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The Songling system has exactly four limit cycles

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ABSTRACT

Determining how many limit cycles a planar polynomial system of differential equations can have is a remarkably hard problem. One of the main difficulties is that the limit cycles can reside within areas of vastly different scales. This makes numerical explorations very hard to perform, requiring high precision computations, where the necessary precision is not known in advance. Using rigorous computations, we can dynamically determine the required precision, and localize all limit cycles of a given system. We prove that the Songling system of planar, quadratic polynomial differential equations has exactly four limit cycles. Furthermore, we give precise bounds for the positions of these limit cycles using rigorous computational methods based on interval arithmetic. The techniques presented here are applicable to the much wider class of real-analytic planar differential equations.

1. Introduction

In 1900, at the International Congress of Mathematics held in Paris, David Hilbert presented ten open problems in mathematics, and later published a more comprehensive list of 23 problems [1] aimed to challenge the mathematical community. Throughout the 20th century, these problems have received great attention, and still do. Today, most of the Hilbert problems have been resolved (two of them were deemed to be unresolvable), but a few ones still remain unsolved: one of these is Hilbert's 16th problem.

Hilbert's 16th problem has two distinct parts: one in real algebraic geometry, and one in dynamical systems. We will address the latter which asks for $H(n)$ – the maximal number of limit cycles (isolated periodic orbits) the family of two-dimensional polynomial vector fields of degree n can display. This problem has been highlighted in Steven Smale's list of challenging problems for the 21st century [2] (it appears as number 13 there), and is phrased roughly as follows:

Consider the differential equation in $\mathbb{R}^2$*:*

$$(\star)\begin{cases}\dot{x} = P_n(x,y),\\ \dot{y} = Q_n(x,y).\end{cases}$$

where P_n *and* Q_n *are polynomials of degree at most n. Is there a bound* $H(n)$ *on the number of limit cycles the system* $(\star)$ *can have, that only depends on the degree n?*

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Note that the bound $H(n)$ should be uniform, that is, it should not depend on the particular polynomial vector field, only on its degree n. Hilbert's 16 problem has a remarkable history, and finding upper bounds for $H(n)$ in the general case appears to be extremely hard. Indeed, very little progress has been made since Hilbert's seminal talk. As of today, this question is not resolved even in the simplest, quadratic, case ($n = 2$). What is known, is that any given *single* polynomial vector field can have only a finite number of limit cycles; this was (independently) proved in [3] and [4]. Partial results for the quadratic case, and a general introduction to the bifurcation theory of planar polynomial vector fields can be found in [5]. See also [6] for an overview of the problem. In terms of rigorous numerical studies of limit cycles of planar vector fields, an early paper is [7]. It uses a *rotated* vector field to form an annulus containing a given (approximate) limit cycle. The existence of a true limit cycle follows by verifying that the original vector field is transverse to the boundary of the annulus—this can be achieved by local (rigorous) computations, rather than by integrating the system. Note that the method presented in [7] cannot be used to prove the uniqness of a limit cycle in a specified region.

Even finding realistic *lower* bounds for $H(n)$ appears to be very hard; for some of the best known lower bounds, see [8,9] ($n = 2$), [10] ($n = 3$), and [11] ($n \geq 4$).

In this paper, we will focus on the Songling system; the three-parameter family of quadratic planar vector fields discussed in [8]. The system is defined by

$$\begin{aligned}\dot{x} &= \lambda x - y - 10x^2 + (5+\delta)xy + y^2,\\ \dot{y} &= x + x^2 + (-25 + 8\varepsilon - 9\delta)xy,\end{aligned} \tag{1}$$

where $\delta = -10^{-13}$, $\varepsilon = -10^{-52}$, and $\lambda = -10^{-200}$. In [8], it is proved that this system has *at least* four limit cycles. Normal form theory is applied in [12] to prove the uniqueness of periodic orbits in a neighborhood of the singular point (0,0).

The main goal of this work is to confirm that the system (1) supports *exactly* four limit cycles and to give precise bounds for positions of initial points of these limit cycles. Additionally, we would like to show that rigorous computational methods based on interval arithmetic [13] can be useful in studies related to the 16th Hilbert problem [4,6,14,15].

Theorem 1. *The Songling system (1) has exactly four limit cycles.*

Proof. From [12] it follows that we only have to prove that no limit cycles of (1) intersect the line segment $(x, y) \in \{0\} \times [0.004, 0.04]$. This is achieved by combining Lemma 8 with the enclosure of the largest limit cycle, given in Lemma 2. □

Computations reported in this work are carried out using the CAPD library [16]. The MPFR library [17] is used for the multiple-precision support. Computation times are reported for a single core 3.5 GHz processor.

