COMPARISON OF INTERVAL METHODS FOR FINDING PERIODIC ORBITS

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Abstract — In this paper we compare several interval methods for identification of periodic orbits in discrete–time dynamical systems. We consider the interval Newton method and methods based on the Krawczyk operator and Hansen–Sengupta operator. We also test the global versions of these three methods. As an example we consider the Hénon map. We find all cycles with period $n \leq 27$ belonging to the trapping region of the map.

I. INTRODUCTION

The recent development of new methods for proving the existence and uniqueness of zeros of nonlinear functions have opened the possibility of rigorous investigations of chaotic systems in terms of unstable periodic orbits. Periodic orbits are important in analysis of chaotic systems as the structure of the strange attractor is built on an infinite set of unstable periodic orbits. Periodic orbits are ordered hierarchically, longer orbits give better approximations to the chaotic attractor.

We compare several interval methods, which can be used for finding all low–period cycles of a chaotic map. We test methods based on the interval Newton operator, the Krawczyk operator and the Hansen–Sengupta operator. We also consider the so–called global versions of these methods, where the problem of existence of periodic orbits is translated to the problem of existence of zeros of a higher–dimensional map. We introduce a modification where the dimension of the search space for the global version is reduced to the dimension of the original dynamical system. We also describe improvements useful especially if the map is invertible and if we know a trapping region of the system.

As an example we study the existence of periodic orbits for the Hénon map [1] defined by $h(x, y) = (1 + y - ax^2, bx)$, where $a = 1.4$ and $b = 0.3$.

It is well known [1] that the set Ω defined as a quadrangle $ABCD$, where $A = (-1.33, 0.42)$, $B = (1.32, 0.133)$, $C = (1.245, -0.14)$ and $D = (-1.06, -0.5)$ is a trapping region, i.e. $h(\Omega) \subset \Omega$ (compare Fig. 1). In our study we search for periodic solution in the trapping region Ω, which encloses the strange attractor observed numerically.

II. SELF-VALIDATING METHODS

In this study we test the performance of different interval methods for finding periodic solutions of discrete–time dynamical systems. For the introduction to the interval arithmetic underlying this technique see [2]. We consider methods based on three interval operators, namely the interval Newton operator, Krawczyk operator and Hansen–Sengupta operator [3, 4], which allow to prove with computer assistance the existence and uniqueness of periodic orbits within a given interval. First let us recall briefly the definitions of these operators.

A. Interval Newton operator

Let us consider a function $\mathbb{R}^m \ni x \mapsto f(x) \in \mathbb{R}^m$. In order to investigate the existence of zeros of f in an m-dimensional interval vector $\mathbf{x}$ one evaluates the *interval Newton operator*

$$\mathbf{N}(\mathbf{x}) = x_0 - (\mathbf{D}f(\mathbf{x}))^{-1} f(x_0), \tag{1}$$

where $\mathbf{D}f(\mathbf{x})$ is the interval matrix containing all Jacobian matrices of the form $\mathbf{D}f(x)$ for $x \in \mathbf{x}$ and x_0 is an arbitrary point belonging to the interval vector $\mathbf{x}$. One usually chooses x_0 to be the center of $\mathbf{x}$.

The following theorem [3] can be used to prove the existence and uniqueness of zeros of f.

Theorem 1 *If* $\mathbf{N}(\mathbf{x}) \subset \mathrm{int}(\mathbf{x})$ *then* $f(x) = 0$ *has a unique solution in* $\mathbf{x}$*. If* $\mathbf{N}(\mathbf{x}) \cap \mathbf{x} = \emptyset$ *then there are no zeros of* f *in* $\mathbf{x}$*.*

The interval Newton operator can be used only when the interval matrix $\mathbf{D}f(\mathbf{x})$ is regular, i.e. all real matrices in $\mathbf{D}f(\mathbf{x})$ are nonsingular. The following two operators can be used for a wider class of systems.

B. Krawczyk and Hansen–Sengupta operators

Krawczyk operator is defined as

$$\mathbf{K}(\mathbf{x}) = x_0 - Cf(x_0) - (C\mathbf{D}f(\mathbf{x}) - I)(\mathbf{x} - x_0), \tag{2}$$

where x_0 is an arbitrary point belonging to $\mathbf{x}$ (usually one uses the center of $\mathbf{x}$) and C is a preconditioning matrix. It is usually chosen as the inverse of $\mathbf{D}f(x_0)$.

