PERIODIC BEHAVIORS IN A DIGITAL FILTER WITH TWO'S COMPLEMENT ARITHMETIC

Xinghuo Yu, Senior Member, IEEE
Faculty of Informatics and Communication
Central Queensland University, Rockhampton QLD 4702, Australia
E-mail: x.yu@cqu.edu.au

Zbigniew Galias*, Member, IEEE
Institute for Nonlinear Science, University of California, San Diego
San Diego, CA 92093-0402, USA
e-mail: galias@ucsd.edu

June 11, 2001

Indexing terms: Chaos, Filters (circuits), Periodicity

Abstract

It is well known that the second order digital filter with two's complement arithmetic may exhibit chaotic behaviors [1, 2]. It is also known that for certain class of filter parameters, the second order filter exhibits periodic behaviors. This paper studies the relation between the period of periodic trajectories and the period of trajectory traveling patterns for the particular class of filter parameters. A complete classification of periodic behaviors is given and the underlying relations are fully explored. The shape and layout of the regions in the state space displaying periodic behavior of the same type are fully examined. The mathematical analysis is accompanied by considerable simulation results.

1 Introduction

In recent years, fast advances in semiconductor devices, integrated circuits and computer technology have made it possible to have a wide range of applications of digital filtering techniques in areas such as speech and image processing, consumer electronics, digital communications and control systems [2]. Digital filtering techniques provide an easy-to-use and efficient digital representation for signal processing and transmission. Digital filtering is about transformations of the input data in the form of a sequence of numbers

*Permanent address: Dept. of Electrical Engineering, University of Mining and Metallurgy, al. Mickiewicza 30, 30–059 Kraków, Poland, e-mail: galias@zet.agh.edu.pl

(discrete in nature) into another data set representing a sequence of numbers at the output. Due to the nonlinearities introduced in real world hardware implementations, complex behaviors such as oscillations and irregular behaviors may occur.

In the well known paper [1], chaos in the following second order digital filter with two's complement arithmetic was studied

$$x(k+1) = F(x(k)) = Ax(k) + Bs_k \tag{1}$$

where $x = (x_1,\ x_2)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2}$$

$$s_k = \begin{cases} -1 & \text{if } bx_1(k) + ax_2(k) \geq 1 \\ 1 & \text{if } bx_1(k) + ax_2(k) < -1 \\ 0 & \text{otherwise} \end{cases}$$

The system behaviors with parameters a and b on the stability margin $|a| < 2,\ b = -1$ were of interest. Under the condition $s_k = 0$ the equation (1) is linear and there exists a linear transformation matrix

$$T = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{pmatrix}, \tag{3}$$

such that the matrix A becomes

$$A = T \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} T^{-1}, \tag{4}$$

where $\cos\theta = a/2,\ 0 < \theta < \pi$. The chaotic region was shown to be located on the boundary of the stable region of the filter. It was proved that for $\theta = 2\pi r$ where r is an irrational number, the system exhibits various chaotic behaviors such as ellipse like fractals in the region $\rho(x) \geq 1$ where

$$\rho(x) = \sqrt{\frac{(x_1 + x_2)^2}{2+a} + \frac{(x_1 - x_2)^2}{2-a}}.$$

It was shown that for certain values of initial conditions the symbolic sequence is periodic, which corresponds to quasi–periodic trajectory filling densely a finite number of ellipses or a periodic orbit visiting centers of these ellipses [1, 3]. This is due to the property that for an irrational r, $A^K \neq I$ for any integer K. Although it is known that for rational r the digital filter may exhibit periodic behaviors [2] the relation between the rational r, the periods of the trajectories and their traveling patterns have not been fully explored.

In this paper, we investigate the relation between the rational r, the period of the periodic trajectories, and the period of the trajectory traveling pattern. A complete classification of periodic behaviors is given and mathematical analysis of the underlying relations is presented. The shape and layout of the regions in the state space corresponding to a given traveling pattern are examined. Considerable simulation studies are presented to show the intriguing behaviors.

2 Simulation Results

It was shown in [1] that for irrational r, the chaos starts when the parameter setting is beyond its stability region. It is similar for rational r as well because the overflow nonlinearity activates in the same way.

Because of the switching value s, the phase plane is divided into three regions D_0, D_+, D_- defined as follows

$$\begin{aligned} D_0 &= \{x : \ -1 \leq -x_1 + ax_2 < 1\}, \\ D_+ &= \{x : \ -x_1 + ax_2 \geq 1\}, \\ D_- &= \{x : \ -x_1 + ax_2 < -1\}, \end{aligned}$$

which correspond to s_k taking value $0, \ +1, \ -1$ respectively. For a given initial conditions the system (1) can be viewed as a linear system driven by the s sequence. There is a well-defined map between the phase plane $\mathbb{R}^2$ and the sequence space $\Sigma = \{s = (s_0, s_1, \cdots) : s_k = -1, 0, 1; \ k = 0, 1, 2, \cdots\}$ [1]. Let us define $L(s)$ as the period of the trajectory traveling pattern, that is the symbolic sequence, s. For example, if the trajectory moves in a complex periodic pattern, say

$$s = (\cdots +1 +1000 -1 -100 \ \ +1 +1000 -1 -100 \cdots)$$

then we say the period of the symbolic sequence is $L(\bar{s}) = 9$ and we represent the period–L symbolic sequence as

$$s = (++000--00).$$

We also denote $r = q/p$ where q and p are positive integers satisfying $2q < p$ and $\gcd(p, q) = 1$ (gcd stands for greatest common divisor). Note that since $\cos\theta = \cos(2\pi - \theta)$ and the only parameter considered is $a = 2\cos\theta$, it is sufficient to study $\theta \in (0, \ \pi)$, that is $0 < 2q < p$.

It can be shown that for arbitrary initial conditions the trajectory of the system (1) for $b = -1$ and $a \in [-2, 2]$ after a finite number of steps enters the region

$$I^2 = [-1, 1] \times [-1, 1], \tag{5}$$

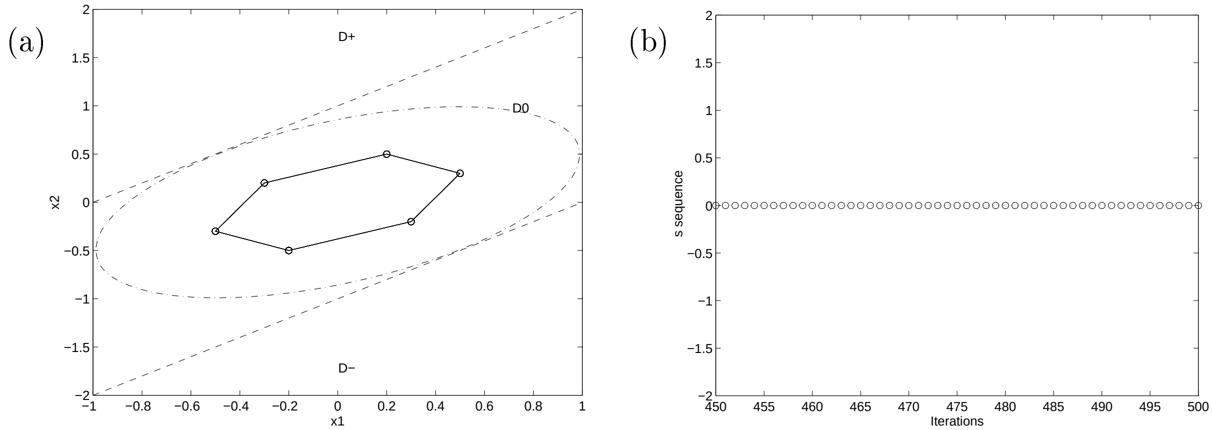

Figure 1: $r = 1/6$, $x(0) = (0.5,\ 0.3)^{\mathrm{T}}$,$s = (0)$, $L = 1$, (a) The trajectory travels on an ellipse in D_0 with period 6, (b) The s sequence of the last 50 iterations.

