ON A DISCRETE TIME NONLINEAR SYSTEM ASSOCIATED WITH THE SECOND-ORDER DIGITAL FILTER *

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Abstract. This paper presents analysis of extremely simple second-order digital filter. In our analysis we take into account the nonlinearity caused by the overflow effect. This digital filter corresponds to a nonlinear semidynamical system on the plane depending on two parameters. The symbolic theory for analysing this system is introduced. Several theorems on existence and stability of periodic orbits are given. With these theorems all periodic solutions with low periods are found. The properties of limit sets of the system for different values of parameters are discussed. The limit sets are classified. We single out the regions in the parameter space with various kind of asymptotic trajectory behaviour. In particular we prove the existence of periodic and quasi-periodic limits sets only. In the last section the layout of Arnold tongues on the parameter plane is discussed. Several properties of Arnold tongues are stated.

Key words. discrete dynamical system, digital filter, circle map, limit set, rotation number, Arnold tongue

AMS subject classifications. 39A10, 58F21, 58F22, 26A03

1. Introduction. In this paper we consider a discrete semidynamical system $(\Omega, \boldsymbol{F})$. Ω is a square on the real plane defined by

$$\Omega := \{\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2 : x_1, x_2 \in [-1, 1]\} \tag{1}$$

and $\boldsymbol{F} : \Omega \ni \boldsymbol{x} \longmapsto \boldsymbol{F}(\boldsymbol{x}) \in \Omega$ is given by

$$\boldsymbol{F}(\boldsymbol{x}) := (x_2, f(bx_1 + ax_2))^T, \tag{2}$$

where $f(x) = \frac{1}{2}(|x+1| - |x-1|)$ is the saturation characteristic shown in Fig. 1. We choose Ω as the domain of the map $\boldsymbol{F}$ because $\boldsymbol{F}^2(\mathbb{R}^2) \subset \Omega$, hence all trajectories after two iterations and limit sets are enclosed in the square Ω. The topology on Ω considered in this paper is the one inherited from $\mathbb{R}^2$.

The analysed system describes the dynamics of the second-order digital filter, which is the fundamental building block of the cascade and parallel realizations of recursive digital filters. Usually in the design and analysis digital filter is treated as a discrete-time linear system. In practice however it is necessary to consider nonlinear effects caused by finite wordlength representation in the analysis of filter's behaviour. In this paper we consider the nonlinearity corresponding to overflow effects produced by addition operation. Because of its wide applications this system has received much attention from researchers. Chua and Lin [2, 3] have studied similar system with "modular" characteristic ($f(x) = (x+1) mod\ 2 - 1$). They have shown that the phase portrait of the system for certain values of parameters is self-similar and has fractal geometry. Wilson [9] applying Lyapunov function technique has shown that the nonlinear system under consideration is asymptotically stable for all parameters (a, b) inside the triangle T – the stability region of the linearised system (compare Fig. 2). It is our goal here to study system behaviour for the parameter range outside the stability region. Such an analysis is important for understanding behaviour of a real

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FIG. 1. *Nonlinear saturation-type characteristic f.*

electronic circuit. Some of the results presented in this paper have been published before [5, 6]. Here we consider the digital filter as a discrete-time semi-dynamical system and extend several results.

The following notations are used in this paper. Let
$O := (0,0)^T$, $A := (1,1)^T$, $B := (1,-1)^T$, $C := (-1,-1)^T$, $D := (-1,1)^T$.
So O is the center of square Ω and A, B, C, D its corners. We also define matrices:

$$(3) \qquad \boldsymbol{A} := \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \qquad \boldsymbol{D} := \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix}, \qquad \boldsymbol{b} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and function $g(\boldsymbol{x}) := bx_1 + ax_2$.
By $\boldsymbol{G}(\boldsymbol{x}) := \boldsymbol{A}\boldsymbol{x} = (x_2, g(\boldsymbol{x}))^T = (x_2, bx_1 + ax_2)^T$ we denote the linear function associated with the nonlinear map $\boldsymbol{F}$.

Let us introduce the following definitions:
Let (X, d) be a metric space, and $h : X \longmapsto X$ a map. We say that a point $x \in X$ is *periodic* with *period* $n > 0$ if n is the smallest integer such that $h^n(x) = x$.
$\omega(x) := \{y \in X : \exists n_k \longrightarrow \infty : h^{n_k}(x) \longrightarrow y\}$ is called the *limit set* of point x.
We say that a set $A \subset X$ is *invariant* if $h(A) = A$. One should notice that we have adapted the standard definition of invariance ($h(A) \subset A$) for our needs.
An invariant set A is called:

- *stable* if $\forall \varepsilon > 0\ \exists \delta > 0 : (d(A, x) < \delta \Rightarrow \forall n > 0\ \ d(A, h^n(x)) < \varepsilon)$.
- *asymptotically stable* if A is stable and there exists U – a neighbourhood of A such that $\forall x \in U\ \ d(h^n(x), A) \to 0$ for $n \to \infty$.
- *absolutely stable* if there exists U, a neighbourhood of A and $n > 0$ such that $h^n(U) \subset A$.

2. Analysis of linear system. Before analyzing the nonlinear system, let us examine the dynamics of its associated linear model (i.e. $f(x) = x$). The linear system can be characterized using eigenvalues of matrix $\boldsymbol{A}$:

$$(4) \qquad z_1 = \frac{a - \sqrt{a^2 + 4b}}{2}\ , \quad z_2 = \frac{a + \sqrt{a^2 + 4b}}{2}.$$

Let us divide the parameter space into the subsets corresponding to the position of eigenvalues with respect to the unit circle (compare Fig. 2):

$$\begin{aligned}
T &= \{(a,b) : b > -1, b < 1 - |a|\},\\
Q_1 &= \{(a,b) : b > 1 - a, b < 1 + a\},\\
Q_2 &= \{(a,b) : b < 1 - a, b > 1 + a\},\\
Q_3 &= \{(a,b) : b < -1, b < 1 - |a|\},\\
Q_4 &= \{(a,b) : b > 1 + |a|\}.
\end{aligned}$$

FIG. 2. *Eigenvalues z_1, z_2 of the linear system.*

For $(a,b) \in T$ both eigenvalues lie inside the unit circle, so O is a sink. For $(a,b) \in Q_1 \cup Q_2$ one eigenvalue lies outside and one inside the unit circle, so O is a saddle. For $(a,b) \in Q_3 \cup Q_4$ both eigenvalues lie outside the unit circle, so O is a source. For other values of parameters the system is degenerate in the sence that one or two eigenvalues lie on the unit circle and the linear system is not hyperbolic.

3. Symbolic theory of the map F. We want to introduce a map from Ω into the sequence space Σ involving three symbols $\{-1, 0, +1\}$; namely $\Sigma = \{(s_0 s_1 \ldots) :$

$s_k \in \{-1, 0, +1\}; k = 0, 1, 2, \ldots\}$. Entries $s_0, s_1, s_2, \ldots$ are extracted from the successive iterations of $\boldsymbol{x}$: $\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}), \boldsymbol{F}^2(\boldsymbol{x}), \ldots$.