Following the convention in the literature on interval arithmetic, we denote intervals by bold letters. For example the closed interval with the endpoints $x_l \leq x_r$ is denoted by $\mathbf{x} = [x_l, x_r]$. For the sake of brevity, we use a short notation to define intervals. For example 9.80749^{8}_{6} denotes the interval [9.807496,9.807498].

2. Preliminaries

The system (1) has two equilibria (0,0) and (0,1). Let us consider the line $\Sigma = \{(x, y) : x = 0\}$ containing both equilibria. For $(x, y) \in \Sigma$ we have

$$\begin{aligned}\dot{x} &= -y + y^2,\\ \dot{y} &= 0.\end{aligned} \tag{2}$$

From (2) it follows that within Σ the derivative $\dot{x}$ is negative for $y \in I = (0, 1)$ and positive for $y \notin [0, 1]$. It is known (by index theory) that each limit cycle has to surround a singular point. It follows that each periodic orbit of (1) has to intersect the interval $(0, I) \subset \Sigma$, where $(0, I)$ denotes a two-dimensional interval vector, with an equivalent notation $\{0\} \times I$.

Let us consider the return map $P : I \mapsto I$ defined as follows. For $y \in I$ the image $P(y)$ is defined by $(0, P(y)) = \phi(\tau(0, y), (0, y))$, where $\phi(t, (x, y))$ is a trajectory of (1) starting at (x, y) and $\tau(x, y)$ is the smallest positive number t such that $\phi(t, (x, y)) \in (0, I)$. If the trajectory $\phi(t, (0, y))$ does not return to $(0, I)$ then P is not defined on y.

3. The existence of limit cycles

In this section we show that there exist at least four fixed points of P, and we provide very tight enclosures of their positions.

3.1. Non-rigorous computations

First, let us study the dynamics of P using non-rigorous computations. During these computations a non-rigorous Taylor integration method with the order 100 and the absolute tolerance 10^{-300} is used. We select points $y_k = 10^{-k/3}$ for $k = 1, 2, \ldots, 300$, evaluate $P(y_k)$ and compute the difference $f(y_k) = y_k - P(y_k)$. The results are plotted in Fig. 1 in the logarithmic scale. One can see that f changes the sign four times in the interval $y \in [10^{-100}, 10^{-1/3}]$. The sign changes are observed in the intervals $[4.64 \cdot 10^{-75}, 10^{-74}]$, $[2.15 \cdot 10^{-21}, 4.65 \cdot 10^{-21}]$, $[4.64 \cdot 10^{-8}, 10^{-7}]$, $[2.15 \cdot 10^{-2}, 4.64 \cdot 10^{-2}]$. The first three sign changes are continuous and the jumps seen in the picture are caused by computing $f(y) = y - P(y)$ only at discrete values and using the logarithmic scale.

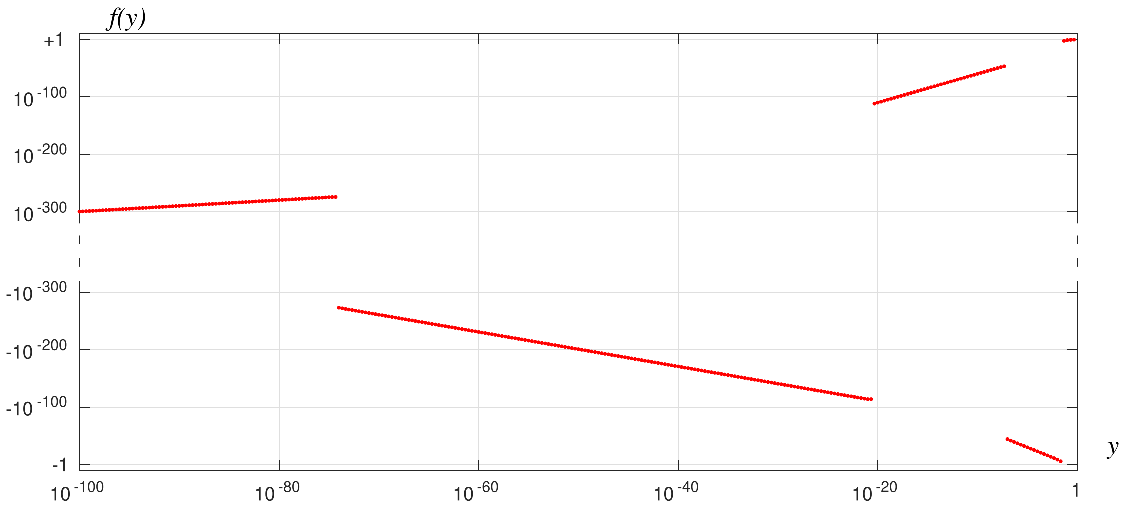

Fig. 1. Plot of the difference $f(y) = y - P(y)$ for $y \in [10^{-100}, 10^{-1/3}]$ in the logarithmic scale. Note that $f(y)$ changes sign four times.