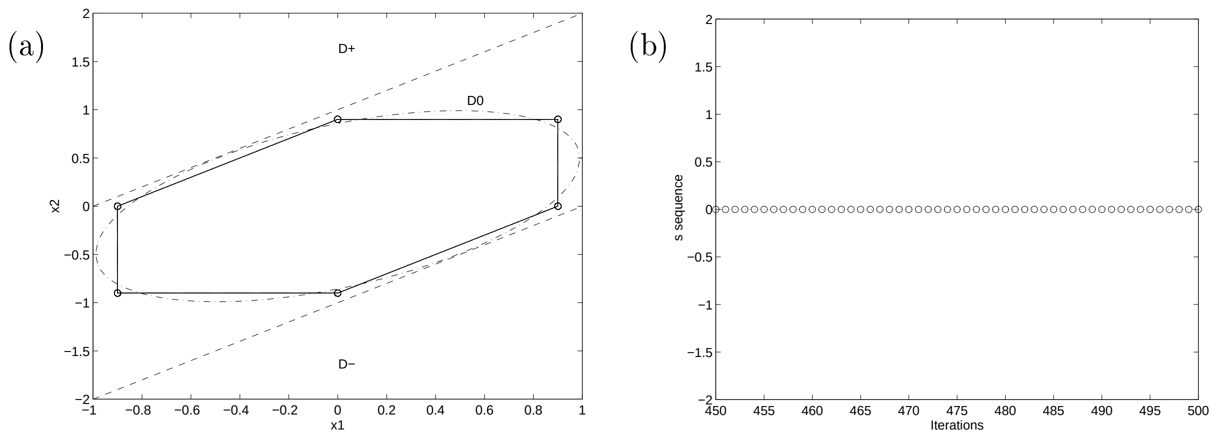

Figure 2: $r = 1/6$, $x(0) = (0.9,\ 0.9)^{\mathrm{T}}$, $s = (0)$, $L = 1$, (a) The trajectory travels on an ellipse in D_0 with period 6 but $\rho > 1$, (b) The s sequence of the last 50 iterations.

and remains in this set. Obviously all periodic orbits must be enclosed within I^2. Hence in our study we limit ourselves to the set I^2 of initial conditions and consider the system (1) as a map from I^2 to I^2.

Numerous simulations were performed (with 500 iterations). Some typical behaviors are depicted in Figures 1–8. (The dashed lines show the boundaries between D_0, D_+ and D_-. The dot-dashed line represents the ellipse $\rho(x) = 1$).

Figures 1 and 2 depict an interesting periodic behavior within D_0 (i.e., $s_k = 0$ for all k). The trajectory travels on a finite set of points resembling an elliptic shape with period equal to p. The period of the symbolic sequence is $L = 1$. It is interesting to note that Figure 2 shows that the periodic trajectory actually travels in a finite set of states outside the ellipse $\rho(x) \leq 1$ but still stays in D_0, in contrast to the irrational case where $\rho(x(0)) > 1$ implies that an overflow $s_k \neq 0$ will occur for some positive k.

Figures 3–8 present much intriguing behaviors where the trajectory travels within D_0, D_+ and D_-. Figures 3 and 4 illustrate that the trajectory travels in D_+ and D_-

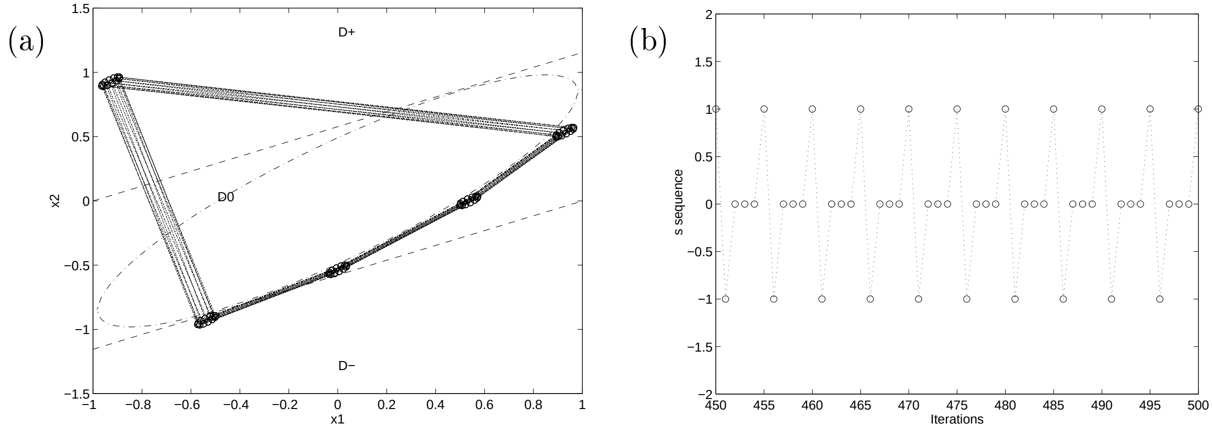

Figure 3: $r = 1/12$, $x(0) = (0.9, 0.5)^{\mathrm{T}}$, $s = (+-000)$, $L = 5$, (a) The trajectory travels in D_0, D_+ and D_- periodically with period $5 \times 12 = 60$, (b) The s sequence of the last 50 iterations.

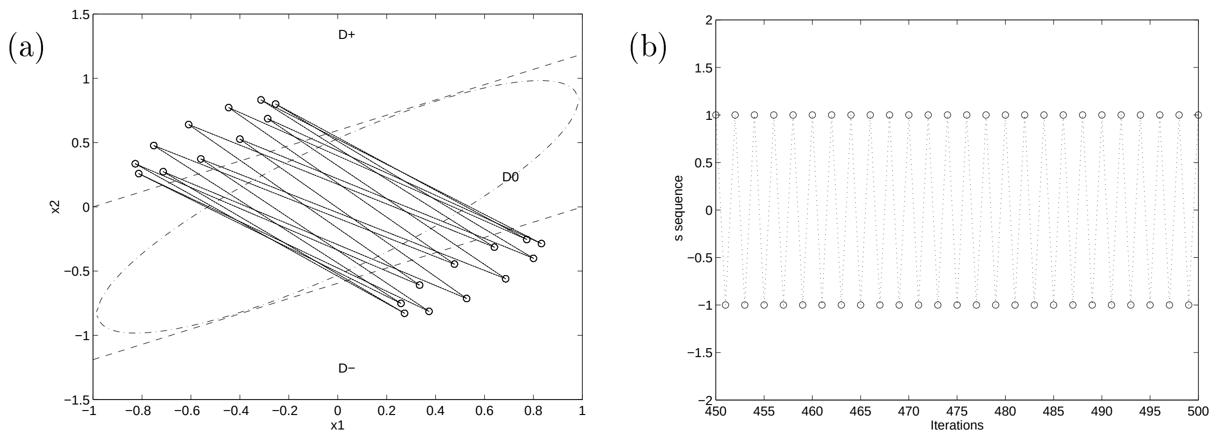

Figure 4: $r = 1/11$, $x(0) = (0.8, -0.4)^{\mathrm{T}}$, $s = (-+)$, $L = 2$, (a) The trajectory travels in D_+ and D_- periodically with period $2 \times 11 = 22$, (b) The s sequence of the last 50 iterations.

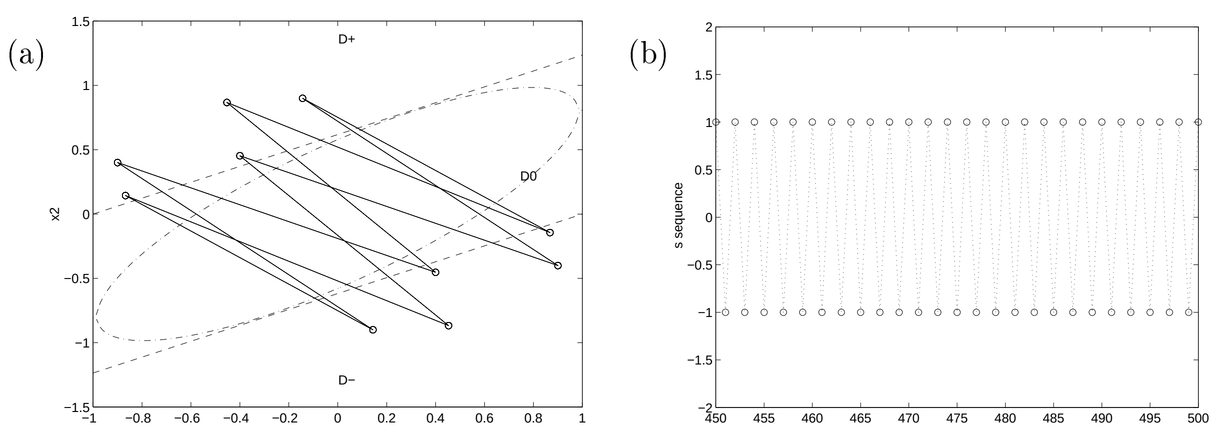

Figure 5: $r = 1/10$, $x(0) = (0.9, -0.4)^{\mathrm{T}}$, $s = (+-)$, $L = 2$, (a) The trajectory travels in D_+ and D_- periodically with period 10, (b) The s sequence of the last 50 iterations.