DEFINITION 3.1. *The map* $\boldsymbol{S} : \Omega \ni \boldsymbol{x} \longmapsto \boldsymbol{S}(\boldsymbol{x}) \in \Sigma$ *is defined by:*

$$s_k = \boldsymbol{S}(\boldsymbol{x})_k := \begin{cases} -1 & if \;\; g(\boldsymbol{F}^k(\boldsymbol{x})) < -1 \\ 1 & if \;\; g(\boldsymbol{F}^k(\boldsymbol{x})) > \;\; 1 \\ 0 & otherwise \end{cases} . \tag{5}$$

The map $\boldsymbol{S}$ provides a powerful tool for analyzing the dynamics of the map $\boldsymbol{F}$. Symbolic theory is based on elementary observations summarized in Lemmas 3.2 and 3.3 below.

LEMMA 3.2. *Let* $\boldsymbol{x} \in \Omega$, $k \geq 0$, $s_k = \mathbf{S}(\boldsymbol{x})_k$, $\boldsymbol{x}^k = \boldsymbol{F}^k(\boldsymbol{x})$, $\boldsymbol{A}_k = \boldsymbol{A} - \boldsymbol{D}|s_k|$, *where* $\boldsymbol{A}$ *and* $\boldsymbol{D}$ *are defined in (3). Then*

$$\boldsymbol{x}^{k+1} = \boldsymbol{A}_k \boldsymbol{x}^k + \boldsymbol{b} s_k. \tag{6}$$

Proof. We have

$$\boldsymbol{x}^{k+1} = \boldsymbol{F}(\boldsymbol{x}^k) = \left[\begin{array}{c} x_2^k \\ f(bx_1^k + ax_2^k) \end{array} \right] = \left[\begin{array}{c} x_2^k \\ (bx_1^k + ax_2^k)(1 - |s_k|) + s_k \end{array} \right] =$$

$$= \left[\begin{array}{c} x_2^k \\ bx_1^k + ax_2^k \end{array} \right] - |s_k| \left[\begin{array}{c} 0 \\ bx_1^k + ax_2^k \end{array} \right] + s_k \left[\begin{array}{c} 0 \\ 1 \end{array} \right] =$$

$$= \boldsymbol{A}\boldsymbol{x}^k - |s_k| \boldsymbol{D}\boldsymbol{x}^k + s_k \boldsymbol{b} = \boldsymbol{A}_k \boldsymbol{x}^k + \boldsymbol{b} s_k.$$

□

Lemma 3.2 shows that the system (1,2) can be also considered as a linear, nonstationary (A_k depends on k) discrete system driven by an input sequence $s_0, s_1, \ldots$. As a consequence of Lemma 3.2 we obtain the following:

LEMMA 3.3. *Let* $\boldsymbol{x} \in \Omega$, $k \geq 0$, $s_k = \mathbf{S}(\boldsymbol{x})_k$, $\boldsymbol{A}_k = \boldsymbol{A} - \boldsymbol{D}|s_k|$. *Then*

$$\boldsymbol{F}^{k+1}(\boldsymbol{x}) = \boldsymbol{A}_k \boldsymbol{A}_{k-1} \ldots \boldsymbol{A}_0 \boldsymbol{x} + (\boldsymbol{A}_k \ldots \boldsymbol{A}_1 s_0 + \cdots + \boldsymbol{A}_k s_{k-1} + \mathbf{I} s_k)\boldsymbol{b}. \tag{7}$$

3.1. Periodic orbits. We would like to use the symbolic theory for detection and studies the stability of periodic orbits existing in the system. The following theorem specifies the relations between periodic orbits and sequences of symbols.

THEOREM 3.4. *Let* $k \geq 0$. *The following conditions are equivalent:*

1. $\boldsymbol{x}$ *is periodic with period k+1.*

2. there exists a sequence $s_0, s_1, \ldots, s_k \in \{-1, 0, 1\}$ *such that* $\boldsymbol{x}$ *satisfies the equation:*

$$\boldsymbol{x} = \boldsymbol{A}_k \boldsymbol{A}_{k-1} \ldots \boldsymbol{A}_0 \boldsymbol{x} + (\boldsymbol{A}_k \ldots \boldsymbol{A}_1 s_0 + \cdots + \boldsymbol{A}_k s_{k-1} + \mathbf{I} s_k)\boldsymbol{b}, \tag{8}$$

where $\boldsymbol{A}_j = \boldsymbol{A} - \boldsymbol{D}|s_j|$
and $\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ *defined by*

$$\boldsymbol{x}^0 := \boldsymbol{x}, \;\; \boldsymbol{x}^{j+1} := \boldsymbol{A}_j \boldsymbol{x}^j + \boldsymbol{b} s_j \quad for \; j \in \{0, \ldots, k-1\} \tag{9}$$

fulfill the conditions:

$$\left\{\begin{array}{lccl} |g(\boldsymbol{x}^j)| \leq & 1 & if & s_j = \ \ 0 \\ \ g(\boldsymbol{x}^j) > & 1 & if & s_j = \ \ 1 \\ \ g(\boldsymbol{x}^j) < & -1 & if & s_j = -1 \end{array}\right. \qquad for\ j \in \{0,\dots,k\}, \tag{10}$$

$$\boldsymbol{x} \neq \boldsymbol{x}^j \ \ for\ j \in \{1,\dots,k\}. \tag{11}$$

Proof. $(2 \Rightarrow 1)$ From (9) and (10) it follows that $\boldsymbol{x}^j = \boldsymbol{F}^j(\boldsymbol{x})$ and $s_j = \boldsymbol{S}(\boldsymbol{x})_j$ for $j = 0,\dots,k$. From (8) and Lemma 3.3 it can be concluded that $\boldsymbol{F}^{k+1}(\boldsymbol{x}) = \boldsymbol{x}$. From (11) it is clear that the period of $\boldsymbol{x}$ is $k+1$.
$(1 \Rightarrow 2)$ Let $s_j = \boldsymbol{S}(\boldsymbol{x})_j$. From Lemma 3.3 it is clear that $\boldsymbol{x}$ fulfills (8). From Lemma 3.2 it follows that $\boldsymbol{x}^0, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{k-1}$ defined by (9) fulfill condition (10). The period of $\boldsymbol{x}$ is $k+1$. Hence condition (11) is satisfied. □

Remark 3.5. *Utilizing Theorem* 3.4 *it is possible to find all periodic trajectories for given period $k+1$ by checking all sequences $s_0, s_1, \dots, s_k$, solving the equation* (8) *and checking conditions* (10, 11).
The number of checked sequences can be significantly reduced using the following two remarks.

