LACK OF SYNCHRONIZATION AND THE EXISTENCE OF INDEPENDENT SYMBOLIC DYNAMICS

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ABSTRACT

We investigate the behavior of coupled chaotic systems. We show that if the coupling is small then there exists independent symbolic dynamics for every coupled subsystem. As an example we consider coupled Hénon maps. We compute the value of the coupling strength for which the symbolic dynamics in every subsystem is sustained.

1. INTRODUCTION

It is well known that when chaotic systems are coupled, they may demonstrate synchronized behavior. Recently there has been a considerable interest in using the concept of synchronization of chaos for solving technical problems. For the applications it is very important to find techniques for investigation of the phenomenon of synchronization of chaotic systems.

There are several methods for investigating the synchronization problem. It was shown in various papers that a very important role is played by the transversal Lyapunov exponents of the synchronized trajectories. It was shown that the criterion based on conditional Lyapunov exponents calculated along a typical trajectory of the system not sufficient and one has to take into account transversal Lyapunov exponents computed along all periodic orbits. Other methods are based on local transversal Lyapunov exponents.

In this paper we describe the method of investigation of coupled systems using topological methods. We are interested in the case when synchronization of chaotic systems cannot be observed due to the existence of independent symbolic dynamics in coupled subsystems. We show that if the coupling is small then one may prove that there exists independent symbolic dynamics for every coupled subsystem. This means that for two different sequences of symbols one may found trajectory of the coupled system which realizes these two sequences in the coupled subsystems. In consequence we obtain the coexistence of different periodic solutions in different subsystems. In this context the existence of independent symbolic dynamics for different subsystems implies the lack of synchronization.

Research supported by Polish Committee of Scientific Research KBN, grant no. 2/P03A/021/15.

As an example we consider the Hénon map [3] defined by the following equation:

$$h(x,y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$ are the "classical" parameter values for which the famous Hénon attractor is observed.

In Section 2 we recall results on the existence of symbolic dynamics for the Hénon map. In Section 3 we study the robustness of symbolic dynamics on parameters and disturbance added to the system. In Section 4 we analyze coupled Hénon maps using the results from Section 4.

2. SYMBOLIC DYNAMICS FOR h^2 AND h^7

In [2] it was shown that there exist symbolic dynamics embedded in h^2 corresponding to the golden subshift on two symbols. The sets N_i and E_i are shown in Fig. 1a. For the exact definition see [2]. In [2] it was shown that the images of vertical edges of N_0 under h^2 are enclosed in E_0 and E_2 on the opposite sides of $N_0 \cup N_1$ and that the images of vertical edges of N_1 under h^2 are enclosed in E_0 and E_1 on the opposite sides of N_0. It was also shown that images of horizontal edges of N_0 and N_1 under h^2 are enclosed in the interior of the topological stripe $E_0 \cup N_0 \cup E_1 \cup N_1 \cup E_2$. For details see [2]. It follows that for every sequence of symbols $(a_0, a_1, \ldots, a_{n-1})$, from the set $\{0, 1\}$ which does not contain the subsequence $(1, 1)$ there exists a point $z = (x, y)$ such that $h^{2i}(z) \in N_{a_i}$ for $i = 0, \ldots, n-1$ and $h^{2n}(z) = z$.

In particular for every positive integer n there exists a periodic point of h^2 with period n. In this way we have also proved that the subshift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is embedded in h^2.

In [4] it was shown that there exist symbolic dynamics embedded in h^7 corresponding two the full shift on two symbols.