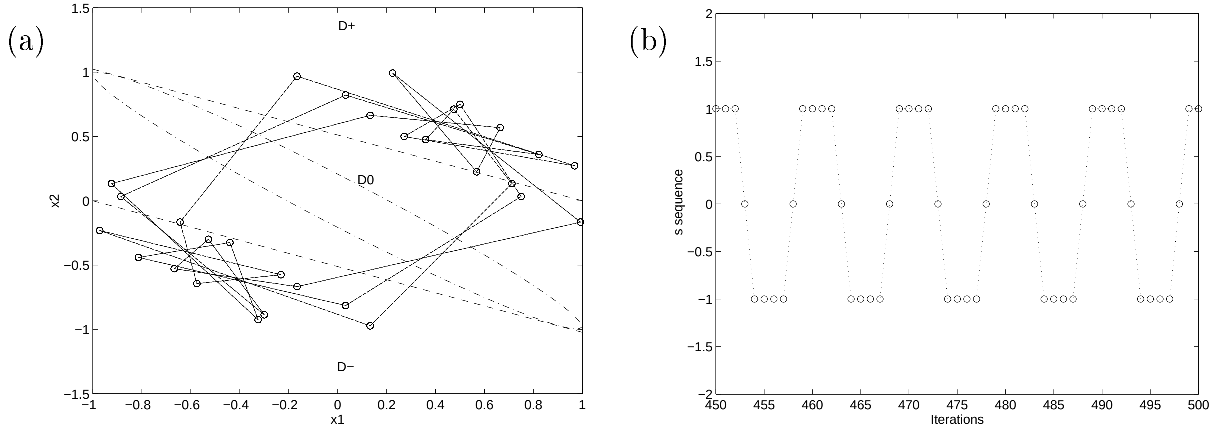

Figure 6: $r = 7/15$, $x(0) = (0.5, 0.75)^{\mathrm{T}}$, $s = (----0++++0)$, $L = 10$, (a) The trajectory travels in D_0, D_+ and D_- periodically with period 30, (b) The s sequence of the last 50 iterations.

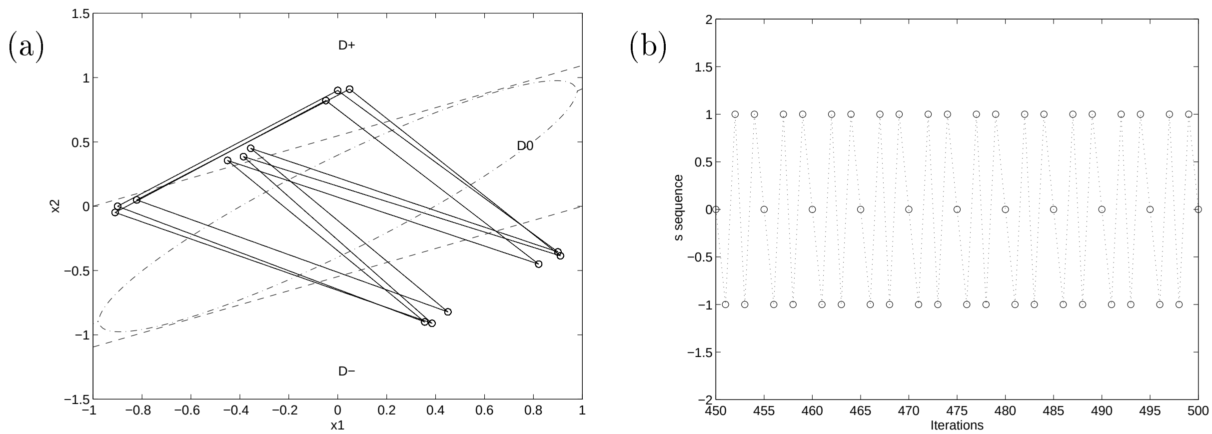

Figure 7: $r = 1/15$, $x(0) = (0.85, -0.4)^{\mathrm{T}}$, $s = (-+-+0)$, $L = 5$, (a) The trajectory travels in D_0, D_+ and D_- periodically with period $3 \times 5 = 15$, (b) The s sequence of the last 50 iterations.

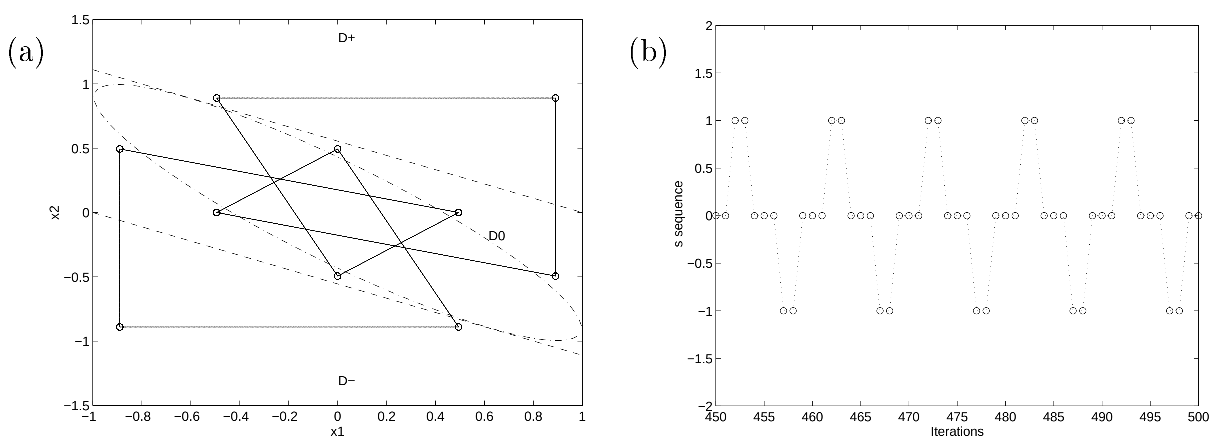

Figure 8: $r = 3/7$, $x(0) = (-0.4940, 0.8901)^{\mathrm{T}}$, $s = (++\,000--000)$, $L = 10$, (a) The trajectory travels in D_0, D_+ and D_- periodically with period 10, (b) The s sequence of the last 50 iterations.

periodically with periods 60 and 22 respectively which are the multiple of p and L. Figure 5 shows the trajectory traveling in D_+ and D_- periodically with period 10 which is the least common multiple of p and L. This particular behavior can be identified in Figures 6 and 7 where the trajectory travels in D_0, D_+ and D_- periodically with periods 30 and 15 respectively which are the least common multiples of p and L. Figure 8 describes an even more interesting case because the period of trajectory is neither a multiple of p nor a divisor of p.

The question to be asked is, what is the relation between the period of the system periodic behaviors, the period of their traveling patterns L and the parameter q/p? This question will be addressed in the following sections.

3 Analysis of Periodic Behaviors

In the following we assume that the rotation number of the considered system is rational, i.e. $a = 2\cos(2\pi q/p)$ and q and p do not have common factors larger than one ($\gcd(q,p) = 1$).