In the interval $[2.15 \cdot 10^{-2}, 4.64 \cdot 10^{-2}]$ the return map is discontinuous. This interval contains a point $y \approx 0.03689$ whose trajectory escapes to infinity. However, the map f changes sign within a smaller interval $[4 \cdot 10^{-2}, 4.64 \cdot 10^{-2}]$ where f is continuous. Therefore, one may expect that $f(y)$ has four zeros in the interval $y \in [10^{-100}, 10^{-1/3}]$ which means that P has four fixed points in this interval. The results presented in Fig. 1 provide approximate positions of the fixed points which may be used as starting points for the Newton method to obtain better approximations.

3.2. Topological approach

The existence of a fixed point can be proved using the following topological lemma.

Lemma 1. *Let g be a continuous map defined on an interval $\mathbf{x}$. If either $g(\mathbf{x}) \subset \mathbf{x}$ or $\mathbf{x} \subset g(\mathbf{x})$, then g has a fixed point in $\mathbf{x}$.*

In order to carry out the proof of existence of a fixed point of P in $\mathbf{y} = [y_l, y_r]$, one has to find enclosures of $P(y_l)$, $P(y_r)$, and show that certain inequalities regarding these enclosures and the endpoints of $\mathbf{y}$ are satisfied. Additionally, one has to prove that P is well defined on $[y_l, y_r]$. This can be done by finding an enclosure of $P([y_l, y_r])$. All these computations can be carried out using interval arithmetic tools for the rigorous integration of nonlinear vector fields.

The existence of four fixed points of P is formulated in the following lemma.

Lemma 2. *Each of the intervals*
$[y_{1l}, y_{1r}] = 0.0426896038820^{85}_{75}$,
$[y_{2l}, y_{2r}] = 6.666660148^{2}_{1} \cdot 10^{-8}$,
$[y_{3l}, y_{3r}] = 2.24780594^{8}_{7} \cdot 10^{-21}$, *and*
$[y_{4l}, y_{4r}] = 7.07106781186547524^{5}_{4} \cdot 10^{-75}$
contains a fixed point of P.

Proof. The following inequalities are verified:
$P(y_{1l}) - y_{1l} < -5.06 \cdot 10^{-15}$, $P(y_{1r}) - y_{1r} > 4.93 \cdot 10^{-15}$,
$P(y_{2l}) - y_{2l} > 2.58 \cdot 10^{-57}$, $P(y_{2r}) - y_{2r} < -2.32 \cdot 10^{-57}$,
$P(y_{3l}) - y_{3l} < -5.05 \cdot 10^{-123}$, $P(y_{3r}) - y_{3r} > 1.29 \cdot 10^{-123}$,
$P(y_{4l}) - y_{4l} > 5.03 \cdot 10^{-295}$, and $P(y_{4r}) - y_{4r} < -6.23 \cdot 10^{-293}$.
For each $i = 1, 2, 3, 4$ an enclosure of $P([y_{il}, y_{ir}])$ is found which proves that P is well defined on $[y_{il}, y_{ir}]$. The assertion follows. □

The computations are carried out using the CAPD library. Multiple-precision interval computations with the precision of up to 1024 bits are used. The total computation time per fixed point varies from 4 s to 14 s. We illustrate the associated limit cycles in Fig. 2.

3.3. The interval Newton method approach

In this section, we present results on the existence and uniqueness of fixed points of P obtained using the interval Newton operator.