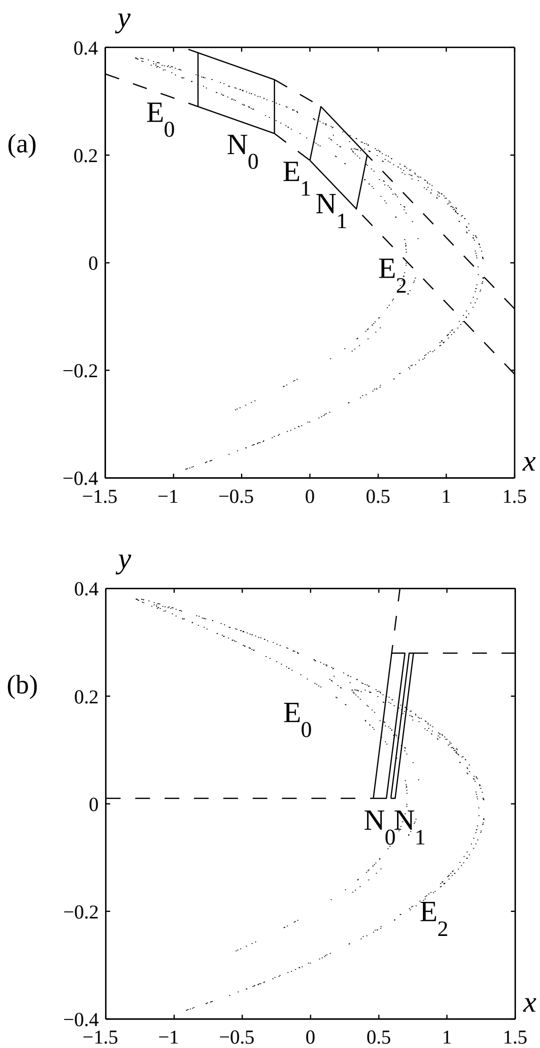

Figure 1: (a) definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^2, (b) definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^7.

The sets N_i and E_i are shown in Fig 1b. It was shown that for $i = 0, 1$ the images under h^7 of vertical edges of N_i lie on the opposite sides of $N_0 \cup N_1$ (are enclosed in E_0 and E_2). It was also shown that the images under h^7 of horizontal edges are enclosed in the interior of topological stripe defined by the sets N_i and E_i (for the details see [4] or [2]. It follows that for every sequence of symbols $a = (a_0, a_1, \ldots, a_{n-1})$ from the set $\{0, 1\}$ there exists a point $z = (x, y)$ such that $h^{2i}(z) \in N_{a_i}$ for $i = 0, \ldots, n-1$ and $h^{2n}(z) = z$. In other words the symbolic dynamics corresponding to the full shift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is embedded in h^7.

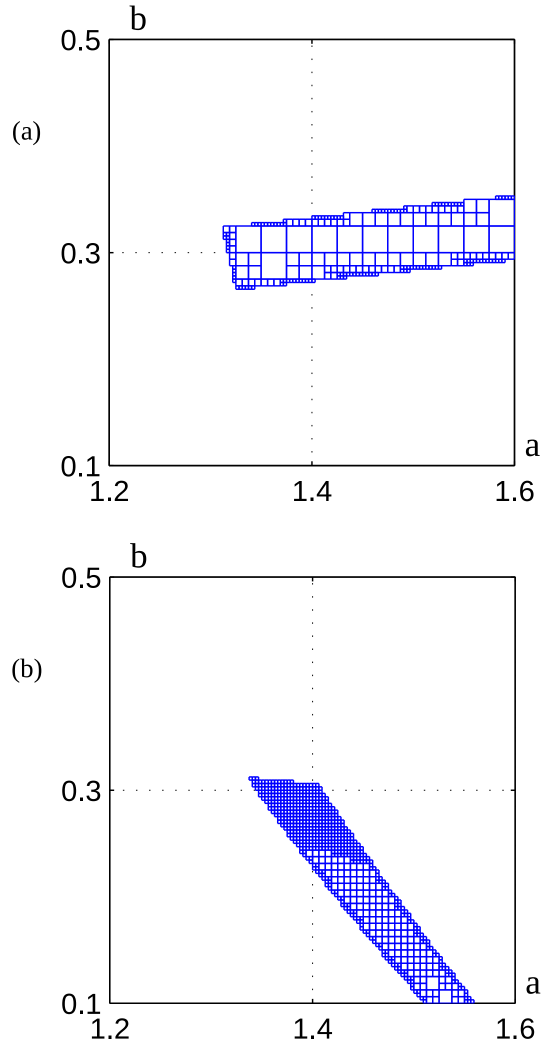

Figure 2: Regions in the (a, b) plane for which symbolic dynamics exists, (a) for h^2, (b) for h^7.