Let us define three affine maps:

$$F_s(x) = Ax + bs, \text{ for } s = -1, 0, +1. \tag{6}$$

Let us consider a trajectory starting at the initial point $x(0)$ and assume that $x(0)$ corresponds to the symbolic sequence $s = (s_0, s_1, s_2, \dots)$. First let us observe that the kth iteration of the system starting from $x(0)$ can be computed as

$$x(k) = (F_{s_{k-1}} \circ \cdots \circ F_{s_1} \circ F_{s_0})(x(0)) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i} B s_i, \quad k \geq 0. \tag{7}$$

See that for the fixed symbolic sequence $F_{s_{k-1}} \circ \cdots \circ F_{s_1} \circ F_{s_0}$ is an affine map. We will show that periodic orbits of the system (1) are closely related to the situation when this map becomes identity for particular choices of symbolic sequence $s = (s_0, s_1, \dots, s_{k-1})$. The equation (7) can be rewritten in the following form:

$$x(k) = A^k x(0) + \Psi_k \cdot (s_0, s_1, \dots, s_{k-1})^{\mathrm{T}}, \tag{8}$$

where

$$\Psi_k = \left(A^{k-1}B, \dots, A^2B, AB, B\right). \tag{9}$$

Now let us assume that $x(0)$ is a periodic point, i.e. there exist $n > 0$ such that $x(n) = x(0)$. We say that n is the minimum period if $x(k) \neq x(0)$ for all $k = 1, 2, \dots, n-1$. Obviously if $x(0)$ is periodic then the symbolic sequence is also periodic. Its period can in general be smaller than n. The period of the symbolic sequence, L, must be a divisor of n. In theory, all period–n orbits for given values of parameters can be obtained by solving the equation

$$x(0) = A^n x(0) + \sum_{i=0}^{n-1} A^{n-i-1} B s_i$$

for all symbolic sequences $s = (s_0, s_1, \cdots, s_{L-1})$, where L is a divisor of n. This method for finding periodic solutions is very ineffective. In order to find all period–n orbits one needs to solve 3^n linear equations.

In this section we investigate the relation between the period n of the orbit, the period L of the symbolic sequence and parameters of the system (specifically the parameter p). We show that given L and p we can fully classify periodic orbits.

This will allow us to analyze the behavior shown in Figures 1–8. Note that in all the simulations $L \neq p$. Furthermore, the discrete states appear to be grouped into clusters and the number of clusters is the same as the period of the symbolic sequence (apart from the case presented in Fig. 8). In the following, we will present theorems which can be used to describe all these behavior types.

The first theorem states that if a point has periodic symbolic sequence with period L then it is periodic with period n being the least common multiple (lcm) of L and p.

Theorem 1. *Let us assume that the symbolic sequence s of the trajectory starting at $x(0)$ has period L ($s = (s_0, s_1, \ldots, s_{L-1})$). Let us define n as the least common multiple of p and L, i.e. $n = \mathrm{lcm}(p, L)$, and let us extend s periodically so that the sequence s has length n. Then*

$$y_s = \Psi_n s^{\mathrm{T}} = \sum_{i=0}^{n-1} A^{n-i-1} B s_i = 0. \tag{10}$$

The map $F_{s_{n-1}} \circ \cdots \circ F_{s_1} \circ F_{s_0}$ is identity and $x(0)$ is periodic with period n (not necessary minimum).

Proof. First let us observe that

$$A^k = T \begin{pmatrix} \cos(2\pi qk/p) & \sin(2\pi qk/p) \\ -\sin(2\pi qk/p) & \cos(2\pi qk/p) \end{pmatrix} T^{-1}. \tag{11}$$

It follows that $A^p = I$ and since n is a multiple of p we have $A^n = I$. The symbolic sequence s is periodic and then according to (8) we have

$$x((k+1)n) = A^n x(kn) + \Psi_n s^{\mathrm{T}} = x(kn) + y_s, \text{ for } k \geq 0. \tag{12}$$

Hence

$$x(kn) = x(0) + ky_s, \text{ for all } k \geq 0.$$

Now suppose $y_s \neq [0,0]^{\mathrm{T}}$. Then it is clear that there exists $k > 0$ such that $x(kn)$ is arbitrary far from $x(0)$ and in particular $x(kn) \notin [-1,\ 1] \times [-1,\ 1]$, which is a contradiction. Thus $y_s = 0$. According to (7) $F_{s_{n-1}} \circ \cdots \circ F_{s_1} \circ F_{s_0}$ is identity and $x(n) = x(0)$. □

The above theorem tells that if the symbolic sequence is periodic the trajectory also has to be periodic and the period must be a multiple of L not larger than $n = \mathrm{lcm}(L, p)$. This is all we can say in the general case. Further description is possible when we consider two cases. The first case, which is observed in most experiments (and in all of the examples shown in Figures 1–8) takes place when L is not a multiple of p. It is studied in the following section. The second case, when L is a multiple of p happens very infrequently. It will be studied later.

3.1 L is not a multiple of p

We first derive results for the case when L is not a multiple of p. We will prove the results on the minimum period of the trajectory in the case. We will show that in this case the trajectory consists of points lying on L ellipses, which centers are defined by the symbolic sequence. Before we proceed, we introduce the concept of elliptic sets. Consider the difference equation

$$z(k+1) = Az(k),$$

where $z(k) \in \mathbb{R}^2$ and A is the same as defined in (2) with $b = -1$ and $|a| < 2$. One can easily check that for the energy type function

$$V(z) = z_2^2 - az_1z_2 - bz_1^2 = (z_2 - \frac{a}{2}z_1)^2 + (-b - \frac{a^2}{4})z_1^2, \tag{13}$$

and for any positive constant $c > 0$, $V(z) = c$ represents an ellipse for $x \in \mathbb{R}^2$ [5]. In fact, this is the orbit where the system state travels for $s = (0)$. One can also easily verify that $V(z(k+1)) = V(z(k)) = V(z(0))$. If there is a set of states $\{z(0), \cdots, z(m-1)\}$ such that each member $z(k)$ of the set satisfies $V(z(k)) = V(z(0))$, then we call the set an elliptic set. It is clear that the points $z(k)$ lie on the ellipse centered at the origin. The size of the ellipse is defined by $z(0)$. In the following we will also consider elliptic sets centered at points other than the origin. This concept will be used in the following sections.

Theorem 2. *Let us assume that the symbolic sequence s of the initial point starting at $x(0)$ has period L, which is not a multiple of p. Let us define the points*

$$z_i = (I - A^L)^{-1}\Psi_L s^i, \text{ for } i = 0, \ldots, L-1 \tag{14}$$

where Ψ_L is defined in (9) and s^i is the symbolic sequence obtained by shifting s to the right i–times, i.e., $s^i = (s_i, s_{i+1}, \cdots, s_{L-1}, s_0, \cdots, s_{i-1})^{\mathrm{T}}$.

1. *If $x(0) \neq z_0$ then the trajectory starting at $x(0)$ has the minimum period $n = \mathrm{lcm}(p, L)$. Furthermore, the points of the periodic trajectory are grouped into L elliptic sets of points centered at z_i and each of the elliptic sets is composed of n/L points.*

2. *If $x(0) = z_0$ then the trajectory starting at $x(0)$ has the minimum period L and $x(i) = z_{i(\bmod L)}$.*

Proof. First let us observe that $\det(I - A^L) = 2(1 - \cos(2\pi qL/p)) \neq 0$, since $\gcd(p, q) = 1$ and L is not a multiple of p. Let us denote by $(z_0, z_1 \ldots, z_{L-1})$ the unique solution of the set of equations

$$\begin{aligned} z_1 &= Az_0 + Bs_0 \\ &\vdots \\ z_{L-1} &= Az_{L-2} + Bs_{L-2} \\ z_0 &= Az_{L-1} + Bs_{L-1} \end{aligned} \tag{15}$$

Straightforward algebraic manipulations of (15) yield

$$\begin{aligned} (I - A^L)z_0 &= \Psi_L(s_0, s_1, \cdots, s_{L-1})^{\mathrm{T}} = \Psi_L s^0, \\ (I - A^L)z_1 &= \Psi_L(s_1, \cdots, s_{L-1}, s_0)^{\mathrm{T}} = \Psi_L s^1, \\ &\vdots \\ (I - A^L)z_{L-1} &= \Psi_L(s_{L-1}, s_0, \cdots, s_{L-2})^{\mathrm{T}} = \Psi_L s^{L-1}. \end{aligned} \tag{16}$$

Since $\det(I - A^L) \neq 0$ it follows that the solutions z_i are uniquely determined as

$$z_i = (I - A^L)^{-1}\Psi_L s^i, \text{ for } i = 0, \ldots, L-1.$$

Let n be the least common multiple of p and L. Define two integers $n_p = n/p$ and $n_L = n/L$.