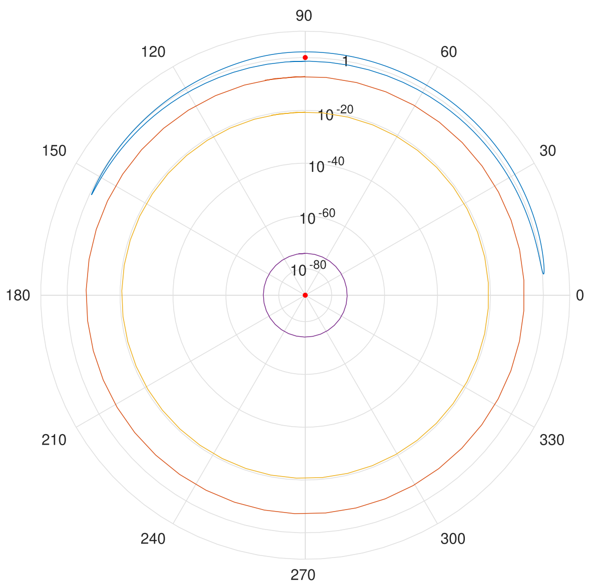

Fig. 2. Polar plot of the four limit cycles of Lemma 2. The radius is in logarithmic scale. The two equilibria are plotted as red dots. Note that three of the four limit cycles surround the equilibrium at the origin. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The interval Newton operator for the (continuously differentiable) map $f : \mathbb{R} \mapsto \mathbb{R}$ and the interval $\mathbf{y}$ is defined as

$$N(\mathbf{y}) = y - f(y)/f'(\mathbf{y}), \tag{3}$$

where $y \in \mathbf{y}$. The most important property of the interval Newton operation states that if $N(\mathbf{y})$ is enclosed in the interior of $\mathbf{y}$ then the interval $\mathbf{y}$ contains exactly one zero of f. To prove the existence of fixed points of P one applies the interval Newton operator to the map $f(y) = y - P(y)$ and verifies that the condition $N(\mathbf{y}) \subset \operatorname{int}(\mathbf{y})$ holds. In this case the interval Newton operator has the form $N(\mathbf{y}) = y - (y - P(y))/(1 - P'(\mathbf{y}))$. Let us note that in this approach one needs rigorous enclosures of both $P(y)$ and $P'(\mathbf{y})$. Based on an enclosure of $P'(\mathbf{y})$ one may state what is the stability type of the fixed point. If $|P'(\mathbf{y})| < 1$ ($|P'(\mathbf{y})| > 1$) then the fixed point is stable (unstable). Once the existence of a fixed point is proved one may iterate the interval Newton operator to obtain very accurate enclosures for the position of this fixed point.

The following lemma presents results on stability types and bounds for positions of four fixed points of P.

Lemma 3. *The interval*

$\mathbf{y}_1 = 0.04268960388208006198429597530554296558700142980749^{8}_{6}$

contains a single (stable) fixed point of P. $P'(\mathbf{y}_1) \subset 9.11^{9}_{8} \cdot 10^{-5}$.

The interval

$\mathbf{y}_2 = 6.66660148152650573950698517996479316281685290427336417127406^{91}_{88} \cdot 10^{-8}$ *contains a single (unstable) fixed point of P.*

$P'(\mathbf{y}_2) \subset 1.000000000000000000000000000000000000000049^{2}_{1}$.

The interval

$\mathbf{y}_3 = 2.24780594779613058605838861895742013017443794174379153303325244559752998777079469200937 80^{3}_{1} \cdot 10^{-21}$

contains a single (stable) fixed point of P.

$P'(\mathbf{y}_3) \subset 0.99999999999999999999999999999999999999999999999999999999999999999999999999999999999999999999999936^{6}_{5}$

The interval

$\mathbf{y}_4 = 7.07106781186547524400844362104849039284835937688474036588339868995366239157720186091861933830487 45303417359328082824292486479850624247482966280728718679061243 5^{6}_{4} \cdot 10^{-75}$ *contains a single (unstable) fixed point of P.*

$P'(\mathbf{y}_4) \subset 1.00062^{9}_{8}$.

One can see that for fixed points in $\mathbf{y}_{2,3,4}$ the derivative of P is very close to one. As a consequence a very accurate integration method has to be used. For the numerical integration the rigorous Taylor integration method with the order 100 is used. During the proof the computations are carried out using multiple-precision arithmetic with up to 2048 bits. The computation time for a single fixed point varies from 45 s to 40 min.

4. The uniqueness of limit cycles

In [12], it is shown that there are exactly four fixed points of P outside the segment [0.004,0.04]. The existence of exactly 3 fixed points of P in the segment (0,0.004) is proved using the normal form theory. The existence of a single fixed point in the segment (0.04,1) follows from the fact that the line $1+x-25y=0$ which intersects the y axis at the point (0,0.04) is transversal to the vector field and the property of the Songling system that it has at most one limit cycle around one of the equilibria (for more details see [12]).

