Robustness of symbolic dynamics and synchronization properties

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Abstract

In this paper we introduce the method for investigation of coupled chaotic systems using topological methods. We show that if the coupling is small then there exists independent symbolic dynamics for every coupled subsystem and in consequence the systems are not synchronized. As an example we consider coupled Hénon maps. Using computer interval arithmetic we find parameter mismatch and perturbation range for which the symbolic dynamics in the Hénon system is sustained. For coupled Hénon maps we compute the value of coupling strength for which the symbolic dynamics in every subsystem survives.

1 Introduction

It is well known that when chaotic systems are coupled, they may demonstrate synchronized behavior. Recently there has been a considerable interest in using the concept of synchronization of chaos for solving technical problems. For the applications it is very important to find techniques for investigation of the phenomenon of synchronization of chaotic systems.

There are several methods for studying the synchronization problem. It was shown in various papers that a very important role is played by the transversal Lyapunov exponents of the synchronized trajectories [Pecora & Carroll, 1990, Ogorzałek, 1993, Heagy et al., 1994]. It was shown that the criterion based on conditional Lyapunov exponents calculated along a typical trajectory of the system is not sufficient and one has to take into account transversal Lyapunov exponents computed along all periodic orbits [Heagy et al., 1995, Pecora et al., 1995]. Other methods are based on local transversal Lyapunov exponents [Pecora et al., 1995, Galias, 1998*a*].

In this paper we describe the method of investigation of coupled systems using topological methods. We are interested in the case when synchronization of chaotic systems is not observed due to the existence of independent symbolic dynamics in coupled subsystems. We show that if the coupling is small then one may prove that there exists independent symbolic dynamics for every coupled subsystem. This means that for two different sequences of symbols one may found trajectory of the coupled system which realizes these two sequences in the coupled subsystems. As consequence we obtain the coexistence of different periodic solutions in different subsystems. In this context the existence of independent symbolic dynamics for different subsystems implies the lack of synchronization.

In Sec. 2 we introduce the method for studying of synchronization properties by means of independent symbolic dynamics. In Sec. 3 we recall results on the existence of symbolic dynamics for the Hénon map. In Sec. 4 we study the robustness of symbolic dynamics on parameters and perturbation added to the system. In Sec. 5 we analyze coupled Hénon maps using the results from Sec. 4.

2 Symbolic dynamics and synchronization

Chaotic systems are often studied in terms of symbolic dynamics and horseshoes which are one of the most important and descriptive tools available. We say that for a given system there exist a symbolic dynamics on n symbols if there are n disjoint sets $N_0 \ldots N_{n-1}$ and a finite type subshift on n symbols $\{0, 1, \ldots, n-1\}$ such that for every sequence $(s_k)_{k=0}^{\infty}$ allowable by this subshift there exist a trajectory $(x_k)_{k=0}^{\infty}$ of the system such that $x_k \in N_{s_k}$ for $k = 0, 1, \ldots$. This corresponds to the existence of a set (invariant part of $N_0 \cup \cdots \cup N_{n-1}$) such that the dynamics of the system restricted to this set is semiconjugate with the given subshift. We are interested in the case when the set of allowable sequences has infinite number of elements. In this case the embedded set is of a Cantor type and the system displays complex dynamics.

One can rigorously prove the existence of symbolic dynamics in nonlinear maps using topological methods. One has to find sets N_i and check that the images of these sets lie properly with respect to the initial sets. Hence, in order to show the existence of symbolic dynamics one has to prove that the images of certain sets in the phase space are enclosed in certain regions in this space. This can be done by means of interval arithmetic implemented on a computer.

In our earlier work we have used computer interval arithmetic to perform a rigorous computer assisted proof of the existence of a (partial) horseshoe for discrete time systems (for example the Hénon map [Zgliczyński, 1997, Galias, 1998*b*]) and also for continuous time systems (Chua's circuit [Galias, 1997], Lorenz equations [Galias & Zgliczyński, 1998], Rössler equations [Zgliczyński, 1997]). For flows we first reduce the problem to discrete–time by means of a Poincaré map technique.

