the case of stabilizing a fixed point of the Poincaré map.

In the OPF method, the control signal is computed using only one variable, for example $\xi_1$:

$$p(\xi) = p_0 + c(\xi_1 - \xi_{F1}). \quad (6)$$

We want to find values of $c$ for which $\xi_F$ is a stable fixed point of the system $\xi \mapsto \mathbf{P}(\xi, p(\xi))$.

**Theorem 1** *Let*

$$\mathbf{f}(\xi, p) = \mathbf{A} \cdot \xi + \mathbf{w}p, \quad (7)$$

*where* $\mathbf{A}$ *is a two-dimensional square matrix,* $\xi = (\xi_1, \xi_2)^T$,

$\mathbf{w} = (w_1, w_2)^T$ *and* $p \in I\!R$. *Let us denote*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{8}$$

*Let* $\mathrm{tr}\mathbf{A} = a_{11} + a_{12}$ *and* $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ *denote the trace and determinant of matrix* $\mathbf{A}$ *respectively. If*

$$\begin{aligned} 1 + \mathrm{tr}\mathbf{A} + \det\mathbf{A} + c(w_1 + w_1 a_{22} - w_2 a_{12}) &> 0 \\ 1 - \det\mathbf{A} + c(-w_1 a_{22} + w_2 a_{12}) &> 0 \\ 1 - \mathrm{tr}\mathbf{A} + \det\mathbf{A} + c(-w_1 + w_1 a_{22} - w_2 a_{12}) &> 0 \end{aligned} \tag{9}$$

*then* $(0,0)^T$ *is a stable fixed point of*

$$\mathbf{f}(\xi) \triangleq \mathbf{f}(\xi, p(\xi)) = \mathbf{f}(\xi, c\xi_1) = \mathbf{A}\cdot\xi + \mathbf{w}c\xi_1. \tag{10}$$

**Proof:**

$$\begin{aligned} \mathbf{f}(\xi) &= \mathbf{A}\cdot\xi + \mathbf{w}c\xi_1 \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} c\xi_1 \\ &= \begin{pmatrix} a_{11} + w_1 c & a_{12} \\ a_{21} + w_2 c & a_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \triangleq \mathbf{W}\xi. \end{aligned}$$

Now $(0,0)^T$ is a stable fixed point of $\mathbf{f}$ if all of the eigenvalues of $\mathbf{W}$ lie within the unit circle. Using the Hurwitz criterion one can show that if the following inequalities are satisfied

$$\begin{aligned} 1 + \mathrm{tr}\mathbf{W} + \det\mathbf{W} &> 0 \\ 1 - \det\mathbf{W} &> 0 \\ 1 - \mathrm{tr}\mathbf{W} + \det\mathbf{W} &> 0 \end{aligned} \tag{11}$$

then the eigenvalues of matrix $\mathbf{W}$ lie within the unit circle. For matrix $\mathbf{W}$, the conditions (11) are equivalent to the inequalities (9). □

This theorem is formulated for the case of a linear map $\mathbf{f}$ with the fixed point $\xi_F = (0,0)^T$. The following theorem extends this result to a nonlinear map $\mathbf{P}$ with an arbitrary fixed point $\xi_F$.

**Theorem 2** *Let* $\mathbf{P}$ *be the map defined in* (2) *and let* $\xi_F$ *be a fixed point of* $\mathbf{P}$. *Let the linear approximation of* $\mathbf{P}$ *be of the form* (3). *Define*

$$\mathbf{P}(\xi) \triangleq \mathbf{P}(\xi, c\xi_1) \tag{12}$$

*If conditions* (9) *are satisfied then there exists a neighborhood* $U$ *of* $\xi_F$ *such that* $\mathbf{P}^n(\xi) \stackrel{n\to\infty}{\longrightarrow} \xi_F$ *for all* $\xi \in U$ *(i.e.* $\xi_F$ *is stable for the map* (12)*).*

**Proof:** Apply Theorem 1 to the linear approximation of the map $\mathbf{P}$ at $\xi_F$. □

The above theorem determines for which values of $c$ successful control is possible.

As an example, let us consider the case when the Jacobian matrix is diagonal. Define two special diagonal matrices

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{pmatrix}, \tag{13}$$

$$\mathbf{D}_2 = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{pmatrix}, \tag{14}$$

with one stable eigenvalue and one unstable eigenvalue each. Note that $|\lambda_u| > 1 > |\lambda_s|$.

**Corollary 3** *Assume that* $w_1 \neq 0$. *If Jacobian* $\mathbf{A}$ *is of the form* (13) *then there exists* $c$ *such that* $\xi_F$ *is an asymptotically stable fixed point of the system* $\xi \mapsto \mathbf{P}(\xi, c\xi_1)$.

**Proof:** First let us assume that the stable eigenvalue is positive. From Theorem 1 it follows that stabilization is possible if

$$\begin{aligned} 1 + \lambda_s + \lambda_u + \lambda_s\lambda_u + c(w_1 + w_1\lambda_s) &> 0 \\ 1 - \lambda_s\lambda_u + c(-w_1\lambda_s) &> 0 \\ 1 - \lambda_s - \lambda_u + \lambda_s\lambda_u + c(-w_1 + w_1\lambda_s) &> 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} w_1 c &> -\frac{1 + \lambda_s + \lambda_u + \lambda_s\lambda_u}{1 + \lambda_s} = -1 - \lambda_u \\ w_1 c &< \frac{1 - \lambda_s\lambda_u}{\lambda_s} \\ w_1 c &< \frac{1 - \lambda_s - \lambda_u + \lambda_s\lambda_u}{1 - \lambda_s} = 1 - \lambda_u \end{aligned}$$

Since $w_1 \neq 0$, we can find $c$ for which the above inequalities hold if

$$\frac{1 - \lambda_s\lambda_u}{\lambda_s} > -1 - \lambda_u \tag{15}$$

$$1 - \lambda_u > -1 - \lambda_u \tag{16}$$

Inequality (16) is equivalent to $2 > 0$, which is always true. Inequality (15) is equivalent to $1 + \lambda_s > 0$, which is also true as $|\lambda_s| < 1$. Hence it is always possible to choose value $c$ such that the fixed point will be stabilized.