In order to prove that the Songling system has exactly four limit cycles it remains to show that P has no fixed points in the segment [0.004,0.04]. In the following, using interval arithmetic tools we show that P has a single fixed point in the segment [0.001,1). From Lemma 3 we know that this fixed point is not in [0.004,0.04].

During the computer assisted proof we will use the following technical lemmas.

Lemma 4. *If $\bar{y} < P(\bar{y})$ then there are no fixed points of P in the segment $[\bar{y}, P(\bar{y})]$. If $\bar{y} > P(\bar{y})$ then there are no fixed points of P in the segment $[P(\bar{y}), \bar{y}]$.*

Proof. Let us consider the case $\bar{y} < P(\bar{y})$. Let us assume that $P(y) = y$ for $y \in [\bar{y}, P(\bar{y})]$. It follows that $y \neq \bar{y}$ and trajectories $\phi(t, (0, y))$ and $\phi(t, (0, \bar{y}))$ intersect for $t \in [0, \tau(0, y)]$, which contradicts the uniqueness of solutions of (1). □

A similar result can be formulated for the inverse of P.

Lemma 5. *If $\bar{y} < P^{-1}(\bar{y})$ then there are no fixed points of P in the segment $[\bar{y}, P^{-1}(\bar{y})]$. If $\bar{y} > P^{-1}(\bar{y})$ then there are no fixed points of P in the segment $[P^{-1}(\bar{y}), \bar{y}]$.*

Using Lemma 4 one may construct a sequence of points $y_0 < y_1 < y_2 < \ldots < y_n$ such that the segment $[y_0, y_n]$ does not contain fixed points of P or a sequence of points $y_0 > y_1 > y_2 > \ldots > y_n$ such that the segment $[y_n, y_0]$ does not contain fixed points of P. This is achieved by selecting a point $y_0 \in (0, 1)$ and computing rigorous bounds for its images under P. To illustrate this procedure let us assume that $P(y_0) > y_0$. In the kth step of the procedure using rigorous computations we obtain an enclosure $[y_{k+1,l}, y_{k+1,r}]$ of $P(y_k)$ and we select $y_{k+1} = y_{k+1,l}$. After n steps we obtain the point y_n. It follows from Lemma 4 that the interval $[y_0, y_n]$ does not contain fixed points of P. For the case $P(y_0) < y_0$ we select $y_{k+1} = y_{k+1,r}$. After n steps we obtain the interval $[y_n, y_0]$ with no fixed points of P. A similar approach may be used to construct intervals not containing fixed points using Lemma 5. We will refer to the approach to exclude the existence of fixed points of P based on Lemmas 4 and 5 as the *iteration based method*.

Let us note that using Lemmas 4 and 5 one can exclude the existence of fixed points of P also in intervals containing points where P is not defined. For the Songling system one can show that the trajectories $\phi(t, (0, 0.03689093))$ and $\phi(t, (0, 0.03689096))$ intersect the line Σ at $y < 0$ and $y > 1$, respectively. It follows that the interval $[0.03689093, 0.03689096]$ contains a point y^* such that the trajectory $\phi(t, (0, y^*))$ escapes to infinity (P is not defined at y^*). On the other hand, computing the trajectory starting at (0,0.364) one may show that $P(0.0364) > 0.3766$. From Lemma 4 it follows that there are no fixed points of P in $[0.0364, 0.3766] \supset [0.03689093, 0.03689096] \ni y^*$.

It will be shown that the iteration based method is not efficient for small y. An alternative approach which will be called the *derivative based method* is based on the evaluation of P'. In this approach we will use the following lemma.

Lemma 6. *Let us assume that $1 \notin P'(\mathbf{y})$ where $\mathbf{y} = [y_l, y_r] \subset (0, 1)$. If $(y_l - P(y_l))(y_r - P(y_r)) > 0$ then $P(y) \neq y$ for $y \in \mathbf{y}$. If $(y_l - P(y_l))(y_r - P(y_r)) < 0$ then $\mathbf{y}$ contains a single fixed point of P.*

Proof. From the assumption $1 \notin P'(\mathbf{y})$ it follows that the function $f(y) = y - P(y)$ is strictly monotonic in $\mathbf{y}$. If $f(y_l) = y_l - P(y_l)$ and $f(y_r) = y_r - P(y_r)$ are of the same sign then f has no zeros in $\mathbf{y}$, which means that P has no fixed points in this interval. If $f(y_l)$ and $f(y_r)$ are of opposite signs then from the monotonicity of f it follows that f has a single zero in $\mathbf{y}$ and hence that P has a single fixed point in $\mathbf{y}$. □