A trajectory starting from $x(i + jL)$ (where $i = 0, 1, \ldots, L-1$ and $j \geq 0$) has a periodic symbolic sequence s^i. Hence it can be easily derived from (8) that

$$x(i + (j+1)L) = A^L x(i + jL) + \Psi_L s^i, \text{ for } i = 0, \ldots, L-1, j \geq 0. \tag{17}$$

Denote $y_i(j) = x(i + jL) - z_i$ for $i = 0, 1, \dots, L - 1$. $y_i(j)$ is the coordinate of the point $x(i + jL)$ after moving the origin to z_i. Then we have

$$y_i(j+1) = x(i + (j+1)L) - z_i = A^L x(i + jL) + \Psi_L s^i - z_i \tag{18}$$

From (17), we have $\Psi s^i = z_i - A^L z_i$, then (18) becomes

$$y_i(j+1) = A^L x(i + jL) - A^L z_i = A^L(x(jL) - z_i) = A^L y_i(j) \tag{19}$$

Iterating (19) n_L times yields

$$y_i(j + n_L) = A^{n_L L} y_i(j) = A^n y_i(j) = A^{pn_p} y_i(j) = y_i(j), \tag{20}$$

since $A^p = I$.

Equation (20) demonstrates two facts. First, it indicates that the least common multiple n of p and L is the period of the trajectory. Indeed $x(n) = x(n_L L) = y_0(n_L) + z_0 = y_0(0) + z_0 = x(0)$ which proves that n is the period. Second, if $x(0)$ is not equal to z_0 then equation (20) means that the successive (every L steps) states are on an elliptic set in the x_1–x_2 phase plane centered at z_i. The trajectory is then divided into L elliptic sets

$$\{x(i + jL) = z_i + y_i(j) = z_i + A^j y_i(0) : j = 0, \dots, n_L - 1\}_{i=0}^{L-1}, \tag{21}$$

From (21) one can easily see that for a fixed i the points $x(i + jL)$ are located on the ellipse centered at z_i. Thus we have proved that the points of periodic trajectory are grouped into L elliptic sets centered at z_i $(i = 0, 1, \dots, L - 1)$. Each elliptic set consists of $n_L = n/L$ points.

If $x(0) = z_0$ then $y_i(j) = 0$ for all $i = 0, 1, \dots, L-1$ and $j \geq 0$. In consequence $x(i) = z_i$ for $i = 0, 1, \dots, L-1$ and after L steps the trajectory returns to the initial condition. This case can be considered as the ith elliptic set compressed into its corresponding center. □

Theorem 2 can in fact explain all the behaviors in Figures 1–8. Figures 1 and 2 correspond to a single center (because of $L = 1$) with trajectories traveling on an elliptic set surrounding the center. Figures 3 and 4 are typical behaviors where 5 and 2 elliptic sets are found respectively. Figures 5–7 are another examples. The trajectories travel periodically on L elliptic sets, each of which contains the number of states equal to $\mathrm{lcm}(p, L)/L$. Figure 8 presents a degenerate case, which is the subject of the second part of the Theorem 2, where each elliptic set is shrunk to its center.

3.2 L is a multiple of p

Another case that needs to be explored is the case when L is a multiple of p. As demonstrated in the proof of Theorem 2 in this case $\det(I - A^L) = 2(1 - \cos(2\pi qL/p)) = 0$. It is not possible to find a unique solution of the equation (15). In consequence there is no corresponding elliptic sets. This situation is analyzed in the following theorem.

Theorem 3. *Let us assume that the symbolic sequence s of the trajectory starting at $x(0)$ has period L, which is a multiple of p. Then L is the minimum period of $x(0)$.*

Proof. Since L is a multiple of p it is clear that $L = \mathrm{lcm}(p, L)$. From Theorem 1 it follows that L is the period of $x(0)$. It must be the minimum period since period of the point in the state space cannot be smaller than the period of the corresponding symbolic sequence. □

Although Theorem 3 states that if there exists a period–L symbolic sequence s where L is a multiple of p, then $x(L) = x(0)$, it does not guarantee that such a symbolic sequence does exist. We now look at some constraints that limit the existence of admissible symbolic sequences. Let $n = \mathrm{lcm}(p, L)$ and $n = pn_p$, where n_p is a positive integer. Since $A^p = I$, then we have

$$x(n) = A^n x(0) + \sum_{i=0}^{n-1} A^{n-i-1} B s_i = x(0) + \sum_{i=0}^{n-1} A^{n-i-1} B s_i \tag{22}$$

From Theorem 1 it follows that $x(n) = x(0)$ and

$$y_s = \sum_{i=0}^{n-1} A^{n-i-1} B s_i = 0,, \tag{23}$$

which can be recast as

$$\sum_{i=0}^{n-1} A^{n-i-1} B s_i = \sum_{i=0}^{n_p-1} \sum_{j=0}^{p-1} A^{p-j-1} B s_{j+ip} = \sum_{j=0}^{p-1} A^{p-1-j} B \sum_{i=0}^{n_p-1} s_{j+ip} = 0. \tag{24}$$

Denote $\hat{s}_j = \sum_{i=0}^{n_p-1} s_{j+ip}$, then (24) becomes

$$[A^{p-1}B, A^{p-2}B, \cdots, B](\hat{s}_0, \hat{s}_1, \cdots, \hat{s}_{p-1})^{\mathrm{T}} = 0 \tag{25}$$

We can further explore the characteristic of (25). Because of (3) and (4), we have

$$A^k = T \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} T^{-1} = \sin^{-1}\theta \begin{pmatrix} -\sin(k-1)\theta & \sin k\theta \\ -\sin k\theta & \sin(k+1)\theta \end{pmatrix}$$

where $\theta = 2\pi/p$. Hence since $\sin\theta \neq 0$, then (25) is equivalent to

$$\begin{pmatrix} \sin(p-1)\theta & \sin(p-2)\theta & \cdots & \sin 2\theta & \sin\theta & 0 \\ 0 & \sin(p-1)\theta & \cdots & \cdots & \sin 2\theta & \sin\theta \end{pmatrix} \hat{s}^{\mathrm{T}} = 0 \tag{26}$$

where $\hat{s} = (\hat{s}_0, \hat{s}_1, \cdots, \hat{s}_{L-1})$. Note that $\sin(p-i)\theta = \sin((p-i)(2\pi/p)) = \sin(2\pi - 2\pi i/p) = -\sin i\theta$. If p is odd, then (26) becomes

$$\begin{pmatrix} -\sin\theta & -\sin 2\theta & \cdots & -\sin\frac{p-1}{2}\theta & \sin\frac{p-1}{2}\theta & \cdots & \sin\theta & 0 \\ 0 & -\sin\theta & -\sin 2\theta & \cdots & -\sin\frac{p-1}{2}\theta & \sin\frac{p-1}{2}\theta & \cdots & \sin\theta \end{pmatrix} \hat{s}^{\mathrm{T}} = 0$$

which leads to

$$\sum_{i=1}^{(p-1)/2} (\hat{s}_{i-1} - \hat{s}_{p-1-i}) \sin i\theta = 0, \quad \sum_{i=1}^{(p-1)/2} (\hat{s}_i - \hat{s}_{p-i}) \sin i\theta = 0 \tag{27}$$

If p is even, then (26) becomes

$$\begin{pmatrix} -\sin\theta & \cdots & -\sin\frac{p-2}{2}\theta & 0 & \sin\frac{p-2}{2}\theta & \cdots & \sin\theta & 0 \\ 0 & -\sin\theta & \cdots & -\sin\frac{p-2}{2}\theta & 0 & \sin\frac{p-2}{2}\theta & \cdots & \sin\theta \end{pmatrix} \hat{s}^{\mathrm{T}} = 0$$

which leads to

$$\sum_{i=1}^{\frac{p-2}{2}} (\hat{s}_{i-1} - \hat{s}_{p-1-i}) \sin i\theta = 0, \quad \sum_{i=1}^{\frac{p-2}{2}} (\hat{s}_i - \hat{s}_{p-i}) \sin i\theta = 0 \tag{28}$$

The equations (27) and (28) can be used to investigate the existence of admissible symbolic sequences. Any admissible symbolic sequence has to satisfy either (27) or (28). Generally, it is difficult to solve (27) and (28) to find admissible symbolic sequences. Furthermore, it appears to be few symbolic sequences which satisfy (27) or (28). In the following subsection, the problem of admissible sequences will be examined using a different approach.