Similarly, one can show for the case $\lambda_s < 0$ that existence of $c$ satisfying (9) is equivalent to $\lambda_s < 1$. □

In the same way, we can prove the following corollary:

**Corollary 4** *If the Jacobian matrix of* $\mathbf{P}$ *is diagonal of the form* (14) *(i.e.* $\mathbf{A} = \mathbf{D}_2$*), then there does not exist* $c$ *for which* $\xi_F$ *is an asymptotically stable fixed point of the map* $\xi \mapsto \mathbf{P}(\xi, c\xi_1)$.

From the above corollary, it follows that stabilization of the fixed point is not possible if the Jacobian matrix takes the form of $\mathbf{D}_2$ and we use $\xi_1$ for computing the control signal. However, if we choose the second variable $\xi_2$, stabilization can be achieved.

The above considerations show that better results are obtained if the unstable eigenvector is parallel to the coordinate which we use for computing the control signal.

The most important conclusion which can be drawn from the results presented in this section is that the possibility of control using the OPF technique depends on the form of the linear approximation of the system's behavior which we use in the neighborhood of the periodic orbit.

(a) (b) (c)

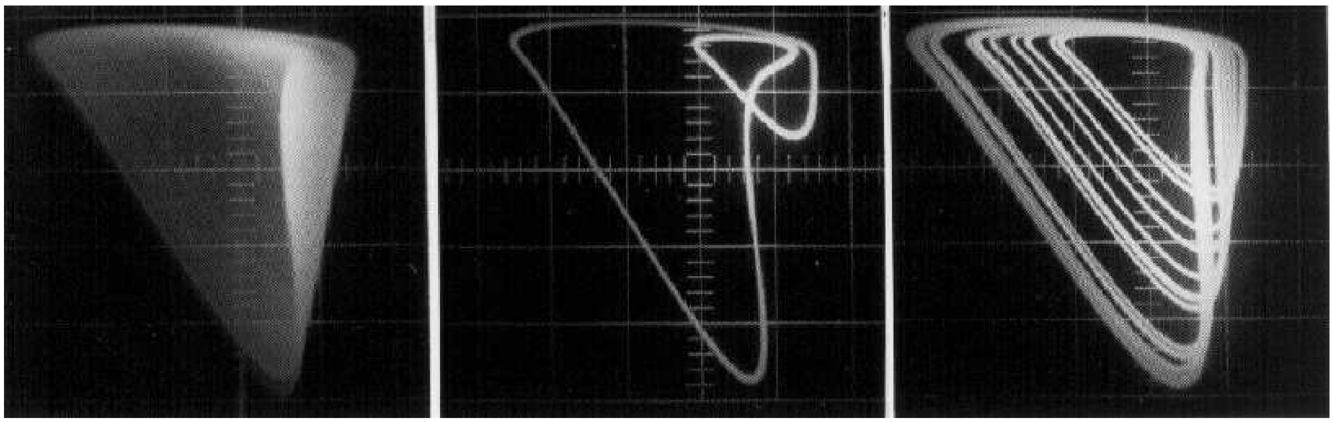

**Figure 4. (a) Chaotic attractor observed in the Colpitts oscillator, (b) stabilized period-1 orbit, (c) stabilized long periodic orbit**

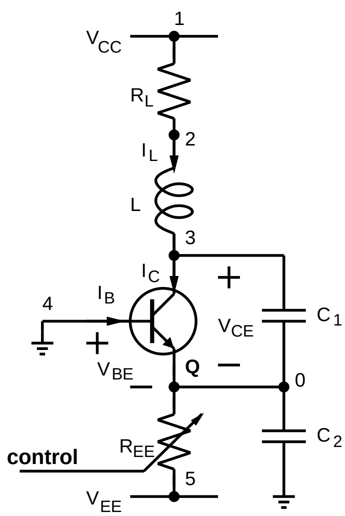


**Figure 3. Colpitts oscillator**

## 4. EXPERIMENTAL RESULTS

We tested our controller using a chaotic Colpitts oscillator [7, 8]. This circuit is shown in Fig. 3.

A typical chaotic attractor which can be observed in the Colpitts oscillator is shown in Fig. 4.a.

By choosing the emitter resistor $R_{EE}$ as the control parameter, we succeeded in stabilizing several periodic orbits of this system. The orbits shown in Fig. 4.b,cwere stabilized using a window of $0.4V$ placed at the bottom left hand side of the attractor. As with Hunt's implementation of OPF control, the orbits were found by trial and error.

## 5. CONCLUSIONS

In this paper, we have described a modified controller for stabilizing periodic orbits in chaotic systems using a modified OPF method. We have developed some analytical results concerning the possibilities for stabilizing periodic orbits in chaotic systems using one-dimensional methods. We stress that the only essential difference between the OGY and OPF methods is the number of variables used to determine the control signal. Furthermore, we have shown that the usual assumption, when using OPF, that the Poincaré map is almost one-dimensional is *unnecessary.* The only assumptions that we need in order to determine if a given periodic orbit can be stabilized are concerned with the dynamics of the system in the neighborhood of the chosen orbit.

## ACKNOWLEDGEMENT

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