RIGOROUS INVESTIGATIONS OF PERIODIC ORBITS IN AN ELECTRONIC CIRCUIT BY MEANS OF INTERVAL METHODS

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ABSTRACT

In this paper we use the combination of the global interval Newton method and the method of close returns for detection and proving the existence of periodic orbits in a continuous–time chaotic dynamical system. We consider a simple third order electronic circuit for which we prove the existence of several unstable periodic orbits. We also find out which of these periodic orbits are symmetric and discuss limitations of this technique.

1. INTRODUCTION

The problem of existence of periodic orbits in nonlinear systems attracts attention of many researchers. The existence and exact position of periodic orbits is a key property in analysis of nonlinear systems and in many applications. Periodic orbits may be located approximately in numerical studies but there is no guarantee that there exists a true periodic trajectory that stays near a computer generated one.

On the other hand periodic orbits may be rigorously studied by means of interval Newton method belonging to the class of self–validating algorithms. In the interval Newton method [5] in order to investigate the existence of zeros of a function $\mathbb{R}^m \ni x \mapsto f(x) \in \mathbb{R}^m$ in an m-dimensional interval $\mathbf{x}$ one computes the so–called interval Newton operator $\mathbf{N}(\mathbf{x})$. If $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ then there exists exactly one zero of f in $\mathbf{x}$.

Here we use a modification of this method called the *global interval Newton method* which may be used for proving the existence of long cycles in continuous–time systems. In this method the interval Newton operator is applied to the generalized Poincaré map. This technique may be automated and used for detection of a large number of periodic orbits in the state space.

In this paper we use bold letters to denote intervals, interval vectors and matrices and usual math italic to denote real quantities. For the introduction to the interval arithmetic underlying the methods discussed here see [5].

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2. GLOBAL INTERVAL NEWTON METHOD

Let us denote by $\varphi_t(x)$ the trajectory of the system starting at x. Let us consider an orbit $\{\varphi_t(\bar{x})\}_{t\in[0,T]}$ and let us choose p planes $\Sigma_1, \ldots, \Sigma_p$ which are transversal to this orbit. Let us denote by Σ the union of the planes Σ_i. We assume that $\bar{x} \in \Sigma$ and that the orbit does not intersect any of the sets $\Sigma_i \cap \Sigma_j$ for $i \neq j$. Let n be the number of points at which the trajectory intersects Σ. Let us define a *generalized Poincaré map* $H : \Sigma \ni x \mapsto H(x) = \varphi_{\tau(x)}(x) \in \Sigma$, where $\tau(x)$ is the time needed for the trajectory $\varphi_t(x)$ to reach Σ.

The trajectory $\{\varphi_t(\bar{x})\}_{t\in[0,T]}$ is periodic if and only if $H^n(\bar{x}) = \bar{x}$. To prove this fact one may apply the interval Newton method to the map $\mathrm{id} - H^n$. For longer orbits however, usually the Jacobian matrix $\mathbf{D}H^n(\bar{x})$ has large diameter, and one cannot check the assumptions of the existence theorem and the method fails (compare [2]).

In order to overcome this problem one may use the interval Newton method for the global map $F : (\mathbb{R}^m)^n \mapsto (\mathbb{R}^m)^n$ defined by

$$[F(z)]_k = x_{(k+1) \bmod n} - H(x_k) \quad \text{for } 0 \leq k < n,$$

where $z = (x_0, \ldots, x_{n-1})$. See that $F(z) = 0$ if and only if x_0 is a fixed point of H^n. In the global interval Newton method the problem of existence of periodic orbits is translated to the problem of existence of zeros of a higher–dimensional function.