To prove that P has no fixed points in the interval $\mathbf{y} = [y_l, y_r]$ two conditions have to be verified. The first condition $(P(y_l) - y_l)(P(y_r) - y_r) > 0$ requires evaluation of P over y_l and y_r and is usually easy to verify. The second condition requires evaluation of the derivative P' over the whole interval $\mathbf{y}$ and verifying that the result does not contain 1. This can be done in a single evaluation of P' only for small intervals. For larger intervals one may split the interval $\mathbf{y}$ into several smaller intervals and verify the condition $P'(y) \neq 1$ separately for each of them.

For the proof that the interval [0.001,1) contains a single fixed point of P we use the combination of the iteration based method, the derivative based method and the Lyapunov function method. First, using the Lyapunov function method, we show that there are no fixed points in the segment [0.98,1). More specifically, we show that P^{-1} is increasing in the segment [0.98,1).

Lemma 7. $P^{-1}(y) > y$ *for* $y \in [0.98, 1)$.

Proof. The change of variables $z = y - 1$ shifts the equilibrium (0,1) to the origin. In these variables the Songling system has the form:

$$\dot{x} = cx + z - 10x^2 + axz + z^2,$$
$$\dot{z} = -dx + x^2 + bxz,$$

where $a = 5 + \delta$, $b = -25 + 8\varepsilon - 9\delta$, $c = 5 + \delta + \lambda$, $d = 24 - 8\varepsilon + 9\delta$. Let us define the Lyapunov function

$$V(x, z) = 0.5d((d + 1)x + cz)^2 + 0.5\big(c^2 + (d + 1)^2\big)z^2.$$

The Lyapunov function is nonnegative and vanishes only at the origin. The derivative of V with respect to t is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= cd(d+1)(x^2+z^2) + d(d+1)(c - 10(d+1))x^3 + cd(d+1)z^3 \\ &\quad +((b+d)(d+1)^2 + (bc^2+acd)(d+1))xz^2 + (d(d+1)(bc + (a+c^2-10cd)(d+1)) + (d+1)^2)x^2z.\end{aligned}$$

Let us assume that $x^2 + z^2 = r^2$. The first term is equal to $cd(d + 1)r^2$. The absolute value of the remaining terms can be bounded by er^3, where $e = 1.482 \cdot 10^5$. It follows that if $r < cd(d + 1)/e \approx 0.02026$ then $\frac{\mathrm{d}V}{\mathrm{d}t}$ is positive and the Lyapunov function $V(x, z)$ is increasing along trajectories.

Let us come back to the Songling system. It can be verified that $P^{-1}(0.98) > 0.995$ and that during the evaluation of $P^{-1}(0.98)$ the trajectory does not leave the circle centered at (0,1) with the radius 0.02026. It follows that $V(x, y - 1)$ decreases during the evaluation of $P^{-1}(y)$ for each $y \in [0.98, 1)$ and hence $V(0, y - 1) > V(0, P^{-1}(y) - 1)$, i.e.,

$$(y - 1)^2 > (P^{-1}(y) - 1)^2. \tag{4}$$

It follows that $P^{-1}(y) > y$ for $y \in [0.98, 1)$. □

Lemma 8. *The segment* $[0.027, 0.99]$ *contains a single fixed point of* P.

Proof. Applying the iteration based method for the map P with $y_1 = 0.98$ and $n = 4$ we obtain $y_4 < 0.042689603882$. It follows from Lemma 4 that the interval [0.042689603882,0.99] does not contain fixed points of P.

Applying the iteration based method for the map P with $y_1 = 0.027$ and $n = 351$ we obtain $y_{351} > 0.042689603881$. It follows that the interval [0.027,0.042689603881] does not contain fixed points of P.

It remains to show that the segment 0.04268960388^2_1 contains a single fixed point of P. Let us consider the segment 0.0426^{90}_{89}. One can show that $P(0.042689) > 4.26896038$ and that $P(0.042690) > 4.26896039$. It follows that the segment 0.0426^{90}_{89} contains at least one fixed point of P. One can also show that $1 \notin P'(0.0426^{90}_{89})$. From Lemma 6 it follows that the segment 0.0426^{90}_{89} contains a single fixed point of P. □

Let us note that the iteration based method can be used to handle regions very close to $y = 1$. For example applying this procedure for the map P^{-1} with $y_1 = 0.99$ and $n = 10$ we obtain $y_{10} > 0.9999999999999999993$. It follows from Lemma 5 that the interval [0.99,0.9999999999999999993] does not contain fixed points of P.