Remark 3.6. *It is sufficient to check sequences defining different cycles. (We say that two sequences $t_0, t_1 \dots, t_{k-1}$ and $s_0, s_1 \dots, s_{k-1}$ define the same cycle if there exists a natural number j such that sequences $t_0, t_1 \dots, t_{k-1}$ and $s_j, s_{j+1} \dots, s_{k-1}, s_1, \dots, s_{j-1}$ are equal.)*

Proof. If we check two sequences defining the same cycle we will obtain points belonging to the same periodic orbits. □

Remark 3.7. *It is sufficient to check sequences with the number of $+1$'s greater or equal to the number of -1's, find all periodic points and take points symmetric to them with respect to the origin.*

Proof. It is clear that if $\boldsymbol{x} = (x_1, x_2)^T$ is periodic then $-\boldsymbol{x} := (-x_1, -x_2)^T$ is also periodic. This is a consequence of odd symmetry of the map $\boldsymbol{F}$ ($\boldsymbol{F}(\boldsymbol{x}) = -\boldsymbol{F}(-\boldsymbol{x})$). If $s_0, \dots, s_{k-1}$ leads to the periodic point $\boldsymbol{x}$ then $-s_0, \dots, -s_{k-1}$ leads to the periodic point $-\boldsymbol{x}$. This fact follows from Theorem 3.4 and symmetry of the map $\boldsymbol{S}$. Now one can notice that it is sufficient to consider only sequences with the number of $+1$'s greater or equal to the number of -1's, and take points symmetric to the periodic points found. □

3.2. Stability of periodic orbits. Now we present theorems on stability of periodic orbits. In the first theorem we consider the case when two subsequent symbols of a periodic point are non-zero (then one of the vertices of the square Ω belongs to the periodic orbit).

Theorem 3.8. *Let $\boldsymbol{x}$ be a periodic point, $s_j = \mathbf{S}(\boldsymbol{x})_j$. If there exists $p \geq 0$ such that $|s_p| = |s_{p+1}| = 1$ then periodic orbit of $\boldsymbol{x}$ is absolutely stable.*

Proof. We prove the theorem for the case when $s_p = s_{p+1} = 1$. For other cases the proof is similar. From Lemma 3.2 it follows that for $s_k = 1$ we have $\boldsymbol{x}^{k+1} = \boldsymbol{A}_k \boldsymbol{x}^k + \boldsymbol{b} s_k = (x_2^k, 1)^T$. Hence $\boldsymbol{x}^{p+1} = (x_2^p, 1)^T$ and $\boldsymbol{x}^{p+2} = (x_2^{p+1}, 1)^T = (1,1)^T = A$. Let us denote $\boldsymbol{y} = \boldsymbol{x}^p$. From Definition 3.1 it follows that $g(\boldsymbol{x}^p) = g(\boldsymbol{y}) = by_1 + ay_2 > 1$ and $g(\boldsymbol{x}^{p+1}) = g(y_2, 1) = by_2 + a > 1$. Let U be an open neighbourhood of $\boldsymbol{y}$ in Ω such that for $\boldsymbol{z} = (z_1, z_2)^T \in U\ :\ bz_2 + a > 1$ and $bz_1 + az_2 > 1$. It is clear that $\boldsymbol{F}^2(U) = A$. Let $\boldsymbol{w}$ belong to the periodic orbit of $\boldsymbol{x}$.Then there exists k such that $\boldsymbol{F}^k(\boldsymbol{w}) = \boldsymbol{y}$.

$V := \boldsymbol{F}^{-k}(U)$ is an open neighbourhood of $\boldsymbol{w}$ in Ω. $\boldsymbol{F}^{k+2}(V) = A$. Thus the periodic orbit containing $\boldsymbol{x}$ is absolutely stable. □

Now we consider the case when the map $\boldsymbol{F}$ is affine in the neighbourhood of the periodic orbit.

THEOREM 3.9. *Assume that $\boldsymbol{K} = (\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^{k-1})$ is a periodic orbit, $s_j = \mathbf{S}(\boldsymbol{x}^0)_j$, $\boldsymbol{A}_j = \boldsymbol{A} - \boldsymbol{D}|s_j|$, $|g(\boldsymbol{x}^j)| \neq 1$ for $j = 0, \ldots, k-1$. Then:*

1. *if both eigenvalues of matrix $\boldsymbol{A}_{k-1} \ldots \boldsymbol{A}_1 \boldsymbol{A}_0$ lie inside the unit circle then $\boldsymbol{K}$ is asymptotically stable.*
2. *if at least one eigenvalue of matrix $\boldsymbol{A}_{k-1} \ldots \boldsymbol{A}_1 \boldsymbol{A}_0$ lies outside the unit circle then $\boldsymbol{K}$ is not stable.*

Proof. Condition $|g(\boldsymbol{x})| \neq 1$ means that the map $\boldsymbol{F}$ is affine in the neighbourhood of $\boldsymbol{x}$. From the assumption that $|g(\boldsymbol{x}^j)| \neq 1$ for $j = 0, \ldots, k-1$ follows that there exists U, a neighbourhood of $\boldsymbol{x}^0$ such that $\boldsymbol{F}^k$ is affine on U. Hence $O \in \mathbb{R}^2$ is a fixed point of the linear map $U \ni \boldsymbol{y} \longrightarrow \boldsymbol{F}^k(\boldsymbol{y} + \boldsymbol{x}_0) - \boldsymbol{x}_0$. Stability of the fixed point O depends on the eigenvalues of matrix $\boldsymbol{A}_{k-1} \ldots \boldsymbol{A}_1 \boldsymbol{A}_0$ and is equivalent to the stability of the orbit $\boldsymbol{K}$. □

REMARK 3.10. *Let $\boldsymbol{K}$ be a periodic orbit and $\boldsymbol{x} \in \boldsymbol{K}$. If in every neighbourhood of $\boldsymbol{x}$ there exists point belonging to another periodic orbit then $\mathbf{K}$ is not asymptotically stable.*

3.3. Low period orbits. By means of Theorem 3.4 we have found all periodic orbits with periods $1, \ldots, 4$ for arbitrary values of parameters. Using theorems of section 3.2 we have been able to establish their stability. The results are summarized below (compare also Fig. 3).

Fixed points

- $s_0 = 0$
  $(a, b) \in T : O$ – asymptotically stable
  $(a, b) \notin T : O$ – not asymptotically stable
  $a + b = 1 : (x, x)^T$, for $x \in [-1, 1]$, $x \neq 0$ – not asymptotically stable
- $s_0 = +1$
  $a + b > 1 : A$ – absolutely stable
- $s_0 = -1$
  $a + b > 1 : C$ – absolutely stable

Period-2 orbits

- $(s_0, s_1) = (0, 0)$
  $(a, b) = (0, 1) : ((x_1, x_2)^T, (x_2, x_1)^T)$, for $x_1, x_2 \in [-1, 1]$, $x_1 \neq x_2$
  $b = a + 1 : ((x, -x)^T, (-x, x)^T)$, for $x \in [-1, 1]$, $x \neq 0$
- $(s_0, s_1) = (+1, -1)$
  $b > a + 1 : (B, D)$ – absolutely stable
- $(s_0, s_1) = (0, -1)$
  $b > |a| + 1 : ((\frac{a}{b-1}, -1)^T, (-1, \frac{a}{b-1})^T)$ – unstable
- $(s_0, s_1) = (0, +1)$
  $b > |a| + 1 : ((\frac{a}{1-b}, 1)^T, (1, \frac{a}{1-b})^T)$ – unstable