Now we describe the procedure for detection of many periodic orbits in chaotic systems. First we extract periodic orbits using the method of close returns [4]. We monitor a trajectory and look for parts of the trajectory coming closely to the initial point. Then using the standard (non-interval) Newton method we sharpen the approximation obtaining a quasi–periodic trajectory of the generalized Poincaré map. We create an interval vector $\mathbf{x}$ centered at the approximate position of the orbit with the same diameter at all points along the orbit. Finally we check whether the image of $\mathbf{x}$ under the interval Newton operator is enclosed in $\mathbf{x}$. If this condition holds the existence of periodic orbit is proven. In the opposite case we can modify the interval vector and repeat the computations.

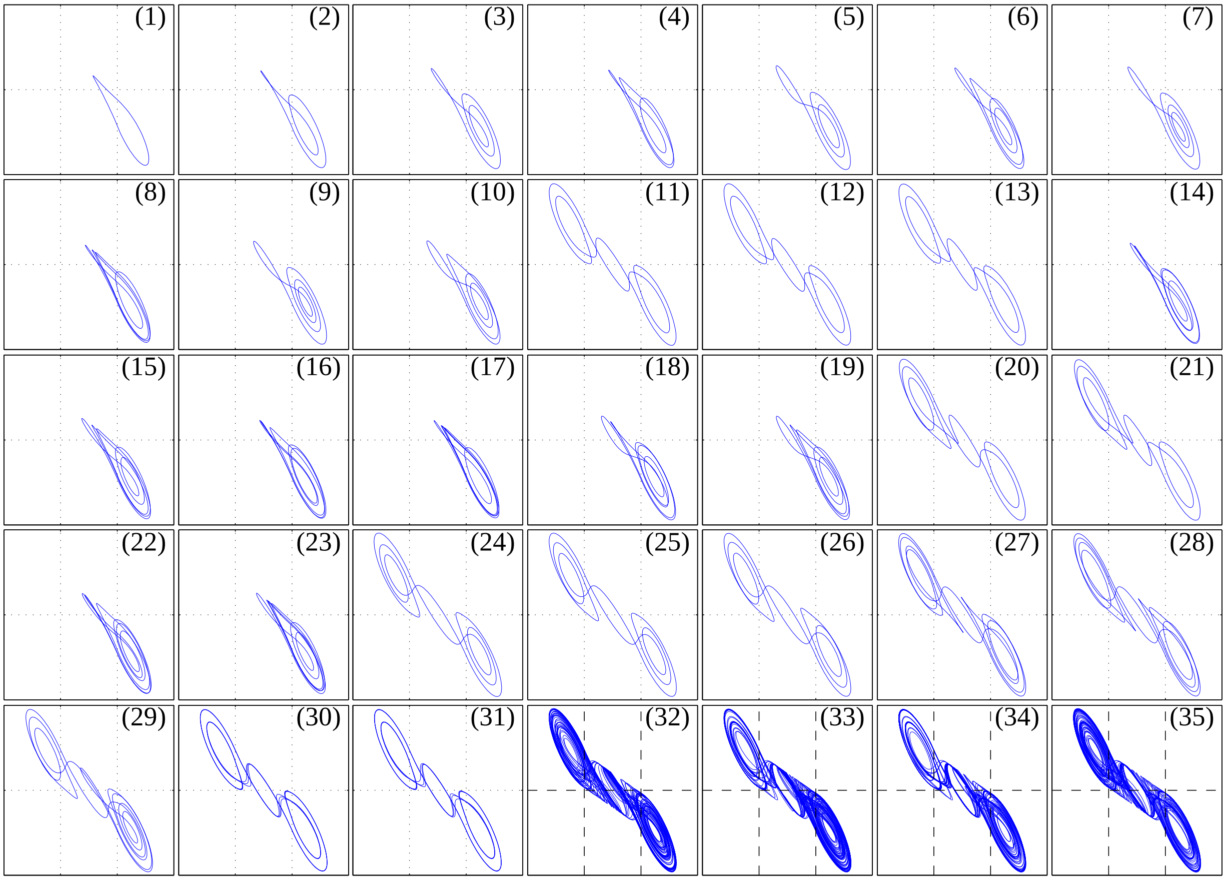


Figure 1: Periodic orbits of the Chua's circuit.