In current work we perform a sensitivity analysis. We consider a perturbed chaotic system and show that symbolic dynamics is not destroyed by a small enough perturbation. For the applications we want to prove the existence of symbolic dynamics for as large perturbation as possible.

For finding the allowable perturbation we adapt the computer assisted proof of the existence of a (partial) horseshoe. The perturbation is represented by an interval vector which modifies the dynamics of the map. We propose to start with a large interval vector and then using the method of generalized bisection to find regions in the perturbation space for which the symbolic dynamics exists. For a given perturbation we try to prove the existence of symbolic dynamics. If the proof fails we divide the interval vector representing the perturbation into several smaller interval vectors and try to complete the proof again.

One should notice that this method allows to find sufficient conditions for the existence of symbolic dynamics. It is possible that the computer assisted proof fails to show the existence of symbolic dynamics of a given type but one continues to exist.

Another approach to the problem of the existence of symbolic dynamics and its robustness was reported in [Mischaikow et al., 1999], where the authors show symbolic dynamics in experimental time-series from a magnetoelastic ribbon under the assumption that the experimental error and the noise are bounded.

The method for finding the perturbation level not destroying the symbolic dynamics described above is a very general technique and may find applications in many different areas. Here we use it for investigation of behavior of coupled chaotic systems. If chaotic systems are coupled and the coupling strength is small, then we expect that the symbolic dynamics is not destroyed by the coupling. The method for studying of coupled systems consists of two steps. First for every subsystem we find the perturbation range for which we can verify that the symbolic dynamics is not destroyed. Then for each subsystem we check that the perturbation introduced by the coupling is smaller than the maximum allowable perturbation. If this is true then the independent symbolic dynamics in every subsystem exists. In consequence, the subsystems are not uniformly synchronized in the sense that there exist trajectories of the whole system realizing arbitrary allowable sequences in every subsystem. We would like to stress that the embedded symbolic dynamics is usually associated with unstable Cantor–like chaotic saddles of zero measure. Hence we can make no conclusions concerning synchronization for the typical initial conditions.

In the subsequent sections we use this method for analysis of coupled chaotic systems. For the sake of simplicity we will consider a discrete–time system, namely the Hénon map. This method can be also used for continuous–time systems, but in this last case the method can be computationally expensive due to the necessity of evaluation of the Poincaré map. Application of this technique to continuous–time systems will be reported elsewhere.

3 Symbolic dynamics for h^2 and h^7

As an example we consider the Hénon map [Hénon, 1976] defined by the following equation:

$$h(x, y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$ are the "classical" parameter values for which the famous Hénon attractor is observed.

In this section we recall the results on the existence of symbolic dynamics for h^2 and h^7.

3.1 Symbolic dynamics for h^2

In [Galias, 1998*b*] it was shown that there exists symbolic dynamics embedded in h^2 corresponding to the golden subshift on two symbols (partial or deformed horseshoe). The sets N_i and E_i are shown in Fig. 1(a). For the exact definition see [Galias, 1998*b*]. It was shown that the images of vertical edges of N_0 under h^2 are enclosed in E_0 and E_2 on the opposite sides of $N_0 \cup N_1$ and that the images of vertical edges of N_1 under h^2 are enclosed in E_0 and E_1 on the opposite sides of N_0. It was also shown that images of horizontal edges of N_0 and N_1 under h^2 are enclosed in the interior of the topological stripe $E_0 \cup N_0 \cup E_1 \cup N_1 \cup E_2$. We say that $h^2(N_0)$ covers N_0 and N_1 horizontally and $h^2(N_1)$ covers N_0 horizontally. It follows that for every sequence of symbols $(a_0, a_1, \ldots, a_{n-1})$, from the set $\{0, 1\}$ which does not contain the subsequence $(1, 1)$ there exists a point $z = (x, y)$ such that $h^{2i}(z) \in N_{a_i}$ for $i = 0, \ldots, n-1$ and $h^{2n}(z) = z$. One should notice that we do not check hyperbolicity on the sets N_i. Therefore we cannot state that each infinite symbolic sequence identifies exactly one trajectory. There may exist many orbits in the phase space which project onto a given symbolic sequence. For details see [Galias, 1998*b*].