Period-3 orbits

- $(s_0, s_1, s_2) = (0, 0, 0)$
  $(a, b) = (-1, -1) : ((x_1, x_2)^T, (x_2, -x_1 - x_2)^T, (-x_1 - x_2, x_1)^T)$, for $x_1, x_2, x_1 + x_2 \in [-1, 1]$, $(x_1, x_2)^T \neq O$
- $(s_0, s_1, s_2) = (0, +1, -1)$
  $(a, b) \in W_1 = \{(a, b) : (a < -1, b < -1, b < \frac{a^2+1}{a-1}, a < \frac{b^2+1}{b-1}\}$

FIG. 3. *Periodic orbits with period length 1,...,4.*

$(B, (-1, b-a)^T, (b-a, 1)^T)$ – absolutely stable

- $(s_0, s_1, s_2) = (0, -1, +1)$
  $(a, b) \in W_1 : (D, (1, a-b)^T (a-b, -1)^T)$ – absolutely stable
- $(s_0, s_1, s_2) = (0, 0, +1)$
  $(a, b) \in W_2 = \overline{W}_1 \setminus \{(-1, -1)\} : ((\frac{b+a^2}{1-ab}, 1)^T, (1, \frac{a+b^2}{1-ab})^T, (\frac{a+b^2}{1-ab}, \frac{b+a^2}{1-ab})^T)$ – unstable
- $(s_0, s_1, s_2) = (0, 0, -1)$
  $(a, b) \in W_2 : ((-\frac{b+a^2}{1-ab}, -1)^T, (-1, -\frac{a+b^2}{1-ab})^T, (-\frac{a+b^2}{1-ab}, -\frac{b+a^2}{1-ab})^T)$ – unstable

Period-4 orbits

- $(s_0, s_1, s_2, s_3) = (0, 0, 0, 0)$
  $(a, b) = (0, 1) : ((x_1, x_2)^T, (x_2, -x_1)^T, (-x_1, -x_2)^T, (-x_2, x_1)^T)$, for $(x_1, x_2)^T \neq O$
- $(s_0, s_1, s_2, s_3) = (-1, -1, +1, +1)$
  $b < -1 - |a| : (A, B, C, D)$ – absolutely stable
- $(s_0, s_1, s_2, s_3) = (0, -1, 0, +1)$
  $b \leq -1 - |a| : ((-\frac{a}{1+b}, 1)^T, (1, \frac{a}{1+b})^T, (\frac{a}{1+b}, -1)^T, (-1, -\frac{a}{1+b})^T)$ – unstable

All asymptotically stable orbits are also absolutely stable. But this is not a gen-

TABLE 1

Periodic orbits (period length 1..4) for different values of parameters. For $(a,b) \notin T$ fixed point O is unstable. * *Ω denotes that all points are periodic, $\overline{AC}$ denotes that points belonging to the interval $\overline{AC}$ are periodic, etc.*

Range of parameters	stable		unstable	
	period length	number of orbits	period length	number of orbits *
T	1	$1:O$		
Q_1	1	$2:A,C$		
Q_2	2	$1:(B,D)$		
Q_4	1	$2:A,C$	2	2
	2	$1:(B,D)$		
$T_1 \cup R_1 \cup P_1$			1	$\overline{AC}$
$T_2 \cup R_2 \cup P_2$			2	$\overline{BD}$
P_3	1	$2:A,C$	2	$\overline{BD}$
P_4	2	$1:(B,D)$	1	$\overline{AC}$
R_1			2	$\Omega \setminus \overline{AC}$
			1	$\overline{AC}$
$R_{1/3} = \{(-1,-1)\}$			3	*hexagon*
$Q_{1/3} = W_1$	3	2	3	2
$P_{1/3} = \partial Q_{1/3} \setminus R_{1/3}$			3	2
$R_{1/4} = \{(0,-1)\}$			4	Ω
$Q_{1/4} = \{(a,b) : b < -1-\lvert a\rvert\}$	4	1	4	1
$P_{1/4} = \partial Q_{1/4} \setminus R_{1/4}$			4	1

eral rule. We have found higher period orbits which are asymptotically stable but not absolutely stable. An example could be the orbit of point $(1, \frac{b^3-ab+a}{1-ab^2})^T$ for $(a,b) = (0.69, -1.1)$. Its trajectory is asymptotically stable period-5 orbit containing no corners of the square Ω. Hence it is not absolutely stable. Periodic orbits found for different subsets of parameter space are summarized in Table 1.

As it has been mentioned we can find all periodic orbits with period k by checking 3^k sequences. This number can be reduced by utilizing Remarks 3.6, 3.7 and excluding non-admissible sequences. An example of sequence which is non-admissible for every pair (a,b) is $(1,1,1,0)$ or $(1,1,1,-1)$. For given a,b a lot of sequences can be excluded by means of methods presented in [7]. As for a given pair (a,b) conditions $(8,10,11)$ in Theorem 3.4 are quite easy to verify, it is possible to find all periodic orbits with period less than 30. For greater periods, time needed for calculations increases rapidly. In spite of extensive searches we have not found periodic orbits with period greater than two for $(a,b) \notin Q_3 \cup T_3$ (compare Fig. 2). In $Q_3 \cup T_3$ we have found pairs (a,b) for which there exist periodic points with periods $3,4,5,6,\ldots,30$, but for a given pair of parameters all periodic points found have the same period. These observations will be proved in the next section.

4. Limit sets. In this section we present the classification of limit cycles for different values of system parameters. First we consider the case $(a,b) \in Q_3$. In this region there exist periodic solutions with arbitrary high periods and also non-periodic limit sets dense in a set homeomorphic to a circle. In section 4.2 we classify limit sets for the case $(a,b) \in Q_1, Q_2, Q_4$. In these regions all limit sets are periodic orbits with period 1 or 2.

4.1. Limit sets for Q_3. As it has been suggested earlier the most interesting behaviour of the system has been encountered in the region Q_3. We prove that all nontrivial ($\boldsymbol{x} \neq O$) trajectories eventually enter the set which is the boundary of a polygon with number of sides depending on parameters (a, b). Thus the study of dynamics of our system can be reduced to analysis of a one-dimensional map of a polygon into itself. This map is weakly monotone (see below for a definition) which implies the existence of the unique rotation number for each pair (a, b). As a corollary we obtain the existence of periodic and quasi-periodic limit sets only. Below we prove these results rigorously. Throughout this subsection we assume that $(a, b) \in Q_3$.