3. ROBUSTNESS OF SYMBOLIC DYNAMICS

In this section we study robustness of symbolic dynamics. The first question we address is whether the symbolic dynamics remains if the parameters of the map are modified. Using the sets N_i, E_i plotted in Fig.1 we have checked whether for different values of (a, b) the assumptions of the theorem on the existence of symbolic dynamics hold.

In Fig. 2a we show the rectangles (a, b) for which we have proved the existence of symbolic dynamics for h^2. Similarly in Fig. 2b we show regions in the parameter space for which the symbolic dynamics for h^7 exists. It is interesting to note that the symbolic dynamics is present in the dynamics of the map even for parameter values far from the standard ones. For example the symbolic dynamics for h^2 exists also for $a = 1.6$, $b = 0.35$ and the symbolic dynamics for h^7 exists for $a = 1.55$, $b = 0.1$.

The second problem we investigate is the existence of symbolic dynamics in the case when the dynamics of the map is disturbed by some additive signal. We assume that we only know the upper limit of the absolute value of this disturbance. We consider a system

$$h_d(x,y) = (1 + y - ax^2 + e_1, bx + e_2), \qquad (2)$$

where $|e_1| \leq d_1$ and $|e_2| \leq d_2$. Using interval arithmetic we have found pairs (d_1, d_2) for which the symbolic dynamics is not destroyed by the disturbance. In order to prove the existence of symbolic dynamics for particular values of e_1 and e_2 we check the assumptions of the existence theorem for the map (2) (we check if the images of edges of N_i lie properly with respect to the sets N_i, E_i).

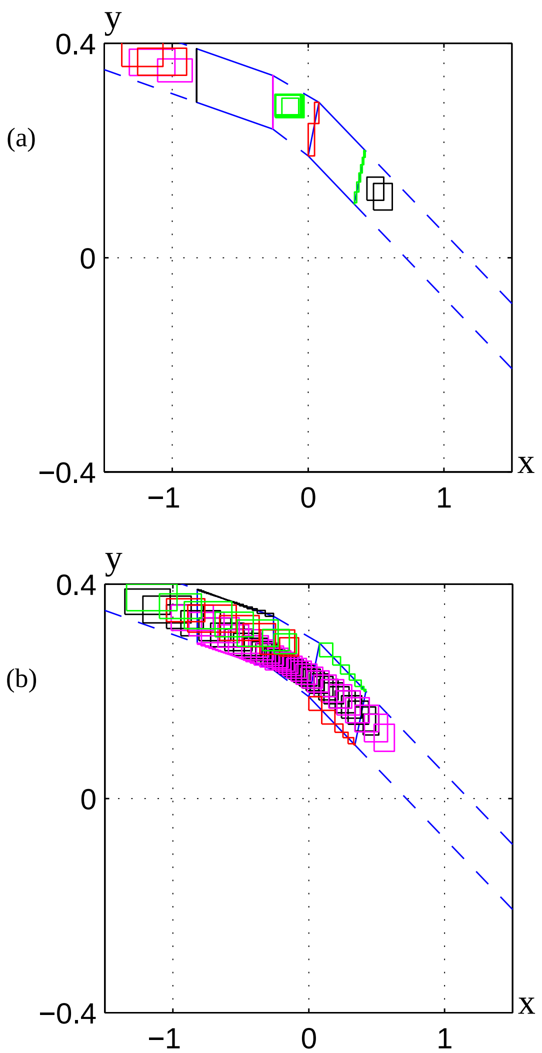

Figure 3: (a) the covering of the vertical edges of N_0 and N_1 with rectangles and its image under h^2, (b) the covering of horizontal edges of N_0 and N_1 and its image under h^2.