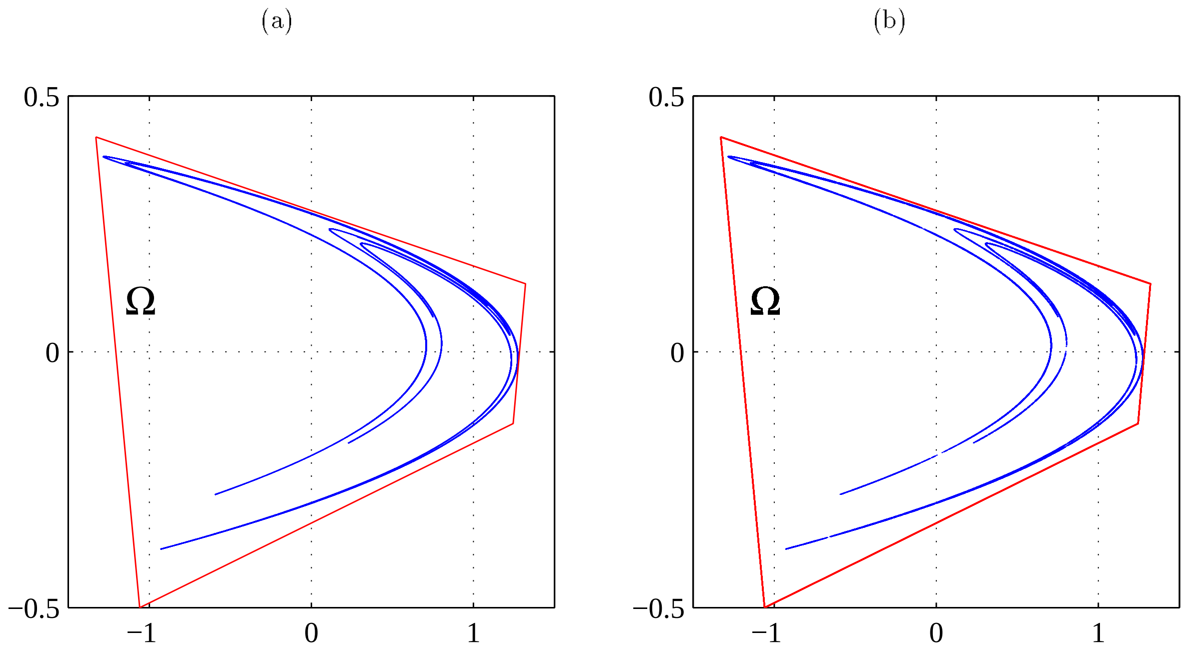

Figure 1: (a) trajectory of the Hénon map consisting of 100000 points and the trapping region Ω, (b) cycles with period $n \leq 27$ for the Hénon map

Hansen–Sengupta operator is defined as

$$\mathbf{H}(\mathbf{x}) = x_0 + \Gamma(C\mathbf{D}f(\mathbf{x}), -Cf(x_0), \mathbf{x} - x_0), \qquad (3)$$

where Γ is the Gauss–Seidel operator [4]. For intervals $\mathbf{a}, \mathbf{b}, \mathbf{x}$ the Gauss–Seidel operator $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is the tightest interval enclosing the set $\{x \in \mathbf{x} \colon ax = b \text{ for some } a \in \mathbf{a}, b \in \mathbf{b}\}$ and for interval matrix $\mathbf{A}$ and interval vectors $\mathbf{b}, \mathbf{x}$ the Gauss–Seidel operator $\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})$ is defined by

$$\begin{aligned} \mathbf{y}_i &= \Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})_i \\ &= \Gamma\left(\mathbf{A}_{ii}, b_i - \Sigma_{k<i}\mathbf{A}_{ik}\mathbf{y}_k - \Sigma_{k>i}\mathbf{A}_{ik}\mathbf{x}_k, \mathbf{x}_i\right). \end{aligned} \qquad (4)$$

For these two operators there are similar theorems on the existence of zeros (see [4]).

C. Standard and global versions

The above three operators can be used to prove the existence of period-n cycles of f by applying the interval operator to the map $\mathrm{id} - f^n$. We shall call this technique a *standard* version of the method.

Another choice, which will be called a *global* version, is to apply the interval operator to the global map $F : (\mathbb{R}^m)^n \mapsto (\mathbb{R}^m)^n$ defined by

$$[F(z)]_k = x_{(k+1) \bmod n} - f(x_k) \qquad (5)$$

for $k = 0, \ldots, n-1$, where $z = (x_0, \ldots, x_{n-1})$. See that $F(z) = 0$ if and only if x_0 is a fixed point of f^n. In this method the problem of existence of periodic orbits is translated to the problem of existence of zeros of a higher–dimensional function.

D. Finding all periodic orbits

In this study we are interested in finding for a given map all period-n cycles in a certain region.