The iteration based method is not efficient for $y < 0.027$ due to the fact that the distance $|y - P(y)|$ decreases fast when y is decreased. For example to carry out the proof for the segment [0.020,0.021] one needs 3210 evaluations of P and the computation time is 15 min. For the segment [0.010,0.011] the number of evaluations exceeds $1.4 \cdot 10^6$ and the computation time is 65 h.

As it has been mentioned before an alternative approach is based on the evaluation of P' (see Lemma 6 and the following discussion). This method allows to reduce the number of evaluations when compared with the iteration based method. For example for the segment [0.01,0.011] the number of evaluations is two times smaller. However the computation time is longer because the evaluation of P' takes longer than the evaluation of P.

A better approach is to use Lemma 6 after a change of coordinates. We will use the polar coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$ (compare also [12]). From $r\dot{r} = x\dot{x} + y\dot{y}$ one obtains

$$\dot{r} = \dot{x}\cos\varphi + \dot{y}\sin\varphi = \lambda r\cos^2\varphi + r^2\cos\varphi\big((6 + d)\cos\varphi\sin\varphi - (14 + 9\delta - 8\varepsilon)\sin^2\varphi - 10)\big).$$

From $\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$ and $\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$ one obtains

$$\dot{\varphi} = \dot{y}\cos\varphi - \dot{x}\sin\varphi = 1 - \lambda\cos\varphi\sin\varphi + r(6 + \delta)\cos^3\varphi + r\big((8\varepsilon - 9\delta - 14)\cos^2\varphi\sin\varphi - \sin\varphi - (5 + \delta)\cos\varphi\big).$$

It follows that the Songling system in the polar coordinates is defined as

$$\begin{aligned}\dot{r} &= r\Lambda_1(\varphi) + r^2R(\varphi),\\ \dot{\varphi} &= 1 - \Lambda_2(\varphi) + r\Phi(\varphi),\end{aligned} \tag{5}$$

where

$$\begin{aligned}&\Lambda_1(\varphi) = \lambda\cos^2\varphi, \quad \Lambda_2(\varphi) = \lambda\cos\varphi\sin\varphi, \quad R(\varphi) = \cos\varphi\big((6+d)\cos\varphi\sin\varphi - (14+9\delta-8\varepsilon)\sin^2\varphi - 10\big),\\ &\Phi(\varphi) = (8\varepsilon - 9\delta - 14)\cos^2\varphi\sin\varphi - \sin\varphi - (5 + \delta)\cos\varphi + (6 + \delta)\cos^3\varphi.\end{aligned}$$

Lemma 9. $P(y) > y$ *for* $y \in [0.001, 0.027]$.

Proof. In the proof we use Lemma 6. First, it is verified that $P(0.027) > 0.027$ and $P(0.001) > 0.001$. It remains to show that $1 \notin P'([0.001, 0.027])$. The evaluation of $P'(y)$ is carried out in the polar coordinates. From (5) it follows that φ grows as long as

$$r < \frac{1 - \max_\varphi \Lambda_2(\varphi)}{\max_\varphi |\Phi(\varphi)|}.$$

One can see that $|\Lambda_2(\varphi)| < 0.5\lambda$. Using interval arithmetic tools we show that $|\Phi(\varphi)| < c = 7.1$. Using the bisection method the interval $t \in [0, 2\pi]$ is divided into 1288 subintervals $\mathbf{t}_i$. Enclosures $\mathbf{z}_i$ of $\Phi(\mathbf{t}_i)$ are computed and it is verified that $|\mathbf{z}_i| < c$. It follows that $\dot{\varphi}$ is positive (i.e. φ grows) when

$$r \le 0.14 < \frac{1 - 0.5\lambda}{c}.$$

Next, we verify that during the evaluation of $P(0.027)$ the trajectory does not leave the circle centered at the origin with the radius 0.14. It follows that for $y \in (0, 0.027]$ the return map $P(y)$ can be computed by integrating the one-dimensional vector field

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{r\Lambda_1(\varphi) + r^2 R(\varphi)}{1 - \Lambda_2(\varphi) + r\Phi(\varphi)} \tag{6}$$

for $\varphi \in [0.5\pi, 2.5\pi]$ with the initial condition $r(\varphi = 0.5\pi) = y$.