3. PERIODIC ORBITS FOR THE CHUA'S CIRCUIT

As an example we consider the Chua's circuit, a simple third–order system defined by the following set of ordinary differential equations:

$$\begin{aligned} C_1\dot{x} &= G(y-x) - g(x), \\ C_2\dot{y} &= G(x-y) + z, \\ L\dot{z} &= -y - R_0 z, \end{aligned} \tag{1a}$$

where $g(\cdot)$ is a three-segment piecewise-linear function

$$g(x) = G_b x + 0.5(G_a - G_b)(|x+1| - |x-1|). \tag{1b}$$

For parameters: $C_1 = 1$, $C_2 = 9.3515$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$ the system (1) exhibits chaotic behavior. We choose the planes $V_\pm = \{\mathbf{x} \in \mathbb{R}^3 : x = \pm 1\}$ separating the three linear regions as the planes defining the generalized Poincaré map ($\Sigma_1 = V_+$, $\Sigma_2 = V_-$). For the details of computation of the generalized Poincaré map, its Jacobian and the Jacobian of the global map F see [3].

Now we report the results of application of the technique described in the previous section for detection and proving the existence of periodic orbits for Chua's circuit. First we have generated the trajectory (consisting of 60000 points) of the Poincaré map $P : V_+ \to V_+$. We have limited ourselves to periodic orbits with length smaller than 150. We have located quasi–periodic trajectories (returning to the neighborhood with radius 0.005 of the initial point) with return time smaller than 150. For most quasi–periodic orbits we have succeeded in proving the existence of a nearby true periodic orbit. In few cases the method failed. We have observed that in all unsuccessful cases the orbit spends a long time in one linear region before returning to one of the transversal planes. In order to overcome this problem one should use more planes defining the generalized Poincaré map. In our study we also address the problem whether the periodic orbit is symmetric. We say that the periodic orbit $(x_0, \ldots x_{n-1})$ of the generalized Poincaré map is symmetric if $x_0 = -x_l$, where $l = \frac{n}{2}$. Due to the fact that the vector field of the system (1) is symmetric with respect to the origin for every non–symmetric periodic orbit there exist a different periodic orbit symmetric to it.

In our investigations we have found 762 different periodic orbits with period shorter than 150. Most of these orbits are not symmetric, which can be easily proved by checking that $\mathbf{x}_0 \cap (-\mathbf{x}_l) = \emptyset$. If this condition is not

a	b	n	n'		period
1	1	2	1		7.38058439_7^9
2	3	4	2		14.38443804_1^5
3	5	4	2		21.330218_2^4
4	7	6	3		21.6768157_0^2
5	9	4	2		24.70392_7^9
6	11	6	3		28.6270866_2^6
7	13	4	2		28.68369_6^8
8	15	8	4		29.0841154_4^6
9	17	4	2		29.52941_2^4
10	19	6	3		31.062129_4^9
11	21	8	2	S	32.83175_0^2
12	22	8	2		32.99894_5^7
13	24	8	2	S	33.73813_1^3
14	25	8	4		35.72253_7^9
15	27	8	4		36.021545_1^3
16	29	10	5		36.0750610_2^4
17	31	10	5		36.4559967_7^9
18	33	8	4		38.731208_2^{93}

a	b	n	n'		period
19	35	8	4		38.779715_3^{91}
20	37	10	2		41.00009_1^4
21	39	10	2		41.79856_5^7
22	41	10	5		42.9797918_5^9
23	43	10	5		43.3956623_2^6
24	45	8	2	S	43.9766674_4^{96}
25	46	8	2		44.1795015_1^6
26	48	8	2	S	44.556774_5^7
27	135	16	4	S	60.168288_4^6
28	136	16	4	S	60.267690_0^2
29	193	12	4		66.802198_0^3
30	567	24	6	S	99.12916_3^8
31	580	24	6	S	99.49633_4^9
32		160	48		790.0381_{89}^{97}
33		186	58		829.6924_5^7
34		204	57		893.61_{599}^{603}
35		246	58		1076.9431_6^92