LEMMA 4.1. *Let $\boldsymbol{x} \in \Omega, \boldsymbol{x} \neq O$. Then there exists $n > 0$ such that $\boldsymbol{G}(\boldsymbol{F}^{n-1}(\boldsymbol{x})) \notin \Omega$ and $\boldsymbol{F}^n(\boldsymbol{x}) \in \overline{DA} \cup \overline{BC}$, where A, B, C, D are corners of square Ω.*

Proof. Let n be the smallest integer such that $\boldsymbol{G}^n(\boldsymbol{x}) \notin \Omega$. Such n exists because for $(a, b) \in Q_3$ both eigenvalues of the linear system $\boldsymbol{G}$ lie outside the unit circle. For $k < n$ $\boldsymbol{G}^k(\boldsymbol{x}) \in \Omega$ and $\boldsymbol{G}^k(\boldsymbol{x}) = \boldsymbol{F}^k(\boldsymbol{x})$. Hence $\boldsymbol{G}(\boldsymbol{F}^{n-1}(\boldsymbol{x})) = \boldsymbol{G}^n(\boldsymbol{x})$. Let $(y_1, y_2)^T := \boldsymbol{F}^{n-1}(\boldsymbol{x})$. $\boldsymbol{G}(\boldsymbol{F}^{k-1}(\boldsymbol{x})) = (y_2, by_1 + ay_2)^T$. As $\boldsymbol{F}^{k-1}(\boldsymbol{x}) \in \Omega$ and $\boldsymbol{G}(\boldsymbol{F}^{k-1}(\boldsymbol{x})) \notin \Omega$ it is clear that $|by_1 + ay_2| > 1$. Thus $\boldsymbol{F}^n(\boldsymbol{x}) = (y_2, f(by_1 + ay_2)) \in \overline{DA} \cup \overline{BC}$. □

Let us define:

$$\Omega^n := \boldsymbol{F}^n(\Omega), \quad \Omega^\infty := \bigcap_{n=0}^{\infty} \Omega^n. \tag{12}$$

LEMMA 4.2. *(Properties of Ω^n and Ω^∞)*

1. *Ω^n and Ω^∞ are symmetric with respect to the origin.*
2. *$\Omega^n \longrightarrow \Omega^\infty$ for $n \longrightarrow \infty$.*
3. *Ω^∞ is invariant.*
4. *$\Omega^\infty \cap \overline{DA} \neq \emptyset$, $\Omega^\infty \cap \overline{BC} \neq \emptyset$.*
5. *$\Omega^\infty \cap int\overline{AB} \neq \emptyset$, $\Omega^\infty \cap int\overline{CD} \neq \emptyset$, where by $int\overline{XY}$ we denote the interval $\overline{XY}$ without its endpoints ($int\overline{XY} := \overline{XY} \setminus \{X, Y\}$).*
6. *Ω^n is an absolutely convex polygon (convex and symmetric with respect to the origin).*
7. *Ω^∞ is absolutely convex.*

Proof.

1. This follows from the symmetry of $\boldsymbol{F}$ and Ω.
2. $\{\Omega^n\}$ is a monotonically decreasing sequence of sets ($\Omega^1 = \boldsymbol{F}(\Omega) \subset \Omega = \Omega^0$, and $\boldsymbol{F}(\Omega^n) \subset \boldsymbol{F}(\Omega^{n-1}) \Rightarrow \boldsymbol{F}(\Omega^{n+1}) \subset \boldsymbol{F}(\Omega^n)$).
3. This is a conclusion from the definition of Ω^∞ .
4. $O \in \Omega^n$ for every $n \geq 0$. Hence $O \in \Omega^\infty$.

Suppose that $\Omega^\infty = \{O\}$. Then all trajectories converge to O which is in contradiction with Lemma 4.1. So there exists $\boldsymbol{x} \neq O$, $\boldsymbol{x} \in \Omega^\infty$. From Lemma 4.1 there exists n such that $\boldsymbol{F}^n(\boldsymbol{x}) \in \overline{DA} \cup \overline{BC}$. As Ω^∞ is invariant then $\boldsymbol{F}^n(\boldsymbol{x}) \in \Omega^\infty$. Hence $\Omega^\infty \cap (\overline{DA} \cup \overline{BC}) \neq \emptyset$ which completes the proof of 4 because Ω^∞ is symmetric.

5. Let us divide set Q_3 into three subsets:

$Q_{30} = \{(a, b) : b < -1, b \leq a - 1, b \leq -a - 1\}$,
$Q_{31} = \{(a, b) : b < -1, |b + a| < 1\}$,
$Q_{32} = \{(a, b) : b < -1, |b - a| < 1\}$.

(a) For $(a, b) \in Q_{30}$ one can easily check that $\boldsymbol{F}(\Omega) = \Omega$.
(b) $(a, b) \in Q_{31}$. From 4 there exists $\boldsymbol{x} = (x, 1)^T \in \Omega^\infty$. $\boldsymbol{F}(\boldsymbol{x}) = (1, f(bx + a))^T$. $bx + a \geq b + a > -1$ for $x \in [-1, 1]$ and $(a, b) \in Q_{31}$. If $bx + a < 1$ then $\boldsymbol{F}(\boldsymbol{x}) \in int\overline{AB}$. If $bx + a \geq 1$ then $A = \boldsymbol{F}(\boldsymbol{x}) \in \Omega^\infty$. Then $\boldsymbol{F}(A) = (1, b + a)^T \in int\overline{AB}$. In both cases $\Omega^\infty \cap int\overline{AB} \neq \emptyset$. From symmetry of Ω^∞ we have $\Omega^\infty \cap int\overline{CD} \neq \emptyset$.
(c) For $(a, b) \in Q_{32}$ the proof is similar.

6. $\Omega^0 = \Omega$ is an absolutely convex polygon. Let us assume that Ω^n is an absolutely convex polygon. Then $\boldsymbol{G}(\Omega^n)$ is also an absolutely convex polygon included in the set $\{(x,y) \in I\!R^2 : |x| \leq 1\}$. From 4 we have $\boldsymbol{G}(\Omega^n) \cap int\overline{AB} \neq \emptyset$ and $\boldsymbol{G}(\Omega^n) \cap int\overline{CD} \neq \emptyset$. Hence $\Omega^{n+1} = \boldsymbol{F}(\Omega^n) = \boldsymbol{G}(\Omega^n) \cap \Omega$ is an absolutely convex polygon as an intersection of two convex polygons.

7. Ω^∞ is an intersection of absolutely convex sets Ω^n .

□

Let $\boldsymbol{W}^\infty$ be the boundary of Ω^∞ ($\boldsymbol{W}^\infty := \partial\Omega^\infty$). From the absolute convexity of Ω^∞ we can derive the invariance of $\boldsymbol{W}^\infty$. Namely, we have:

LEMMA 4.3. *$\boldsymbol{F}(\boldsymbol{W}^\infty) = \boldsymbol{W}^\infty$.*

Proof. We know that $\boldsymbol{F}(\Omega^\infty) = \Omega^\infty$. Set Ω^∞ is obtained from the set $\boldsymbol{G}(\Omega^\infty)$ by orthogonal projection of points of $\boldsymbol{G}(\Omega^\infty)$ lying outside Ω on the straight lines $x_2 = 1$ and $x_2 = -1$. It is clear that the image of the boundary of Ω^∞ is the boundary of Ω^∞. □

LEMMA 4.4. *If $\boldsymbol{z}, \boldsymbol{w} \in \boldsymbol{W}^\infty, \boldsymbol{z} \neq \boldsymbol{w}, \boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{w})$, then*

1. *$\boldsymbol{z}, \boldsymbol{w} \in \overline{DA}$ or $\boldsymbol{z}, \boldsymbol{w} \in \overline{BC}$,*
2. *$\boldsymbol{F}(\overline{\boldsymbol{z}\boldsymbol{w}})$ is one of the corners of the square Ω.*

Proof.