As an example in Fig. 3 we show this images for $|e_1|, |e_2| < 0.012$. Vertical edges of N_i were covered by 2, 2, 2, and 7 rectangles respectively and horizontal edges were covered by 27, 46, 6, 7 rectangles respectively. The images of these rectangles under the map h^2 were computed are we have checked that they lie in a proper way with respect to the sets N_i and E_i. Hence we proved that there exist symbolic dynamics for the disturbed Hénon map if the disturbance has magnitude $|e_i| < 0.012$. The results for vertical edges are shown in Fig. 3a and for horizontal edges in Fig. 3b.

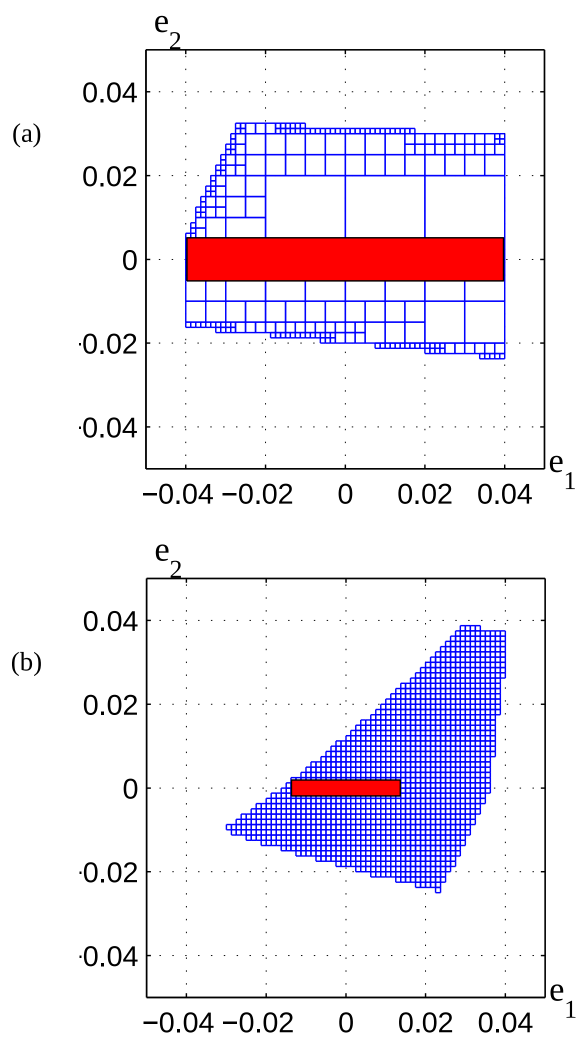

Figure 4: (a) regions in the (e_1, e_2) space for which the symbolic dynamics for h^2 exists, filled rectangle contains error introduced due to the coupling for $|d_1| < 0.0138$. (b) regions in the (e_1, e_2) space for which the symbolic dynamics for h^7 exists, filled rectangle contains error introduced due to the coupling for $|d_1| < 0.0218$.

We have performed similar experiments for different intervals on the (e_1, e_2) plane. In Fig. 4a and Fig. 4b we plot rectangles in the space (e_1, e_2) for which we have proved the existence of symbolic dynamics of h^2 and h^7 respectively.

One can clearly see that for h^7 the symbolic dynamics is less robust. It can be destroyed by the disturbance of smaller amplitude.

4. COUPLED HÉNON MAPS

In this section we analyze the behavior of coupled Hénon maps using the results from the previous section. In order to prove that there exist independent symbolic dynamics in a coupled system we have to estimated the error which is obtained by adding the coupling terms and check if this error is contained in the region for which the symbolic dynamic exists (these regions are plotted in Fig. 4).