In this way it was shown that the subshift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is embedded in h^2.

3.2 Symbolic dynamics for h^7

In [Zgliczyński, 1997] it was shown that there exist symbolic dynamics embedded in h^7 corresponding two the full shift on two symbols (full horseshoe).

The sets N_i and E_i are shown in Fig 1(b). It was shown that for $i = 0, 1$ the images of vertical edges of N_i under h^7 lie on the opposite sides of $N_0 \cup N_1$ (are enclosed in E_0 and E_2). It was also shown that the images of horizontal edges under h^7 are enclosed in the interior of topological stripe defined by the sets N_i and E_i. Each of the sets $h^7(N_0)$ and $h^7(N_1)$ covers N_0 and N_1 horizontally. For the details see [Zgliczyński, 1997] or [Galias, 1998*b*]. It follows that for every sequence of symbols $a = (a_0, a_1, \ldots, a_{n-1})$ from the set $\{0, 1\}$ there exists at least one point $z = (x, y)$ such that $h^{7i}(z) \in N_{a_i}$ for $i = 0, \ldots, n-1$ and $h^{7n}(z) = z$. In other words the symbolic dynamics corresponding to the full shift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is embedded in h^7.

4 Robustness of symbolic dynamics

In this section we study robustness of symbolic dynamics for the Hénon map.

The first question we address is whether the symbolic dynamics survives if the parameters of the map are modified.

Using the sets N_i, E_i plotted in Fig.1 we have checked whether for different values of (a, b) the assumptions of the theorem on the existence of symbolic dynamics hold. In order to perform this task we have used the following generalized bisection procedure. We start with intervals $\mathbf{a} = [1.2, 1.6]$ and $\mathbf{b} = [0.1, 0.5]$ and try to prove the existence of symbolic dynamics for these intervals. If we do not succeed we divide the rectangle $\mathbf{a} \times \mathbf{b}$ into 4 rectangles and repeat the procedure. We do not divide the rectangle if the lengths of its edges are smaller than $\varepsilon = 0.001$.

It follows from the definition of the Hénon map that the images of N_i change continuously with the parameter changes and hence we can use the bisection method.

In Fig. 2(a) we show the rectangles (a, b) for which we have proved the existence of symbolic dynamics for h^2. Similarly in Fig. 2(b) we show regions in the parameter space for which the symbolic dynamics for h^7 exists. It is interesting to note that the symbolic dynamics is present in the dynamics of the map even for parameter values far from the standard ones. For example the symbolic dynamics for h^2 exists also for $a = 1.6$, $b = 0.35$ and the symbolic dynamics for h^7 exists for $a = 1.55$, $b = 0.1$.

The above results may be used for studying synchronization of systems when the synchronization signal is introduced via parameter modification.

The second problem we investigate is the existence of symbolic dynamics in the case when the dynamics of the map is perturbed by some additive signal. We assume that we only know the upper limit of the absolute value of this perturbation. We consider a perturbed system

$$h_p(x, y) = (1 + y - ax^2 + e_1, bx + e_2), \tag{2}$$

Using interval arithmetic we have found pairs (e_1, e_2) for which the symbolic dynamics is not destroyed by the perturbation. In order to prove the existence of symbolic dynamics for particular values of e_1 and e_2 we check the assumptions of the existence theorem for the map (2) (we check that the images of edges of N_i lie properly with respect to the sets N_i, E_i).

As an example in Fig. 3 we show these images for the intervals $\mathbf{e}_1 = \mathbf{e}_2 = [-0.012, 0.012]$. Vertical edges of N_i were covered by 2, 2, 2, and 7 rectangles respectively and horizontal edges were covered by 27, 46, 6, 7 rectangles respectively. The images of these rectangles under the map h^2 were computed are we have checked that they lie in a proper way with respect to the sets N_i and E_i. The results for vertical edges are shown in Fig. 3(a) and for horizontal edges in Fig. 3(b). Hence we proved that there exist symbolic dynamics for the perturbed Hénon map if the perturbation has magnitude $|e_i| < 0.012$.