In order to find fixed points of f^n we use the combination of the interval method and the generalized bisection (see also [5]). We describe the technique for interval Newton operator (for other operators we proceed in the same way). First the region of interest is covered by m–dimensional intervals (the number of them increases with n). For each interval $\mathbf{x}$ the interval operator $\mathbf{N}(\mathbf{x})$ is evaluated. If $\mathbf{N}(\mathbf{x}) \subset \mathrm{int}(\mathbf{x})$ then there is exactly one fixed point of h^n in $\mathbf{x}$. If $\mathbf{N}(\mathbf{x}) \cap \mathbf{x} = \emptyset$ then there are no fixed points of h^n in $\mathbf{x}$. If none of these two conditions is fulfilled we divide the interval vector $\mathbf{x}$ into smaller parts and repeat the computations.

E. Reducing the dimension of the search space for the global version

The problem that arises, when we implement the global version is the dimension of the space, where we are looking for periodic orbits. In order to find all period–n orbits of an m–dimensional map we have to search an mn–dimensional space.

In order to reduce the dimension of the search space we propose to use $\mathbb{R}^m$ as the search space. For the interval vector $\mathbf{x} \in \mathbb{R}^m$ we first produce the sequence $(\mathbf{x}_i)_{i=0}^{n-1}$, where $\mathbf{x}_i = f^i(\mathbf{x})$ and we set $\mathbf{z} = (\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})$. Then we apply the global interval Newton operator to $\mathbf{z}$. If the division is necessary we divide the m–dimensional interval $\mathbf{x}$, instead of mn–dimensional interval $\mathbf{z}$. Although some of the components of $\mathbf{z}$ generated from $\mathbf{x}$ using the procedure described above may by large (due to the wrapping effect and positive Lyapunov exponents of f if f is chaotic) it appears that this method is superior to all the other methods.

F. Further modifications

In order to speed up the algorithm for the case of the Hénon map we add two modifications. First modification uses the fact that we search for periodic solutions in the trapping region and that the Hénon map is invertible. For the interval $\mathbf{x}$ under consideration we compute its inverse $f^{-i}(\mathbf{x})$. If for some positive i the inverse $f^{-i}(\mathbf{x})$ lies outside the trapping region than there is no periodic orbit in $\mathbf{x}$ that also visits the trapping region. One can also check forward iterations in similar way but it gives no improvement due to the nature of the trapping region.

The second modification is based on the fact that we are searching for periodic orbits. For the interval $\mathbf{x}$ under consideration we compute several forward and backward iterations. If any of this iterations is included in the region for which the algorithm was completed then we can skip the interval $\mathbf{x}$, as there are no new periodic orbits in $\mathbf{x}$.

III. PERIODIC ORBITS FOR THE HÉNON MAP

In this section we test the methods described above. First we compare five versions of the interval Newton method: standard version (Newton Standard), standard version with modifications (Newton Standard +), global version with the search space $\mathbb{R}^2$ (Newton Global), global version with the search space $\mathbb{R}^2$ and with modifications (Newton Global +) and global version with $\mathbb{R}^{2n}$ search space (Newton Global N). The results are shown in Fig. 2a. For $n \leq 3$ the standard Newton method is the best one. For $4 \leq n \leq 12$ the standard interval Newton method with modifications is the quickest one. It is not possible however to use the standard Newton method for finding all longer orbits. It appears that the method is not capable of finding all periodic orbits with period n for $n > 17$. For $n = 18$ there are some periodic points x for which one cannot check the assumption $\mathbf{N}(\mathbf{x}) \subset \mathbf{x}$ for any interval vector $\mathbf{x} \ni x$. This is due to the wrapping effect, which causes that $\mathbf{D}h^n(\mathbf{x})$ has a very large diameter [6]. For $N \geq 13$ the global version with reduced search space and other improvements is better.

It is interesting to note that the global version with search space $\mathbb{R}^{2n}$ is the worst one. Although a lot of rectangles can be excluded before evaluation of interval operator, the algorithm is very slow. It is even

