Chaos in digital filters: a global picture

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1 Introduction

The complex behavior in second-order digital filters has attracted much recent interest [1, 2, 4, 6]. In this paper we consider the implications for the filter's dynamics of two rules for the correction of the overflow effect, namely the modular and saturation characteristics. To simplify analysis, we neglect the quantization error which occurs in the finite wordlength representation. Thus, under zero input conditions, the filter can be modeled by a two-dimensional discrete-time dynamical system with the following state equations [1]:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} x_2(k) \\ f[bx_1(k) + ax_2(k)] \end{pmatrix} = \mathbf{F} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \tag{1}$$

where $f(x)$ is the overflow rule. The state space is $I\!R^2$, but we concentrate mainly on the trajectories inside the invariant set $I^2 = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$ since $\mathbf{F}^k(I\!R^2) \subseteq I^2$ for all $k \ge 2$ for the overflow rules we consider.

2 Digital filters with modular nonlinearity

When we use 2's complement arithmetic as the overflow rule, $f(x)$ has the form:

$$f(x) = x - 2n \qquad \text{for } -1 + 2n \le x < 1 + 2n, \quad n \text{ an integer} \tag{2}$$

In this section we shall consider the probabilistic (measure-theoretic) approach in the analysis of chaos in the dynamical system $\mathbf{F}$. Measure-preserving dynamical system can display at least two different levels of chaotic behavior. They can be Bernouilli or exact systems.

B-systems and exact systems both have the mixing property. Mixing means that a set of initial conditions of nonzero measure will eventually spread over the whole phase space as the system evolves. In order for the reader to get an intuitive idea for the difference between exact systems and B-systems, we present the 7-th iterate of 40000 points in the area $|x| \le 0.025$, $|y| \le 0.025$ by a B-system (the map $\mathbf{F}$ with

Figure 1: The 7-th iterate of the map **F** for 40000 points in the block $|x| \leq 0.025$, $|y| \leq 0.025$. (a) $a = 3$, $b = -1$. (b) $a = 2$, $b = 6$.

$a = 3$ and $b = -1$) in Fig. 1a and by an exact system (the map **F** with $a = 2$ and $b = 6$) in Fig. 1b.

In this section we shall prove that in the region $|b| > 1$, if a and b are integers and $b \neq \pm a + 1$, the map **F** is exact. First we introduce some definitions from measure theory.

Definition 1 *Let (M, Ω, μ) be a normalized measure space, and $G : M \to M$ a measure-preserving transformation ($\mu(G^{-1}(A)) = \mu(A)$ for all $A \in \Omega$). G is called mixing if*

$$\lim_{n \to \infty} \mu(A \cap G^{-n}(B)) = \mu(A)\mu(B) \quad \textit{for all } A, B \in \Omega \tag{3}$$

The measure we will use for the map **F** will be the Borel measure.

Lemma 1 *Let $b \neq 0$ be an integer. Then the map* **F** *is measure-preserving.*

Theorem 1 *If $b = -1$, $a > 2$ and a is an integer, then* **F** *is mixing.*

Definition 2 *Let (M, Ω, μ) be a normalized measure space, and $G : M \to M$ a measure-preserving transformation such that for all $A \in \Omega$, $G(A) \in \Omega$. G is called exact if*

$$\lim_{n \to \infty} \mu(G^n(A)) = 1 \quad \textit{for every } A \in \Omega \ , \mu(A) > 0 \tag{4}$$

Theorem 2 *If a,b are integers, such that $b \neq a + 1$, $b \neq -a + 1$ and $b \neq 0, \pm 1$, then the map* **F** *is exact.*

It can be proved that exactness of G implies that G is mixing. The converse is not necessarily true; the mixing map **F** for $b = -1$ and a an integer larger than 2, is not an exact map. Recently [6], the map **F** was proved to be Bernouilli for $b = -1$ and $|a| > 2$.

3 Digital filters with saturation arithmetic

In this section we consider the saturation function for the overflow rule, i.e.:

$$f(x) = \frac{1}{2}(|x+1| - |x-1|) \tag{5}$$

We present the classification of limit sets for different values of system parameters. We consider the case $(a,b) \in Q = \{(a,b) : b < -1, b < a+1, b < -a+1\}$. Previously [3], it was proved that for $(a,b) \notin \overline{Q}$ all limit sets are periodic with period length one or two. Throughout this section we assume that $(a,b) \in Q$.

Let us define:

$$\Omega^\infty := \bigcap_{n=0}^{\infty} \mathbf{F}^n(I^2) \tag{6}$$

Let $\mathbf{W}^\infty$ be the boundary of Ω^∞ ($\mathbf{W}^\infty := \partial\Omega^\infty$).

Lemma 2 *Ω^∞ is an invariant absolutely convex polygon. Its boundary $\mathbf{W}^\infty$ is also invariant.*

Let us define

$$\Phi := \mathbf{F}|\mathbf{W}^\infty : \mathbf{W}^\infty \longmapsto \mathbf{W}^\infty \tag{7}$$

Lemma 3 *Φ is a continuous surjection. If $\mathbf{W}^\infty$ contains no corners of the state space I^2 then Φ is a homeomorphism.*

Theorem 3 *Let $\mathbf{x} \neq 0$. Then there exists $n_0 \geq 0$ such that for every $n \geq n_0$: $\mathbf{F}^n(\mathbf{x}) \in \mathbf{W}^\infty$.*

The above theorem states that every non-trivial trajectory in finite time enters the set $\mathbf{W}^\infty$ and remains in it. Thus we can reduce our study to the analysis of one-dimensional map of $\mathbf{W}^\infty$ into itself. As $\mathbf{W}^\infty$ is homeomorphic to a circle we can define the rotation number of Φ. The map Φ is weakly monotone which implies the existence of a unique rotation number for each pair (a,b).

Theorem 4 *If $(a,b) \in Q_3$, $\mathbf{x} \neq O$, ρ is the rotation number of Φ, then*

1. *If Φ is not a homeomorphism then ρ is rational.*
2. *If ρ is rational ($\rho = p/q$) then the limit set of $\mathbf{x}$ is a period-q orbit contained in $\mathbf{W}^\infty$.*
3. *If ρ is irrational then the limit set of $\mathbf{x}$ is dense in $\mathbf{W}^\infty$.*

In Fig. 2 we present the structure of Arnold tongues on parameter plane. The regions with the same rotation number are shown. Using Lemma 3 it is possible to find parameters (a,b) for which Φ is a homeomorphism. Points (a,b) lying inside the half-circular regions correspond to homeomorphic Φ and for other points the map Φ is not homeomorphic.

Figure 2: The ranges of parameters (a, b) with a given rotation number. (a) Global diagram, (b) Fine structure of Arnold tongues.

4 Conclusions

We considered the implications of two overflow rules for the filter's behaviour. We observed that the overflow rule used is crucial for the filter's dynamics. In the case of modular characteristic we proved the strong chaotic filter's behaviour (mixing, exactness) in a wide range of parameters. On the other hand the filter with saturation characteristic was proved not to be chaotic. In particular we proved in that case the existence of periodic and quasi-periodic limit sets only.

References

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