Similar results for the $\mathbf{e}_1, \mathbf{e}_2 = [-0.0009, 0.0009]$ and for the symbolic dynamics of h^7 are plotted in Fig. 4.

In order to find the regions in the plane (e_1, e_2) for which there exist symbolic dynamics we have used the generalized bisection procedure starting with the intervals $\mathbf{e}_1, \mathbf{e}_2 = [-0.04, 0.04]$. In Fig. 5(a) and 5(b) we plot rectangles in the plane (e_1, e_2) for which we proved the existence of symbolic dynamics for h^2 and h^7 respectively.

We want to say once again that we have found the regions in the plane (e_1, e_2) for which the symbolic dynamics survives. It does not mean that for other (e_1, e_2) there is no symbolic dynamics. The method gives a sufficient conditions and cannot be used to find regions where there is no symbolic dynamics corresponding to the given subshift. Possibly if we could change the positions of sets N_i it would be possible to prove the existence of symbolic dynamics for larger regions.

One can clearly see that for h^7 the proof of the existence of symbolic dynamics is less robust. It is not very surprising. As the Hénon map is chaotic it has sensitive dependence on initial conditions and hence it is easier to check conditions involving the second iterate than the seventh iterate. On the other hand the difference in perturbation range is much smaller than we could have expected. This is caused by the definitions of sets N_i. In the case of h^7 the sets N_0, N_1 are relatively small in the unstable direction and large in the stable direction and hence points in N_i after one iteration are mapped not very far away from each other.

5 Coupled Hénon maps

In this section we analyze the behavior of coupled Hénon maps using the results from the previous section. In order to prove that there exist independent symbolic dynamics in a coupled system we have to estimate the perturbation introduced by adding the coupling terms and check that this perturbation is contained in the region for which the symbolic dynamic exists (these regions are plotted in Fig. 5).

As a first example let us consider two Hénon maps coupled in a master–slave configuration:

$$h_m(x, y) = h(x, y) = (1 + y - ax^2, bx), \tag{3}$$
$$h_s(x', y') = h(x' + d(x - x'), y'). \tag{4}$$

The first system is independent and is called a driving system or a master. The second one is called a response system or a slave. From the results described in the previous section we know that if the response system is perturbed weakly then there exist independent symbolic dynamics in this system. In order to check whether the symbolic dynamics survives we have to check if the perturbation is small enough. The error terms introduced by the coupling can be computed as:

$$e_1 = -2adx'(x - x') - ad^2(x - x')^2,$$
$$e_2 = bd(x - x').$$

As we investigate the existence of symbolic dynamics, we know that $(x, y), (x', y') \in N_0 \cup N_1$. From the definitions of sets N_1 and N_2 it follows that $x, x' \in [-0.82, 0.42]$

and $y, y' \in [0.1, 0.39]$. By means of interval arithmetic one can easily check that if $|d| < 0.0138$ then $|e_1| < 0.0397$ and $|e_2| < 0.00514$. This rectangle is contained in the region where the symbolic dynamics exists [compare Fig. 5(a)]. Hence we are sure that if $|d| < 0.0138$ there exist independent symbolic dynamics in the response system. In other words there are trajectories of the whole system realizing symbolic sequences in the response system independent of the itinerary realized by the orbit in the driving system.

Similarly for the symbolic dynamics on h^7 we have $x, x' \in [0.46, 0.755]$, $y, y' \in [0, 0.28]$. For $|d| < 0.0218$ the perturbation is bounded by $|e_1| < 0.01366$ and $|e_2| < 0.00193$. This rectangle is contained in the region where the symbolic dynamics exists [compare Fig. 5(b)].

It is interesting to note that although we have proved the existence of the symbolic dynamics for h^7 for smaller perturbation we can prove the existence of independent symbolic dynamics for stronger coupling. This is due to the fact that for h^7, the sets N_0 and N_1 have smaller range (in the x direction) and in the estimation of errors e_i we multiply the coupling by smaller intervals.

From the existence of independent symbolic dynamics it follows that the systems are not synchronized. The trajectory in the driving system following an arbitrary symbolic sequence does not influence the symbolic dynamics in the response system and the trajectory in this second system can realize any other symbolic sequence.