3.3 Admissible sequences

In this section we investigate the problem of shape and layout of the sets in the state space corresponding to a given symbolic sequence. Let us assume that we have a period–L symbolic sequence $s = (s_0, s_1, \ldots, s_{L-1})$. We would like to find the set of points in the state space, which produces this symbolic sequence. We call a symbolic sequence admissible if there exists a point in the state space which realizes this sequence.

Let us denote by G_s the inverse of $F_s(x) = Ax + bs$, for $s = -1, 0, +1$. For $b \neq 0$ the maps G_s are well defined and can be computed as $G_s(x) = A^{-1}(x - bs)$. First we present a theorem which allows to effectively find the set in the state space corresponding to a given periodic symbolic sequence.

Theorem 4. *Let $s = (s_0, s_1, \ldots, s_{L-1})$ be the periodic symbolic sequence. Let $n = \mathrm{lcm}(L, p)$. The symbolic sequence s is admissible if and only if the following two conditions are satisfied*

$$y_s = \Psi_n s^{\mathrm{T}} = \sum_{i=0}^{n-1} A^{n-i-1} B s_i = 0. \tag{29}$$

$$W_s = \prod_{i=0}^{n-1} G_{s_0} \circ G_{s_1} \circ \cdots \circ G_{s_i - 1}(I^2) \neq \emptyset, \tag{30}$$

If s is admissible then the set of points which realize s is W_s.

Proof. Let us assume that the periodic symbolic sequence $s = (s_0, s_1, \dots, s_{L-1})$ is admissible. It means that there exists a point $x(0)$ which realizes this sequence. From the Theorem 1 it follows that $\Psi_n s^{\mathrm{T}} = 0$, i.e. the condition (29) is satisfied. The fact that $x(0)$ corresponds to the symbolic sequence $s = (s_0, s_1, \dots, s_{n-1})$ is equivalent to the following set of conditions:

$$x(0) \in I^2, F_{s_0}(x(0)) \in I^2, F_{s_1}(F_{s_0}(x(0))) \in I^2, \dots, F_{s_{n-1}}(\dots F_{s_1}(F_{s_0}(x(0))\dots) \in I^2. \quad (31)$$

This in turn is equivalent to

$$x(0) \in I^2, x(0) \in G_{s_0}(I^2), x(0) \in G_{s_0} \circ G_{s_1}(I^2), \dots, x(0) \in G_{s_0} \circ G_{s_1} \circ \dots \circ G_{s_n-1}(I^2). \quad (32)$$

In consequence the intersection (30) is not empty as it contains $x(0)$.

Now we prove that if the conditions (29) and (30) hold then the sequence is admissible. Let us choose a point $x(0)$ belonging to the product (30). It follows that

$$x(0) \in I^2, F_{s_0}(x(0)) \in I^2, F_{s_1}(F_{s_0}(x(0))) \in I^2, \dots, F_{s_{n-1}}(\dots F_{s_1}(F_{s_0}(x(0))\dots) \in I^2. \quad (33)$$

Hence the first n elements of the symbolic sequence of $x(0)$ are $s_0, s_1, \dots, s_{n-1}$. It follows from (7) and (29) that

$$x(n) = A^n x(0) + \sum_{i=0}^{n-1} A^{n-i-1} B s_i = x(0). \quad (34)$$

The point $x(0)$ is periodic and hence it has periodic symbolic sequence $s = (s_0, s_1, \dots, s_{n-1})$. In this way we proved that s is admissible. $\square$

From the above theorem it follows that the set of points in the state space corresponding to a particular symbolic sequence is the intersection of n sets. Each of these sets is a convex quadrangle (the maps $G_{s_0} \circ G_{s_1} \circ \cdot \circ G_{s_i-1}$ are affine), hence the intersection must be either empty or must be a convex polygon with at most $4n$ edges. I may also happen that the intersection is a degenerate polygon with empty interior (a segment or a point), although we have not observed such cases. This theorem provides an efficient method for finding the set in the state space corresponding to the given symbolic sequence. First we check the condition (29). If $y_s \neq 0$ the symbolic sequence is not admissible. In the opposite case we find the product (30). This whole procedure can be implemented on a computer. Finding intersection of two convex polygons is a standard problem of computational geometry, and algorithms to perform this task are widely available [4].

Using the above theorem we have found sets of points in the state space realizing different symbolic sequences for several values of parameters. We have considered $q/p =$

L	s	area
1	(0)	4.0000
	Total	4.0000

Table 1: Admissible symbolic sequences for $q/p = 1/4$.

L	s	area
1	(0)	3.0000
2	$(+-)$	1.0000
	Total	4.0000

Table 2: Admissible symbolic sequences for $q/p = 1/6$.

$1/4, 1/5, 1/6, 3/7, 2/9, 1/12$. As mentioned before one could find all period–n cycles by considering all period–n symbolic sequences and checking 3^n cases. This procedure is however not very efficient and allows finding only periodic trajectories with a small period. We have followed a different approach. For each parameter values we have generated trajectories starting from many initial conditions uniformly distributed in the state space. For each of the trajectories we have checked whether the symbolic sequence generated is periodic with period smaller than 200. This procedure gave us a set of admissible sequences. For each sequence s found using the Theorem 4 we have found the region W_s in the state space corresponding to this symbolic sequence.

The results are collected in Tables 1–6. The first and second column of each table give the period of the sequence and the sequence itself. The last column gives the area of the set of initial point corresponding to a given symbolic sequence s and other sequences obtained by shifting s (for example if $s = (0++)$, then the area is the sum of areas of sets of initial conditions with $s = (0++)$, $s = (+0+)$ and $s = (++0)$). In the last row we print the sum of the areas for all admissible symbolic sequences found.

These results are also presented in graphic form in Fig. 9, where using different colors we plot sets in the state space corresponding to admissible sequences found.

Let us discuss the results in more detail. As before the situation depends on whether L is a multiple of p or not.

If L is not a multiple of p then from Theorem 2 it follows that the periodic trajectory lies on L ellipses and there are n/L points in each elliptic set. Since $L \neq p$ it follows that $n/L > 1$. So we have at least two points in each elliptic set. Since W_s is convex one can prove that the centers of ellipses also belong to this set. In consequence the polygon W_s contains an ellipse, and in fact each polygon side is tangent to the ellipse which it contains. This is one of the differences between the case of rational rotation number and irrational one. In this last case sets corresponding to admissible periodic sequences are always ellipses.

L	s	area
1	(0)	3.0899
2	$(-+)$	0.3443
3	$(00+)$	0.1155
3	(-00)	0.1155
12	$(00+-+-+-+00+)$	0.0257
12	$(-+-00-00-+-+)$	0.0257
18	$(00+-+-00-00-+-+00+)$	0.1725
78	$(00+-+-00-00-+-+00+00+-+-00-00-+-+00+00+$ $-+-+-+00+00+-+-+-+00+00+-+-00-00-+-+00+)$	0.0094
78	$(0+00+-+-00-00-+-+00+00+-+-00-00-+-+-+-$ $00-00-+-+-+-00-00-+-+00+00+-+-00-00-+-+0)$	0.0094
102	$(00+-+-00-00-+-+-+-00-00-+-+-+-00-$ $00-+-+00+00+-+-00-00-+-+00+00+-+-+-+$ $00+00+-+-+-+00+00+-+-00-00-+-+00+)$	0.0544
	Total	3.9624

Table 3: Admissible symbolic sequences for $p/q = 1/5$.

L	s	area
1	(0)	1.3863
1	$(+)$	0.3285
1	$(-)$	0.3285
4	$(00--)$	0.1937
4	$(++00)$	0.1937
6	$(++0--0)$	0.6809
6	$(++0000)$	0.0238
6	$(0000--)$	0.0238
10	$(++000--000)$	0.1673
10	$(++++0----0)$	0.1675
22	$(++000--0000--0000--000)$	0.0010
22	$(++0000++000--000++0000)$	0.0010
22	$(++000--00--00--00--000)$	0.0128
22	$(++000--000++00++00++00)$	0.0128
22	$(++0--0++0---0++++0--0)$	0.0301
22	$(++0--0++0--0++++0----0)$	0.0301
34	$(++0000++000--00--00--00--000++0000)$	0.0345
34	$(++000--0000--0000--000++00++00++00)$	0.0345
80	$(++0000++000--00--00--00--000++0000++0000+$ $+000--00--00--00--000++0000++0000++0000)$	0.0160
80	$(++000--0000--0000--0000--0000--000++00++00+$ $+00++000--0000--0000--000++00++00++00)$	0.0160
86	$(++000--000++00++00++00++000--0000--0000--000+$ $+00++00++00++000--000++000--000++000--000)$	0.0040
86	$(++0000++000--00--00--00--000++000--000++000-$ $-000++000--000++000--00--00--00--000++0000)$	0.0040
	Total	3.6907

Table 4: Admissible symbolic sequences for $p/q = 3/7$.