Table 1: Some periodic orbits for the Chua's circuit, a is the number of the orbit in Fig. 1, b is the number of the orbit in the list of orbits found, the list is sorted according to the length of the orbit, n is the period of the orbit on the generalized Poincaré map H, n' is the period on the Poincaré map P, S in the fifth column depicts that the periodic orbit is symmetric

fulfilled we suspect that the periodic orbit is symmetric. In order to prove this we consider a map $G : (\mathbb{R}^m)^l \mapsto (\mathbb{R}^m)^l$

$$[G(z)]_k = \begin{cases} x_{(k+1) \bmod n} - H(x_k) & \text{for } k < l-1, \\ x_{(k+1) \bmod n} + H(x_k) & \text{for } k = l-1, \end{cases}$$

where $z = (x_0, \ldots, x_{l-1})$. It is clear that $G(z) = 0$ if and only if $H^l(x_0) = -x_0$. It follows that $H^{2l}(x_0) = x_0$ and x_0 defines a symmetric periodic orbit with period $2l$. Applying the interval Newton operator to the above map we have checked that 8 of the orbits found are symmetric. In the remaining 754 orbits there are 81 pars of mutually symmetric periodic orbits. In other words there are 673 pars of non–symmetric orbits and 8 symmetric orbits which gives the total number of 1354 different orbits with length smaller than 150. For all the orbits by iterating the interval Newton method we have obtained the uncertainty of their position on Σ smaller than 10^{-7}.

Some of the orbits for which the existence was proven are shown in Fig. 1. In particular we show all orbits found with period smaller than 45 (orbits (1)–(26), for each pair of mutually symmetric orbits we show only one) and all symmetric orbits found (orbits (11), (13), (24), (26)–(28), (30), (31)). Their parameters are collected in Table 1.

We have also tried to prove the existence of longer periodic orbits. We were able to prove the existence of several periodic orbits with length larger than 500. Four of them are shown in Fig. 1 (orbits (32)–(35)). The longest periodic orbit found has period $T \approx 1076.94$ which is approximately 146 times larger than the shortest one (with period $T \approx 7.38$). This shows that the technique based on the global interval Newton method can be successfully used for proving the existence of long orbits.

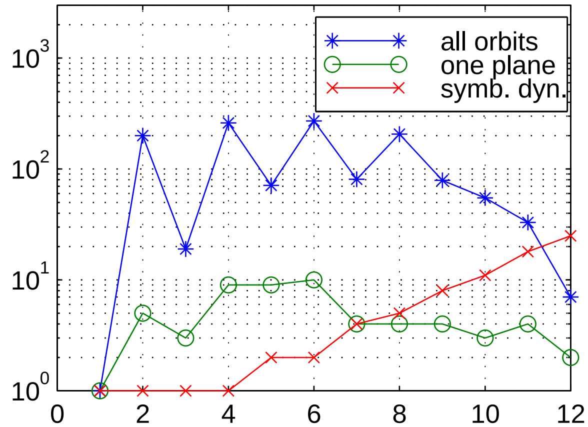
4. NUMBER OF PERIODIC ORBITS AND TOPOLOGICAL ENTROPY

In this section we discuss the results obtained in terms of the number of cycles found for the Poincaré map P.

It is known that there exists a symbolic dynamics for the Chua circuit [1]. Namely there are two sets $N_0, N_1 \subset V_+$ such that for every finite sequence of symbols $a_0, \ldots,$ a_{n-1} from the set $\{0, 1\}$, which does not contain the subsequence $(1, 1)$ there exist a periodic orbit $(x_0, \ldots, x_{n-1})$ such that $x_i \in N_{a_i}$ for $i = 1, \ldots, n-1$. The number $O_s(n)$ of period-n orbits of P corresponding to the existence of symbolic dynamics is shown in Fig. 2(a). All of these orbits have no intersection with the plane V_-. Because the return time of the Poincaré map for points in $N_0 \cup N_1$ is shorter than 9 it follows that all these periodic orbits with period smaller than 17 has period shorter than 150 in the continuous–time system. In the same figure we plot the number of orbits found. We show the total number of periodic orbits $O(n)$ and the number of periodic orbits $O_+(n)$ visiting only the plane V_+. One can see that $O_s(n) > O_+(n)$ for $n > 8$. Hence it is clear that we have not found all periodic orbits with period shorter than 150.