1. $\boldsymbol{z} = (z_1, z_2)^T$, $\boldsymbol{w} = (w_1, w_2)^T$. $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{w}) \quad \Rightarrow \quad w_2 = z_2$ and $f(bw_1 + aw_2) = f(bz_1 + az_2)$. If $|bw_1 + aw_2| \leq 1$ then $\boldsymbol{G}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{F}(\boldsymbol{w}) = \boldsymbol{G}(\boldsymbol{w})$ and $\boldsymbol{w} = \boldsymbol{z}$ because $\boldsymbol{G}$ is invertible for $b \neq 0$. Thus $|bw_1 + aw_2| > 1$.

Now we show that $|w_2| = 1$. Suppose that $|w_2| < 1$.
Let $L_1 := \boldsymbol{W}^\infty \cap \{\boldsymbol{x} \in \Omega \ : \ x_1 = w_2\}$, $L_2 := \boldsymbol{W}^\infty \cap \{\boldsymbol{x} \in \Omega \ : \ x_2 = w_2\}$.
Both L_1 and L_2 contain exactly two points as $|w_2| < 1$, Ω^∞ has nonempty intersection with all sides of Ω and Ω^∞ is absolutely convex with nonempty interior. $L_1 = \boldsymbol{F}(L_2) = \boldsymbol{F}(\{\boldsymbol{z}, \boldsymbol{w}\}) = \{\boldsymbol{F}(\boldsymbol{z}), \boldsymbol{F}(\boldsymbol{w})\} = \{\boldsymbol{F}(\boldsymbol{w})\}$.
But L_1 contains exactly two points. This is a contradiction, so $|w_2| = 1$.

2. From 1 it is clear that $\boldsymbol{F}(\overline{\boldsymbol{z}\boldsymbol{w}})$ is one point. In 1 we have already shown that $|w_2| = 1$ and $|bw_1 + aw_2| > 1$ which completes the proof.

□

Let us define

$$\Phi := \boldsymbol{F}|\boldsymbol{W}^\infty : \boldsymbol{W}^\infty \longmapsto \boldsymbol{W}^\infty. \tag{13}$$

LEMMA 4.5. *Φ is a continuous surjection. If $\boldsymbol{W}^\infty \cap \{A, B, C, D\} = \emptyset$ then Φ is a homeomorphism.*

Proof. Follows from the previous two lemmas and continuity of $\boldsymbol{F}$. □

LEMMA 4.6. *Intersections of $\boldsymbol{W}^\infty$ with sides of Ω are non-degenerate intervals.*

Proof. Let us denote $\overline{A_1D_1} := \boldsymbol{W}^\infty \cap \overline{DA}$, $\overline{B_1C_1} := \boldsymbol{W}^\infty \cap \overline{BC}$, $\overline{A_2B_2} := \boldsymbol{W}^\infty \cap \overline{AB}$, $\overline{C_2D_2} := \boldsymbol{W}^\infty \cap \overline{CD}$. As $\boldsymbol{W}^\infty$ is symmetric it is enough to show that $A_1 \neq D_1$ and $A_2 \neq B_2$.

1. Suppose that $A_1 = D_1$. Let us consider three cases:

(a) $A \neq A_1 \neq D$. Then $A, D \notin \boldsymbol{W}^\infty$ and also $B, C \notin \boldsymbol{W}^\infty$ because $\boldsymbol{W}^\infty$ is symmetric. From the previous lemma Φ is a homeomorphism. Thus

$$\bigcup_{n=0}^{\infty} \Phi^{-n}(\{A_1, B_1\})$$

is countable. Hence there exists $\boldsymbol{x} \in \boldsymbol{W}^\infty$ such that $\boldsymbol{F}^n(\boldsymbol{x}) \notin \{A_1, B_1\}$ for every n. This is in contradiction with Lemma 4.1, as $\{A_1, B_1\} = \Omega^\infty \cap (\overline{DA} \cup \overline{BC})$

(b) $A_1 = A$. $A \in \overline{AB} \cap \Omega^\infty \subset \Phi(\Phi^{-1}(\overline{AB} \cap \Omega^\infty)) = \Phi(\overline{DA} \cap \Omega^\infty) = \Phi(\{A\})$
$\Phi(A) = A \quad \Rightarrow \quad a + b \geq 1 \quad \Rightarrow \quad (a, b) \notin Q_3$
(c) Proof in the case $A_1 = D$ is similar.
From all the cases it follows that $A_1 \neq D_1$.

2. We know that $A_1 \neq D_1$. If $\boldsymbol{F}(A_1) = \boldsymbol{F}(D_1)$ then $\Omega^\infty \cap \overline{AB} = \{A\}$ or $\{B\}$, which is in contradiction with Lemma 4.2. Hence points $\boldsymbol{F}(A_1)$,$\boldsymbol{F}(D_1)$ are different. Both of them belong to the interval $\overline{AB}$. So we have $A_2 \neq B_2$.
□

THEOREM 4.7. *There exists $n \geq 0$ such that $\Omega^\infty = \Omega^n$.*

Proof. Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$ be the same as in the previous lemma. Let us consider two cases.

(a) First let us assume that all points A_1, B_1, C_1, D_1 belong to the set

$$I := \{\boldsymbol{x} \in \Omega : \exists n : \boldsymbol{F}^n(\boldsymbol{x}) \in int\overline{A_1D_1} \cup int\overline{B_1C_1}\} \tag{14}$$

Let $P^n := \Phi^{-n}(int\overline{A_1D_1} \cup int\overline{B_1C_1})$. Sets P^n are open in $\boldsymbol{W}^\infty$ because Φ is continuous. From Lemma 4.1 for every $\boldsymbol{x} \in \boldsymbol{W}^\infty$ there exists $n \geq 0$ such that $\boldsymbol{F}^n(\boldsymbol{x}) \in \overline{A_1D_1} \cup \overline{B_1C_1}$. From assumption about A_1, B_1, C_1, D_1 for every $\boldsymbol{x} \in \boldsymbol{W}^\infty$ there exists $n \geq 0$ such that $\boldsymbol{F}^n(\boldsymbol{x}) \in int\overline{A_1D_1} \cup int\overline{B_1C_1}$. Hence $\{P^n\}_{n=0}^\infty$ is an open covering of $\boldsymbol{W}^\infty$.
As $\boldsymbol{W}^\infty$ is compact there exists $p \geq 0$ such that $\{P^n\}_{n=0}^p$ is a covering of $\boldsymbol{W}^\infty$.
Let us choose U_1, U_2 – open in Ω such that $int\overline{A_1D_1} \subset U_1 \subset \Omega^\infty$; $int\overline{B_1C_1} \subset U_2 \subset \Omega^\infty$.
Let $U := \bigcup_{n=0}^p \boldsymbol{F}^{-n}(U_1 \cup U_2)$. U is open in Ω, $\boldsymbol{W}^\infty \subset U$
($\boldsymbol{W}^\infty \subset \bigcup_{n=0}^p \Phi^{-n}(int\overline{A_1D_1} \cup int\overline{B_1C_1}) \subset \bigcup_{n=0}^p \boldsymbol{F}^{-n}(int\overline{A_1D_1} \cup int\overline{B_1C_1}) \subset \bigcup_{n=0}^p \boldsymbol{F}^{-n}(U_1 \cup U_2) = U$).
Let $V := U \cup \Omega^\infty$, V is an open neighbourhood of Ω^∞.
From properties of U and invariance of Ω^∞ it is clear that $\boldsymbol{F}^p(V) \subset \Omega^\infty$. Because Ω^n converges to Ω^∞ and V is an open neighbourhood of Ω^∞ then there exists $m \geq 0$ such that $\Omega^m \subset V$. Thus $\Omega^{m+p} = \boldsymbol{F}^p(\Omega^m) \subset \boldsymbol{F}^p(V) \subset \Omega^\infty$. On the other hand $\Omega^\infty \subset \Omega^{m+p}$. Hence $\Omega^\infty = \Omega^{m+p}$.