One should also notice that the coupling values for which one observes synchronization ($d > 0.4$) [Galias, 1998a] are of an order of magnitude larger than the values for which we were able to prove the existence of independent symbolic dynamics.

As a second example let us consider a ring of bidirectionally coupled Hénon maps. Every cell is connected with its two nearest neighbors. The dynamics of the kth cell is given by

$$h_d(x_k, y_k) = h(x_k + d(x_{(k+1) \bmod n} - x_k) + d(x_{(k-1) \bmod n} - x_k), y_k), \tag{5}$$

for $k = 0, \ldots, n-1$. The error terms in the kth cell introduced by the coupling can be computed as

$$\begin{aligned} e_{k1} &= -2adx_k z_k - ad^2 z_k^2, \\ e_{k2} &= bdz_k. \end{aligned}$$

where $z_k = x_{(k+1) \bmod n} + x_{(k-1) \bmod n} - 2x_k$. Using interval arithmetic one can show that for $|d| < 0.0068$ the error terms are bounded by $|e_{k1}| < 0.0392$ and $|e_{k2}| < 0.00506$ and there exists the independent symbolic dynamics for h^2 in every cell [compare Fig. 5(b)]. Similarly one can show that for $|d| < 0.0105$ the independent symbolic dynamics for

h^7 exists. These results are independent of the number of cells in a ring, i.e. for any number of coupled Hénon maps if the coupling strength is smaller than the values given above then there exist independent symbolic dynamics in every subsystem.

In the above two examples we have considered coupled Hénon maps. This is however a general technique. We can also use the above method to analyze behavior of coupled systems if they are different. In fact the driving signal can come from any system as long as we know the range of this signal.

6 Conclusions

In this paper we have considered the problem of robustness of symbolic dynamics for chaotic systems. We have shown that the symbolic dynamics is not destroyed if the perturbation is small. For the Hénon map we have found the parameter values and the values of perturbation for which the symbolic dynamics survives. Using these results we have found the values of coupling strength for which there exist independent symbolic dynamics for every coupled subsystem for the case of unidirectionally coupled Hénon maps and a ring of bidirectionally coupled Hénon maps.

Acknowledgments

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(a)

(b)

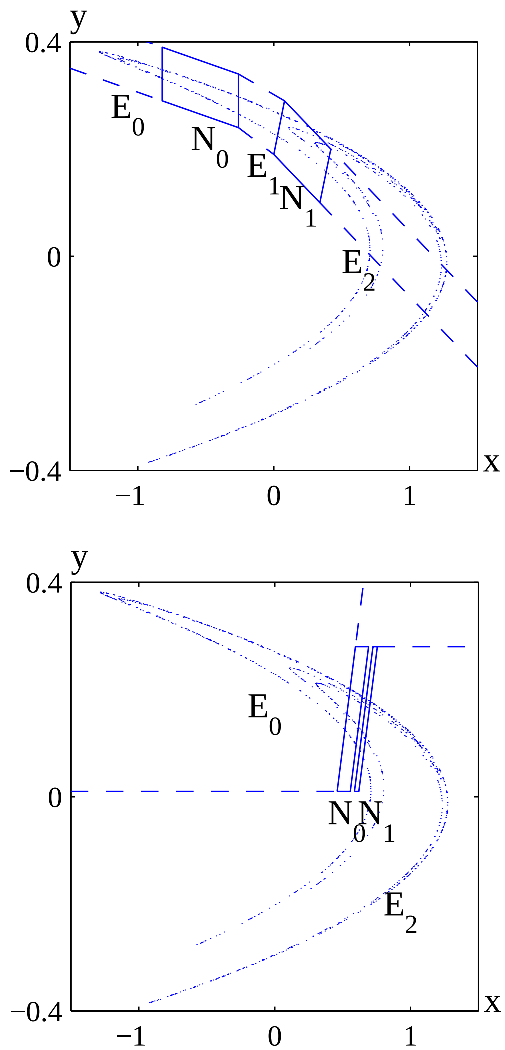

Figure 1: (a) definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^2, (b) definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^7.