Another indication that there may by more periodic orbits with period shorter than 150 is based on the observation that we have found only 81 pars of mutually symmetric orbits and 531 non–symmetric orbits intersecting both of the planes $V_{\pm}$ (so both orbits in the pair could be found from the trajectory of the Poincaré map). Hence it is very likely that there are many other periodic orbits not found.

(a)



(b)

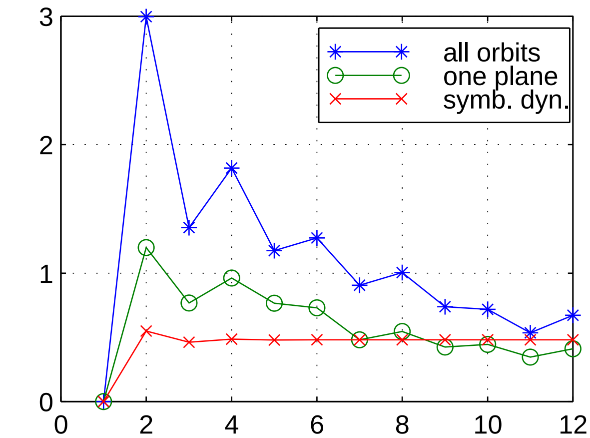


Figure 2: (a) the number of period-n cycles of P: $O(n)$ — all orbits found (stars), $O_1(n)$ — orbits found visiting only plane V_+ (circles), $O_s(n)$ — from the existence of symbolic dynamics for P ($\times$ symbols), (b) estimation of topological entropy based on the number of cycles

This is caused by the inevitable property of the close returns method. In this method longer periodic orbits are less likely to be found especially if they lie in the neighborhood of a shorter orbit.

It is also interesting to note that the number $O(n)$ of periodic orbits found oscillates. For odd n it is much smaller than for n even. In particular we have found only one fixed point of the Poincaré map and no fixed points of P visiting also the plane V_-.

It is well known that the number of period–n cycles can be used for estimation of topological entropy of the map. For example topological entropy of an axiom A diffeomorphism f is equal to

$$h(f) = \lim_{n\to\infty} \frac{\log C(f^n)}{n},$$

where $C(f^n)$ denotes the number of fixed points of f^n. From the existence of symbolic dynamics we know that the topological entropy of the Poincaré map is bounded from below by $\mathrm{h}(P) \geq \log \frac{1+\sqrt{5}}{2} > 0.48$, see Fig. 2(b).

In Fig. 2(b) the expression $\mathrm{h}_n(P) = \frac{1}{n}\log \mathrm{C}(P^n)$ is used for estimation of topological entropy of the Poincaré map. As it can be easily seen the number of periodic orbits found for small n is much larger than the number following from the existence of symbolic dynamics. The estimation $\mathrm{h}_n(P)$ decreases with n and this is caused by the fact that we did not found all periodic orbits. We believe however that the topological entropy of the Poincaré map is much larger than 0.48.

5. CONCLUSIONS

In this paper we have used the global interval Newton method for proving the existence of periodic orbits in a model of an electronic circuit. We have shown that this method is much more powerful that the non-global version of the interval Newton method. It is not limited to short periodic orbits. We have shown that this technique may be combined with the method of close returns for performing an exhaustive search of periodic orbits in the state space. We have found 1372 cycles (including 8 symmetric orbits) with period shorter than 150. We have also found several long cycles. The longest periodic orbit found has period approximately 146 times longer than the shortest one.

6. REFERENCES

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