L	s	area
1	(0)	3.1257
2	$(-+)$	0.1368
3	$(00+)$	0.1105
3	(-00)	0.1105
7	$(0000-00)$	0.0218
7	$(000000+)$	0.0218
10	$(0000-0000+)$	0.1287
24	$(0000-000000-000000-0000+)$	0.0036
24	$(0000-0000+000000+000000+)$	0.0036
38	$(00+-+-00-0000+0000-00-+-+00+0000-0000+)$	0.0107
52	$(0000-00-+-+00+0000-0000+00+-+-00-0000+000000+$ $000000+)$	0.0485
52	$(00+-+-00-0000+0000-00-+-+00+0000-000000-$ $000000-0000+)$	0.0485
66	$(0000-00-+-+00+0000-000000-000000-0000+00+-+$ $-00-0000+000000+000000+)$	0.0190
74	$(0000-0000+0000-0000+0000-0000+0000-0000+0000-$ $0000+0000-0000+000000+000000+)$	0.0010
74	$(0+0000-0000+00+-+-+-+-+-+-+-+00+0000-$ $000000-000000-0000+00+-+-+-+-+-+-+0)$	0.0010
74	$(000000+0000-00-+-+-+-+-+-+-+-00-0000+$ $0000-00-+-+-+-+-+-+-+-00-0000+000000+)$	0.0010
74	$(0+0000-000000-000000-0000+0000-0000+0000-0000+$ $0000-0000+0000-0000+0000-000)$	0.0010
98	$(0000-0000+0000-0000+000000+000000+0000-0000+0000-$ $0000+0000-0000+0000-000000-000000-0000+0000-0000+)$	0.0052
102	$(000000+000000+000000+000000+000000+0000-00-+-$ $+-+-+00+0000-000000-000000-0000+00+-+-+-+$ $-00-0000+000000+)$	0.0005
102	$(00+-+-+-+-00-0000+000000+000000+0000-00-$ $+-+-+-+00+0000-000000-000000-000000-000000-$ $000000-000000-0000+)$	0.0005
102	$(00+-+-+-+-00-0000+000000+000000+000000+000000+$ $0000-00-+-+-+-+00+0000-000000-000000-000000-$ $000000-0000+)$	0.0287
	Total	3.8286

Table 5: Admissible symbolic sequences for $q/p = 2/9$.

L	s	area
1	(0)	1.6073
2	$(-+)$	0.6926
3	$(-0+)$	0.3640
3	$(-+0)$	0.3640
4	$(-00+)$	0.1821
4	$(-+00)$	0.1821
5	$(-000+)$	0.0414
5	$(-+000)$	0.0414
5	$(-0+-+)$	0.0414
5	$(-+-+0)$	0.0414
52	$(-000+-00+-0+-0+-+-+-+-+0-+-+0-+-+-$ $+-+-0+-0+-00+-000+)$	0.0122
52	$(-0+-+-+-+-+0-+0-+00-+000-+000-+00-+0-$ $+0-+-+-+-+-0+-+)$	0.0122
72	$(-000+-000+-000+-00+-0+-0+-+-0+-+-0+-0+$ $-00+-000+-000+-000+-000+-000+-000+)$	0.0078
72	$(-+00-+000-+000-+000-+000-+000-+000-+000-$ $+000-+000-+00-+0-+0-+-+0-+-+0-+0)$	0.0078
74	$(-000+-00+-0+-0+-+-+-+-+0-+0-+00-+000-$ $+000-+00-+0-+0-+-+-+-+-0+-0+-00+-000+)$	0.1318
77	$(-+-+0-+0-+00-+000-+000-+00-+0-+0-+-+-+-$ $+-0+-+-0+-+-0+-+-0+-+-0+-+-0+-+-0+-+-+)$	0.0032
77	$(-+-+0-+-+0-+-+0-+-+0-+-+-+-+-0+-0+-00+$ $-000+-000+-00+-0+-0+-+-+-+-+0-+-+0-+-+0)$	0.0032
84	$(-000+-000+-000+-00+-0+-0+-+-0+-+-0+-0+-00+$ $-000+-000+-00+-0+-0+-+-0+-+-0+-0+-00+-000+)$	0.0247
84	$(-+-+0-+0-+00-+000-+000-+000-+000-+00-+0-+0-$ $+-+0-+-+0-+0-+00-+000-+000-+00-+0-+0-+-+0)$	0.0247
	Total	3.7853

Table 6: Admissible symbolic sequences for $q/p = 1/12$.

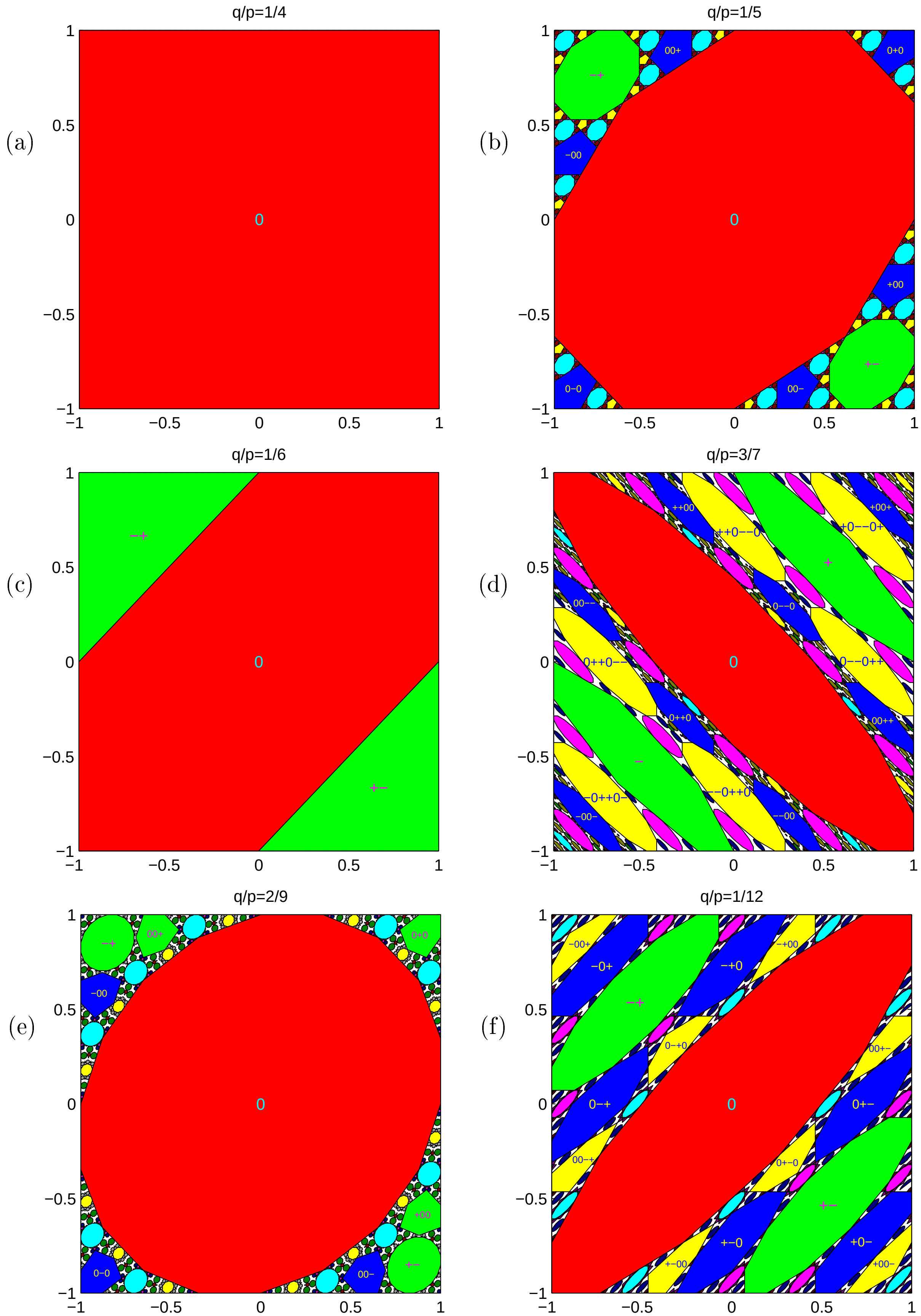

Figure 9: Admissible sequences for (a) $q/p = 1/4$, (b) $q/p = 1/5$, (c) $q/p = 1/6$, (d) $q/p = 3/7$, (e) $q/p = 2/9$, (f) $q/p = 1/12$