As a first example let us consider two Hénon maps coupled unidirectionally:

$$h(x, y) = (1 + y - ax^2, bx), \tag{3}$$

$$h_r(x', y') = h(x' + d_1(x - x'), y' + d_2(y - y')). \tag{4}$$

The first system is independent and is called the driving system. The second one is called the response system. We will consider the case $d_2 = 0$.

From the results described in the previous section we know that if the response system is disturbed weakly then there exist independent symbolic dynamics in this system.

In order to check whether the symbolic dynamics survives we have to check if the disturbance is small. The error terms can be computed as:

$$e_1 = d_2(y - y') - 2ad_1x'(x - x') - ad_1^2(x - x')^2,$$
$$e_2 = bd_1(x - x').$$

We investigate the existence of symbolic dynamics, so we may assume that $(x, y), (x', y') \in N_0 \cup N_1$. From the definitions of sets N_1 and N_2 we know that $x, x' \in [-0.82, 0.42]$ and $y, y' \in [0.1, 0.39]$. By means of interval arithmetics tools using the above formulas one can easily check that if $|d_1| < 0.0138$ then $|e_1| < 0.0397$ and $|e_2| < 0.00514$. This rectangle is contained in the region where the symbolic dynamics exists (compare Fig. 4a).

Similarly for the symbolic dynamics on h^7 we have $x, x' \in [0.46, 0.755]$, $y, y' \in [0, 0.28]$. For $|d_1| < 0.0218$ the disturbance is bounded by $|e_1| < 0.01366$ and $|e_2| < 0.00193$. This rectangle is contained in the region where the symbolic dynamics exists (compare Fig. 4b).

It is interesting to note that although the symbolic dynamics for h^7 disappears for smaller disturbances we can prove the existence of independent symbolic dynamics for stronger coupling. This is due to the fact that in this case the sets N_0 and N_1 has smaller range (in the x direction and in the estimation of errors e_i we multiply the coupling by smaller intervals.

From the existence of independent symbolic dynamics it follows that the systems are not synchronized. The trajectory in the driving system following an arbitrary symbolic sequence does not influence the symbolic dynamics in the response system and the trajectory in this second system can realize any other symbolic sequence.

One should also notice that the coupling values for which one observes synchronization ($d_1 > 0.4$) [1] are of an order of magnitude larger than the values for which there exist independent symbolic dynamics.

As a second example let us consider a ring of bidirectionally coupled Hénon maps. Every cell is connected with two nearest neighbors.

$$h_d(x_k, y_k) = h(x' + d(x_{k+1} - x_k) + d(x_{k-1} - x_k), y_k),$$
$$\text{for } k = 1, \ldots, n$$

where $x_0 \stackrel{\text{df}}{=} x_n$ and $x_{n+1} \stackrel{\text{df}}{=} x_1$. The error terms introduced by the coupling can be computed as

$$e_1 = -2adx_kz_k + ad^2z_k^2,$$
$$e_2 = bdz_k.$$

where $z_k = x_{k+1} + x_{k-1} - 2x_k$ Using the interval arithmetic one can show that for $|d_1| < 0.0068$ the error terms are bounded by $|e_1| < 0.0392$ and $|e_2| < 0.00506$ and the independent symbolic dynamics for h^2 exists.

5. CONCLUSIONS

In this paper we have considered the problem of robustness of symbolic dynamics for chaotic systems. We have shown that the symbolic dynamics is not destroyed if the disturbance is small. For the Hénon map we have found the parameter values and the values of disturbance for which the symbolic dynamics survives. Using these results we have found the values of coupling strength for which there exist independent symbolic dynamics for every coupled subsystem for the case of unidirectionally coupled Hénon map and a ring of bidirectionally coupled Hénon maps.

6. REFERENCES

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