The bisection method is used to split the interval [0.001,0.027] into 20809 subintervals. For each subinterval the map P and its derivative are computed by integrating the vector field (6) for $\varphi \in [0.5\pi, 2.5\pi]$ and it is proved that $P'(y) \neq 1$ for $y \in [0.001, 0.027]$. The total computation time is 40 h. The assertion follows from Lemma 6. □

Let us note that the method to evaluate P' which is used in the proof of Lemma 9 is much faster than the standard approach in which the vector field (1) is integrated. For example in the case of the interval $\mathbf{y} = [0.010, 0.011]$ the number of evaluations needed to prove that $1 \notin P'(\mathbf{y})$ is 181 and the computation time is 8 min compared to more than $1.4 \cdot 10^6$ evaluations and 65 h of computations for the standard approach. The most fine splitting of the interval [0.001,0.027] is necessary close to the endpoint 0.001. For example the calculations involving the interval [0.001,0.002] with the width being less than 4% of the total width took 70% of the computation time.

From Lemmas 7, 8, and 9 it follows that the segment [0.001,1) contains a single fixed points of P.

Additionally, we prove that there are no fixed points of P in a neighborhood of the origin.

Lemma 10. *$P(y) < y$ for $y \in (0, 2 \cdot 10^{-202}]$.*

Proof. The Songling system can be written as:

$$\dot{x} = \lambda x - y - 10x^2 + axy + y^2,$$
$$\dot{y} = x + x^2 + bxy,$$

where $a = 5 + \delta$, $b = -25 + 8\varepsilon - 9\delta$. Let us define the Lyapunov function

$$V(x, y) = (1 + 0.25\lambda^2)y^2 + (x - 0.5\lambda y)^2.$$

The Lyapunov function is nonnegative and $V(x, y) = 0$ only at the origin. The derivative of V with respect to t is

$$\tfrac{\mathrm{d}V}{\mathrm{d}t} = \lambda\big(x^2 + y^2\big) - (20 + \lambda)x^3 - \lambda y^3 + \big(2 + 2b + b\lambda^2 - a\lambda\big)xy^2 + \big(2 + 2a + \lambda^2 - b\lambda + 10\lambda\big)x^2 y.$$

Let $r = \sqrt{x^2 + y^2}$. The first term is equal to λr^2 (recall that λ is negative). The absolute value of the remaining terms can be bounded by $40r^3$. It follows that for $r < |\lambda|/40 = 2.5 \cdot 10^{-202}$ the Lyapunov function $V(x, y)$ is decreasing along trajectories.

In the next step, we verify that $P(2 \cdot 10^{-202}) < 2 \cdot 10^{-202}$ and that during the evaluation of $P(2 \cdot 10^{-202})$ the trajectory does not leave the circle centered at the origin with the radius $2.5 \cdot 10^{-202}$. It follows that for $y \in (0, 2 \cdot 10^{-202}]$ the Lyapunov function $V(x, y)$ decreases during the evaluation of $P(y)$ and hence $V(0, y) > V(0, P(y))$, i.e. $y^2 > P(y)^2$. The assertion follows. □

5. Conclusions

As mentioned in the introduction, this paper has two main goals: (1) to prove that the Songling system has exactly four limit cycles, and (2) to illustrate the powers (and potential future use) of rigorous computations based on interval arithmetic.

For the problem we are considering here, the rigorous computations show their strength in producing coarse and tight enclosures of limit cycles, as illustrated in Lemma 2 and Lemma 3, respectively. The same lemmas also give local uniqueness results within each enclosure, leading to an exact count of the limit cycles. The weakness of the same computational techniques lies in proving *non*-existence of limit cycles. This is a global problem, and as such requires much more computational effort. Small neighbourhoods of 0 and 1 are handled analytically by Lyapunov function methods, as in Lemma 10 and Lemma 7, respectively. But for the remaining sectors, our methods were not practical. As an illustration, Lemma 8 (clearing the interval $[2.7 \cdot 10^{-2}, 9.9 \cdot 10^{-1}]$ from fixed points of P) required 15 min to compute, whereas Lemma 9 (clearing $[10^{-3}, 2.7 \cdot 10^{-2}]$) took 40 h.

Although we could (in principle) exhaust the remaining interval $[2 \cdot 10^{-202}, 10^{-3}]$, it would have required an enormous amount of computing power. Instead, we used the analytical normal form result of [12], which clears the larger interval $[0.0, 4 \cdot 10^{-3}]$ from fixed points of P.

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