(b) Now let us assume that the case (a) is not satisfied. This means that at least one of the points A_1, B_1, C_1, D_1 does not belong to the set I. For example $A_1 \notin I$. From Lemma 4.1 it follows that there exists n such that $\boldsymbol{G}(\boldsymbol{F}^n(A_1)) \notin \Omega$. Let us denote $E := \boldsymbol{F}^n(A_1)$. From the assumption it follows that $\boldsymbol{F}(E) \in \{A_1, B_1, C_1, D_1\}$. $\boldsymbol{G}(\Omega^\infty)$ is absolutely convex. $\boldsymbol{G}(E) \in \partial\boldsymbol{G}(\Omega^\infty)$, $\boldsymbol{F}(E) \in \partial\boldsymbol{G}(\Omega^\infty)$. Hence $\overline{\boldsymbol{G}(E)\boldsymbol{F}(E)} \subset \partial\boldsymbol{G}(\Omega^\infty)$. $\boldsymbol{F}(E) \in \{A, B, C, D\}$. If not then $\partial\boldsymbol{G}(\Omega^\infty)$ would contain three intervals lying on different parallel lines, which contradicts the convexity of $\boldsymbol{G}(\Omega^\infty)$. Hence $\{A_1, B_1, C_1, D_1\} \cap \{A, B, C, D\} \neq \emptyset$. For example $A_1 = A = A_2$. Then $C_1 = C = C_2$ from symmetry of Ω^∞. Let $P^n := \Phi^{-n}(int(\overline{D_1A} \cup \overline{AB_2}) \cup int(\overline{B_1C} \cup \overline{CD_2}))$. The rest of the proof is exactly the same as in the case (a). □

We have shown in the previous theorem that Ω^∞ is an absolutely convex polygon. The examples of Ω^∞ being hexagon and decagon are shown in Fig. 4. We have found sets (a, b) for which Ω^∞ has 4,6,8,10,12,14,16 sides. The results are shown in Fig. 5(a). One can notice that the number of sides of Ω^∞ tends to infinity when point $(a, b) \in Q_3$ approaches one of the straight lines $b = 1 - a$, $b = 1 + a$.

THEOREM 4.8. *Let $\boldsymbol{x} \neq 0$. Then there exists $n_0 \geq 0$ such, that for every $n \geq n_0$: $\boldsymbol{F}^n(\boldsymbol{x}) \in \boldsymbol{W}^\infty$.*

Proof. 1. $\boldsymbol{x} \in \Omega^\infty$. From Lemma 4.1 there exists k such that $\boldsymbol{F}^k(\boldsymbol{x}) \in \overline{DA} \cup \overline{BC}$. $\boldsymbol{F}^k(\boldsymbol{x}) \in \boldsymbol{W}^\infty$ as Ω^∞ is invariant.

2. $\boldsymbol{x} \notin \Omega^\infty$. From the last theorem it follows that there exists m such that $\Omega^m = \Omega^\infty$. Hence $\boldsymbol{F}^m(\boldsymbol{x}) \in \Omega^\infty$ and we can use first part of the proof. $\square$

The above theorem states that every non-trivial trajectory in finite time enters the set $\boldsymbol{W}^\infty$ and remains in it. Thus we can reduce our study to the analysis of the one-dimensional map (specific for chosen (a, b)) of $\boldsymbol{W}^\infty$ into itself. The examples of such maps are shown in Fig. 4.

FIG. 4. *Examples of invariant sets and corresponding one-dimensional maps. The invariant set being the boundary of (a) hexagon ($a = 0.5, b = -1.3$), (b) decagon ($a = 0.5, b = -1.07$).*

One can notice that in both cases the maps are non-decreasing. In the second case the map is homeomorphic, while in the first case it is not. The conditions for the parameters that this map is homeomorphic are given later. $\boldsymbol{W}^\infty$ is the boundary of the absolutely convex polygon Ω^∞ - hence it is homeomorphic to the unit circle S^1.

FIG. 5. *(a) Regions of existence of invariant polygon in the (a, b) parameter space, (b) Invariant set containing corners A, B, C, D of the square Ω.*

An example of such a homeomorphism could be

$$\Psi : \boldsymbol{W}^{\infty} \ni \boldsymbol{x} \longmapsto \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in S^1, \tag{15}$$

where $||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2}$.

We use the well-known theory of circle-maps to analyze our system. Let us recall some important definitions and theorems from the circle-map theory.

Let $f : S^1 \longmapsto S^1$ be a continuous map of S^1 to itself. The map $\Pi : I\!R \longmapsto S^1$ defined by $\Pi(t) = (cos(2\pi t), sin(2\pi t))$ is continuous and onto. Let $F : I\!R \longmapsto I\!R$ be a *lift* of f, i.e. F is continuous, $\Pi \circ F = f \circ \Pi$, and for each $x \in I\!R$, $F(x+1) = F(x) + k$, where k is an integer constant. The integer k is unique for a given continuous map f and is called *the degree of* f. In this paper we are interested in degree-one maps only.

An important concept in the study of degree-one maps is the rotation number. Let f be a degree-one map, and let F be a lift of f. If $x \in I\!R$, then the *rotation number* of x under F is defined by

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}. \tag{16}$$

We say that a map of a circle is *non-decreasing* if its lift is non-decreasing.

PROPOSITION 4.9. *If f is a non-decreasing degree-one map, F is a lift of f, then $\rho_F(x)$ exists for every $x \in I\!R$ and does not depend on initial point x. $\rho_F(x)$ is rational iff f has a periodic point.*

The proof is similar to the proof of the corresponding result for orientation preserving homeomorphisms (see [8], pp.80-81, 2.4.2) and is omitted here. Thus for non-decreasing degree-one maps we can define the rotation number in the following way:

$$\rho_f = \rho_F(x) \ (mod\ 1),$$

where x is an arbitrary real value and F is an arbitrary lift of f. Because the difference between any two lifts is integer and $\rho_F(x)$ does not depend on x then ρ_f is properly defined.