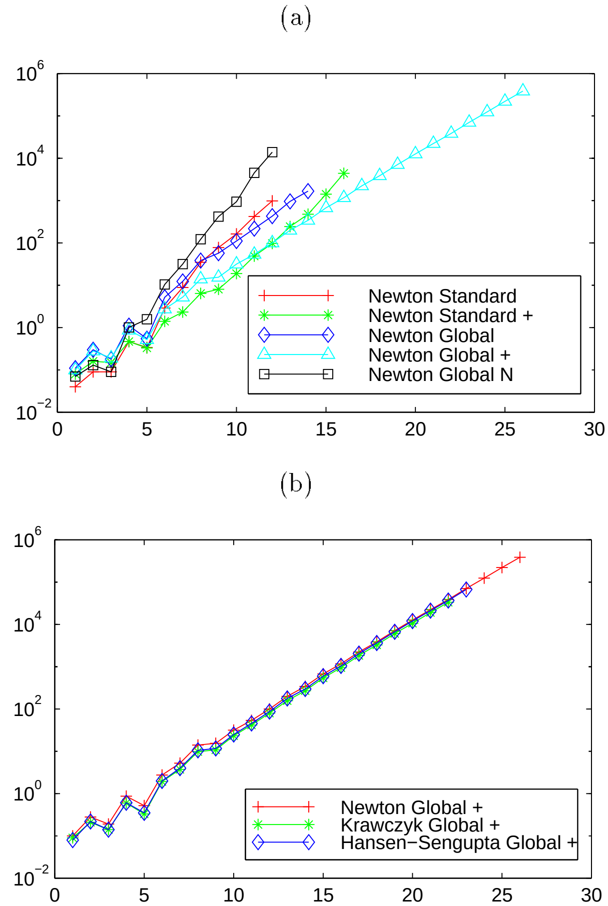

Figure 2: (a) Computation time needed to find all period–n cycles using different versions of the interval Newton method: standard version, standard version with modifications, global version, global version with modifications and global version with $\mathbb{R}^{2n}$ search space (b) Computation time needed to find all period–n cycles using Newton, Krawczyk and Hansen–Sengupta methods

slower than the algorithm based on the standard Newton operator and hence of not much use.

In Fig. 2b we show the computation time for global versions of Newton, Krawczyk and Hansen–Sengupta methods. One can clearly see that there are no significant differences in computation time between these three methods.

Using the Krawczyk method which is slightly better than the two other methods we have found all periodic orbits with period $n \leq 27$. The results are summarized in Table 1. We have proved that there are 30326 periodic orbits with period $n \leq 27$ and there are 760909 points belonging to these orbits. These unstable periodic points shown in Fig.1b give very good approximation of the Hénon attractor. In this picture one can hardly see two very small regions of the attractor not

n	Q_n	P_n	$Q_{\le n}$	$P_{\le n}$	H_n	rectangles	time [s]
1	1	1	1	1	0.0000	47	0.06
2	1	3	2	3	0.5493	57	0.15
3	0	1	2	3	0.0000	71	0.14
4	1	7	3	7	0.4865	173	0.48
5	0	1	3	7	0.0000	148	0.28
6	2	15	5	19	0.4513	413	1.75
7	4	29	9	47	0.4810	493	3.51
8	7	63	16	103	0.5179	994	9.37
9	6	55	22	157	0.4453	1001	10.18
10	10	103	32	257	0.4635	1605	22.24
11	14	155	46	411	0.4585	2175	39.72
12	19	247	65	639	0.4591	3684	75.76
13	32	417	97	1055	0.4641	5649	147.54
14	44	647	141	1671	0.4623	9208	267.57
15	72	1081	213	2751	0.4657	14949	519.55
16	102	1695	315	4383	0.4647	24153	950.66
17	166	2823	481	7205	0.4674	39452	1813.20
18	233	4263	714	11399	0.4643	61809	3188.78
19	364	6917	1078	18315	0.4654	98781	5811.89
20	535	10807	1613	29015	0.4644	159357	10340.88
21	834	17543	2447	46529	0.4654	257437	19122.05
22	1225	27107	3672	73479	0.4640	413477	32870.33
23	1930	44391	5602	117869	0.4653	679721	59594.13
24	2902	69951	8504	187517	0.4648	1133521	104863.78
25	4498	112451	13002	299967	0.4652	1835853	185243.31
26	6806	177375	19808	476923	0.4648	3030141	324122.85
27	10518	284041	30326	760909	0.4651	4870321	570993.28

Table 1: Periodic orbits for the Hénon map. Q_n — number of periodic orbits with period n, P_n — number of fixed points of h^n, $Q_{\le n}$ — number of periodic orbits with period smaller or equal to n, $P_{\le n}$ — number of fixed points of h^i for $i \le n$, $H_n = n^{-1}\log(P_n)$ — estimation of topological entropy based on P_n.

visited by periodic orbits found.

IV. CONCLUSIONS

In this paper we have compared the performance of several interval methods, which can be used for finding all low–period cycles. We have shown that global versions with the reduced search space is superior to all other methods. We have also shown that using Krawczyk or Hansen–Sengupta operators does not reduce the computational time considerably, at least for the map considered in this paper.

Using interval methods we have found all periodic orbits for the Hénon map with period $n \le 27$.

Acknowledgments

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