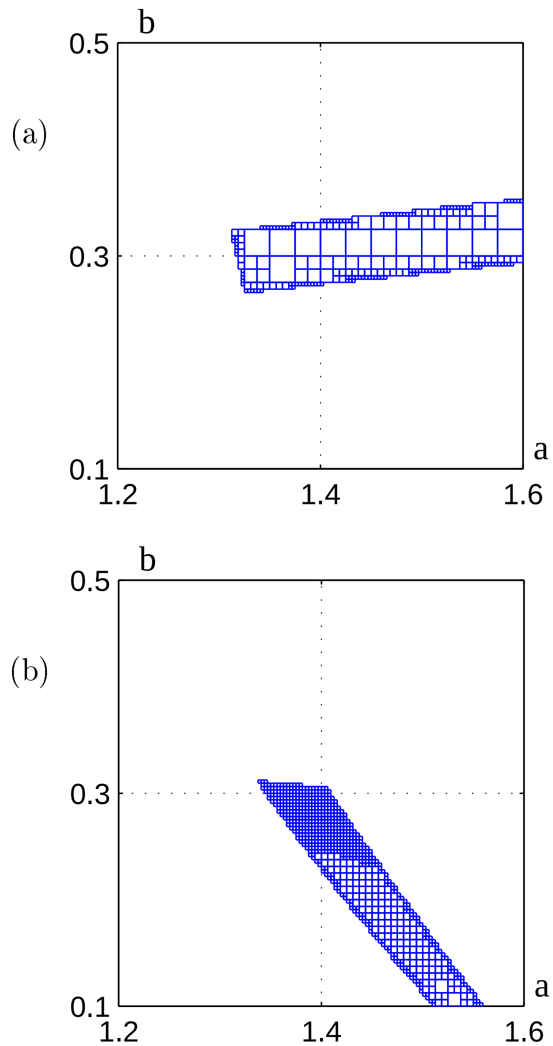

Figure 2: Regions in the (a, b) plane for which symbolic dynamics exists, (a) for h^2, (b) for h^7.

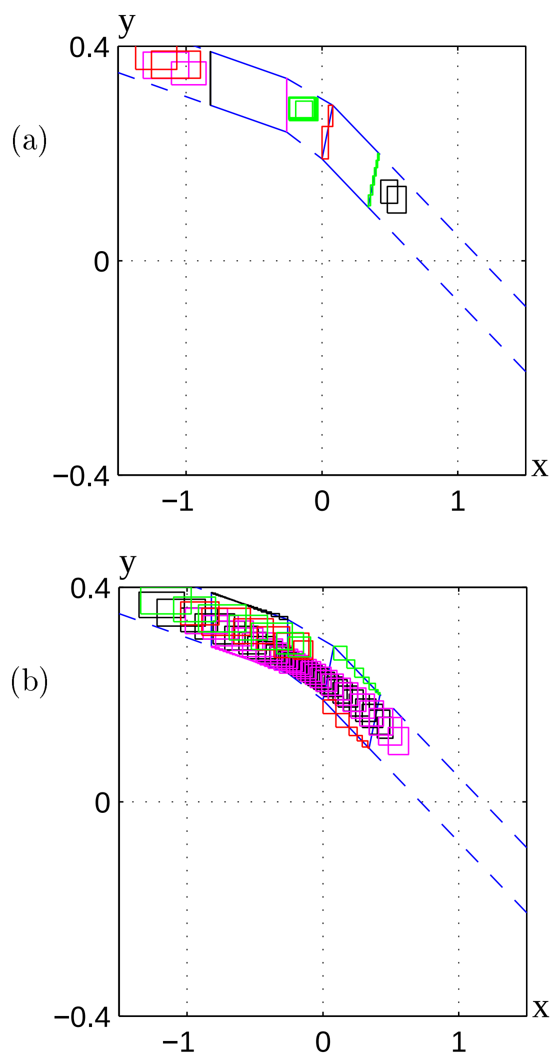

Figure 3: (a) the covering of the vertical edges of N_0 and N_1 with rectangles and its image under h^2, (b) the covering of horizontal edges of N_0 and N_1 and its image under h^2.