If L is a multiple of p there is no corresponding ellipse, but still each set is a convex polygon. Two examples are shown in Fig. 10(a,b). We were not able to find this degenerate case for $p < 12$. For $q/p = 1/12$ (and also for $q/p = 5/12$) such sequences exist. The period of the sequence is $L = 72$ or $L = 84$ (see also Table 6). For this type of solutions there is no corresponding ellipse. Since L is a multiple of p we cannot solve the periodic point equation uniquely to find the ellipse center. The two solutions we have found have very interesting structure. It seems that they are based on two periodic sequences with a smaller period (in this particular case with $L = 5$, $s_1 = (-000+)$ and $s_2 = (0+-+-)$, plotted in Fig. 10(c)). It looks like the orbit spends long time around one of the orbits with low period (several polygons touch the ellipse-like structure) but after certain number of steps it separates from the short orbit (the trajectory goes into a linear region different from the one where the whole ellipse goes), after a short time the trajectory starts to follow the second low period orbit and the procedure repeats.

On basis of the the results and discussion presented above one can make the following observations:

1. The set of points corresponding to a given periodic symbolic sequence is a convex polygon. It contains the ellipse (apart from the case when L is a multiple of p). Usually for large L it approximates the ellipse quite well, although it always has to be a polygon with at most $n = 4 \cdot \mathrm{lcm}(p, L)$ sides.

2. Sometimes ($q/p = 1/4$, $1/6$, and also $1/3$) it is possible to find all admissible sequences. In these cases the corresponding sets of initial conditions cover the set $[-1, 1] \times [-1, 1]$. For $1/4$ all trajectories has symbolic sequence $s = (0)$, and for $1/6$ a trajectory has a symbolic sequence $s = (0)$ or $s = (-+)$, and there are no other admissible symbolic sequences.

3. For other cases it seems that one could find periodic sequences with arbitrarily large period. In such a case full classification of behavior in terms of symbolic sequences would not be possible.

4. Sets corresponding to low–period periodic sequences are large. In all cases considered symbolic sequences with period $L < 200$ allows to classify a significant part of the state space (the corresponding sets occupy more than $3.6/4$ of the area of the state space$[-1, 1] \times [-1, 1]$). Usually sequences with longer period correspond to sets with smaller area, but there are some exceptions (see for example $L = 12, 18$ for $q/p = 1/5$).

5. Periodic admissible symbolic sequences found are rather sparse. The number of

(a)

(b)

(c)

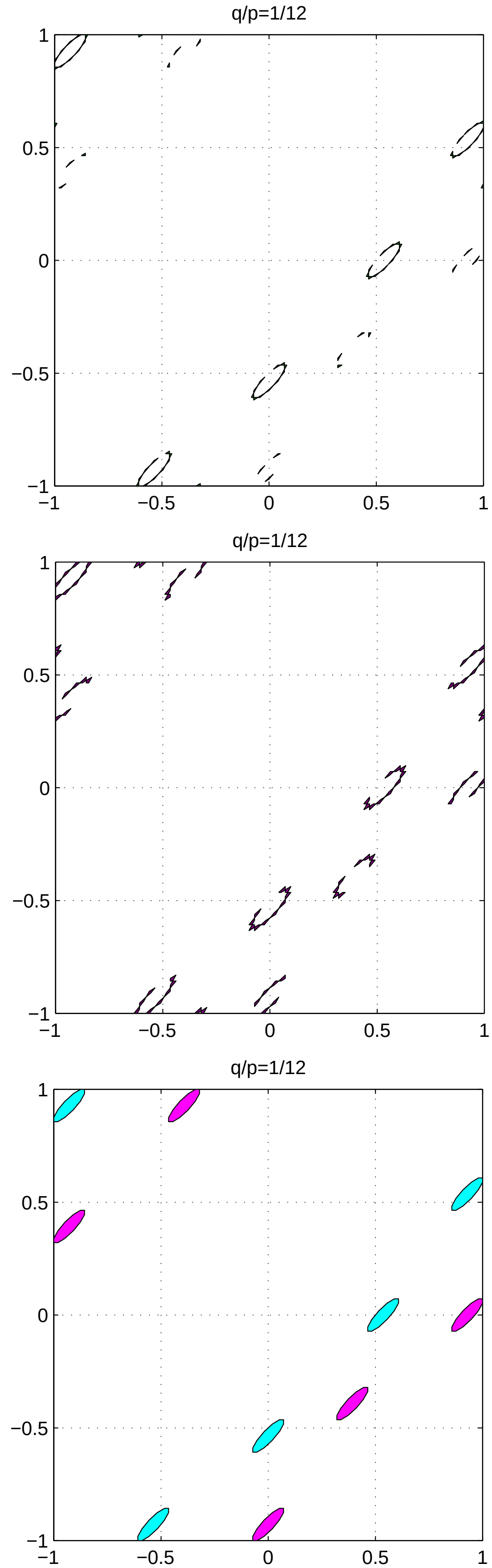

Figure 10: Degenerate admissible sequences for $q/p = 1/12$ (a) period–72 sequence, (b) period–84 sequence, (c) short sequences $(-000+)$ and $(0+-+-)$ serving as a “base” for the degenerate sequences.

admissible periodic sequences with the length $L < 100$ is of order of 20 or smaller, which is a very small number when compared to the total number of symbolic sequences with these periods. This is related to the fact that the admissible periodic symbolic sequences must satisfy equation (29).

6. There exist periodic sequences with L being a multiple of p. This kind of periodic orbits, with no corresponding elliptic sets is specific to rational r. For irrational r all periodic orbits are associated with ellipse centers.

4 Discussion and Conclusion

In this paper, the periodic behaviors of the digital filter with two's complement arithmetic for rational r have been studied. The relation between the period of periodic trajectories and their traveling patterns (symbolic sequences) has been fully explored. To our knowledge, it is the first time such relation in the digital filter with two's complement arithmetic is fully explored for rational rs. However there are still several open questions that remain to be solved.

1. Are there any points with non–periodic symbolic sequences.

2. For $p/q = 1/3, 1/4, 1/6$ we have found all admissible symbolic sequences. Are there any other parameter values for which there are finitely many admissible sequences?

3. Are there any symbolic sequences with $L = p$? This seems to be the simplest degenerate case. We were however not able to find such an example or to prove that it is not possible, although from Equations (27) and (28) it could be shown that indeed in most cases it is not possible.

Solutions to these intriguing problems would certainly further improve the understanding of the periodic behaviors of the digital filter with two's complement arithmetic.

5 Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable comments. They also wish to thank the Associate Editor Professor M. Ogorzałek and Editor-in-Chief Professor M.N.S. Swamy for providing an opportunity to publish this work. The first author wishes to thank the Australian Research Council for a grant. The second author was sponsored by the ARO, grant No. DAAG55-98-1-0269.

References

[1] L.O. Chua and T. Lin, "Chaos in digital filters," *IEEE Trans. Circuits and Systems*, vol. 35, 648–652, 1988.

[2] M.J. Ogorzałek, "Complex behaviors in digital filters," *Int'l J. Bifurcation and Chaos*, vol. 2, no. 1, 11–29, 1992.

[3] Z. Galias, M. Ogorzałek, "On symbolic dynamics of a chaotic second order digital filter," *International Journal of Circuit Theory and Applications,* vol. 20, pp. 401–409, 1992.

[4] J. O'Rourke, "Computational Geometry in C", Cambridge University Press, 1998.

[5] B. Semendyayev, Handbook of Mathematics, p.200, Springer Verlag, 1992.

[6] X. Yu, "Discretization chaos in the switching control system with finite switching values," *International Journal of Chaos and Bifurcation,* vol. 5, no. 6, pp. 1749–1755, 1995.