(a)

(b)

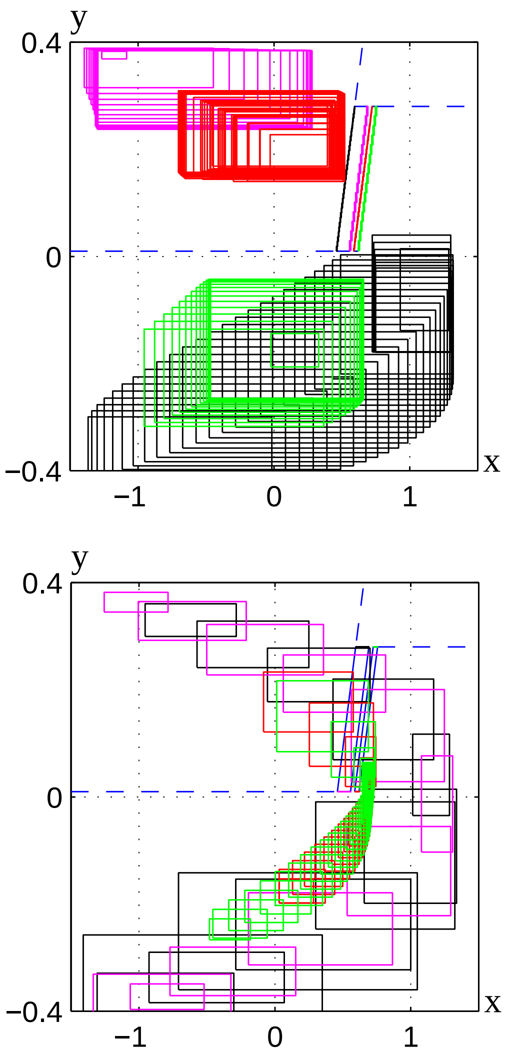

Figure 4: (a) covering of vertical edges of N_0 and N_1 with rectangles and their images under h^7, (b) covering of horizontal edges of N_0 and N_1 and their images under h^7.

(a)

(b)

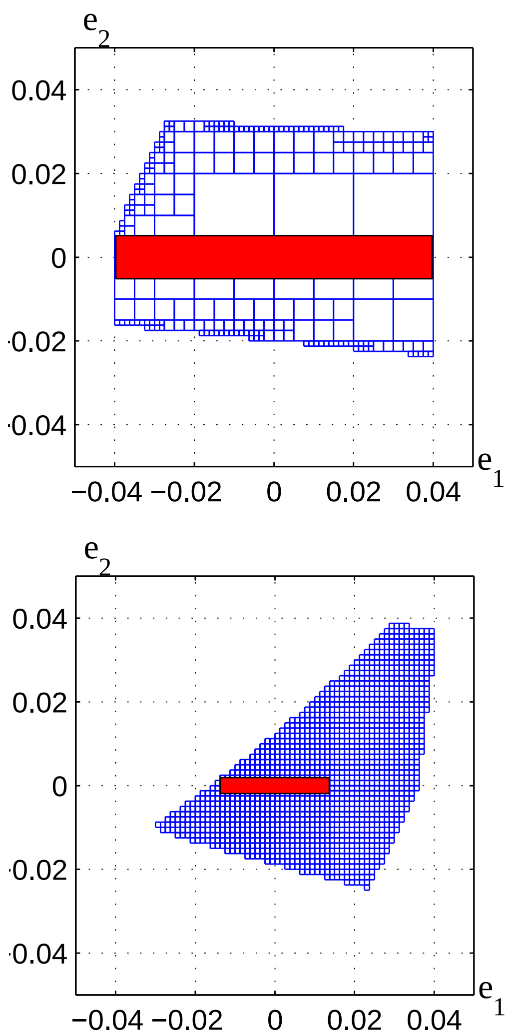

Figure 5: (a) regions in the (e_1, e_2) space for which the symbolic dynamics for h^2 exists, filled rectangle contains error introduced due to the coupling for $|d| < 0.0138$. (b) regions in the (e_1, e_2) space for which the symbolic dynamics for h^7 exists, filled rectangle contains error introduced due to the coupling for $|d| < 0.0218$.