Let $P(f)$ denote the set of periodic points, and $\Omega(f)$ the set of nonwandering points for map f, i.e. $\Omega(f) = \{x : \forall U\text{-neighbourhood of } x\ \exists n > 0 : f^n(U) \cap U \neq \emptyset\}$.

PROPOSITION 4.10. *([1], p.226, IX.21) If $P(f)$ is closed and non-empty, then $\Omega(f) = P(f)$.*

PROPOSITION 4.11. *([1], p.226, IX.22) If $P(f)$ is empty, then every point of the circle is chain recurrent. (We say that a point x is chain recurrent if $\forall \varepsilon > 0\ \exists n > 0\ \exists x_0, \ldots, x_n : d(f(x_i), x_{i+1}) < \varepsilon$ and $x = x_0 = x_n$).*

PROPOSITION 4.12. *If f is a non-decreasing degree-one circle map and the rotation number of f is rational then for every $x \in S^1$ the limit set $\omega(x) \subset P(f)$.*

Proof. We check assumptions of Proposition 4.10. From Proposition 4.9 $P(f)$ is not empty. It is clear that $P(f)$ is closed. Hence $\Omega(f) = P(f)$. Because every limit point is nonwandering then also every limit point is periodic. □

PROPOSITION 4.13. *If f is a non-decreasing degree-one map and f is not injective, then the rotation number of f is rational.*

Proof. Let us assume that ρ_f is irrational. As f is not injective, non-decreasing map then there exists $x \in S^1$ such that $f^{-1}(x)$ is a non-degenerate arc. Let $y \in int f^{-1}(x)$. $f(y) = x$. $P(f)$ is empty. Hence $f^n(x) \notin f^{-1}(x)$ for every $n \geq 0$. Thus y is not chain recurrent. As $P(f)$ is empty, it follows from Proposition 4.11 that every point of the circle is chain recurrent. This contradiction completes the proof. □

Let us go back to our system and construct a semi-dynamical system on a circle (S^1, ϕ) conjugate to the system $(\boldsymbol{W}^\infty, \Phi)$.

$$\phi := \Psi\Phi\Psi^{-1}, \tag{17}$$

where Ψ if defined in (15). As a consequence of Lemma 4.4 we obtain the following:

LEMMA 4.14. *ϕ is a non-decreasing degree-one map.*

For a given point $\boldsymbol{x} \neq O$ we define its rotation number under $\boldsymbol{F}$ denoted by $\rho_{\mathbf{F}}(\boldsymbol{x})$ in the following way: $\rho_{\mathbf{F}}(\boldsymbol{x}) := \rho_\phi(\Psi(\boldsymbol{F}^{n_0}(\boldsymbol{x})))$ where n_0 is the integer from the Theorem 4.8. From Proposition 4.9 and Lemma 4.14 we immidiately obtain the following:

COROLLARY 4.15. *For a given pair* (a, b) *and* $\boldsymbol{x} \neq O$ *the rotation number* $\rho_{\mathbf{F}}(\boldsymbol{x})$ *does not depend on* $\boldsymbol{x}$.

COROLLARY 4.16. $\boldsymbol{x} \neq O$, *if the rotation number of* $\boldsymbol{F}$ *is rational, then the limit set of* $\boldsymbol{x}$ *is a periodic orbit included in* $\boldsymbol{W}^{\infty}$.

Proof. As $\boldsymbol{x} \neq O$ it follows from Theorem 4.8 that there exists n_0 such that for $n \geq n_0 : \boldsymbol{F}^n(\boldsymbol{x}) \in \boldsymbol{W}^{\infty}$. Let $\boldsymbol{y} = \boldsymbol{F}^{n_0}(\boldsymbol{x})$. As ρ_{ϕ} is rational then the limit set of $\boldsymbol{y}$ is periodic. Hence also the limit set of $\boldsymbol{x}$ under $\boldsymbol{F}$ is a periodic orbit. ∎

It would be instrumental to find all pairs (a, b) for which ϕ is a homeomorphism. First we write down the conditions for (a, b) that Ω^{∞} is a polygon with 2n sides containing points A and C.
Let $c_0 := 1,\ c_1 := 1,\ c_{p+2} := bc_{p+1} + ac_p$ for $p \geq 0$.
Let $n \geq 2$. Ω^{∞} is $2n-$polygon containing A and C iff

$$|c_p| < 1 \textit{ for } p = 2, \ldots, n-1, \tag{18}$$

$$|c_n| \geq 1, \tag{19}$$

$$b(c_{n-2} - \frac{(c_{n-1} - c_{n-2})(1 + c_{n-1})}{c_n - c_{n-1}}) - a \leq -1. \tag{20}$$

If the last inequality is strong then ϕ is not a homeomorphism. If the last inequality becomes equality then ϕ is a homeomorphism. Similar conditions can be written for the case $\{B, D\} \cap \Omega^{\infty} \neq \emptyset$. We checked the above conditions for small n. The results are presented in Fig. 5(b). Points (a, b) for which $\{A, B, C, D\} \cap \boldsymbol{W}^{\infty} \neq \emptyset$ and ϕ is homeomorphism lies on the curve l. Points (a, b) for which ϕ is not homeomorphic lie below the curve l. Points (a, b) for which $\{A, B, C, D\} \cap \boldsymbol{W}^{\infty} = \emptyset$ lie above l.

Let us consider the case when ϕ is not homeomorphic, i.e. (a, b) lies below the curve l. From Proposition 4.13 it follows that ρ_{ϕ} is rational. Hence every trajectory starting from $\boldsymbol{x} \neq 0$ tends to a periodic orbit included in $\boldsymbol{W}^{\infty}$. We have observed that in this case corners of Ω belonging to $\boldsymbol{W}^{\infty}$ are periodic. We have also noticed that if a periodic orbit contains no corners of Ω then it is not stable. We were not able to prove these facts for all $(a, b) \in Q_3$ lying below the curve l, but we have obtained some partial results. For example if Ω^{∞} is a square or Ω^{∞} is a hexagon and $-b|a| > 1$, or Ω^{∞} is an octagon and $-b(b + a^2) > 1$ then the above hypotheses hold. We have also performed several simulations confirming that for (a, b) lying below the curve l one of points A, B, C, D is periodic.

The results proved above can be summarized in the following:

COROLLARY 4.17. *If* $(a, b) \in Q_3$, $\boldsymbol{x} \neq O$, ρ *is the rotation number of* $\boldsymbol{F}$, *then*

1. *The trajectory of* $\boldsymbol{x}$ *eventually enters the set* $\boldsymbol{W}^{\infty}$ *and remains in it.*
2. *If* ϕ *is not a homeomorphism (this corresponds to points* (a, b) *lying below the curve* l*) then* ρ *is rational.*
3. *If* ρ *is rational* ($\rho = p/q$) *then the limit set of* $\boldsymbol{x}$ *is a period-q orbit contained in* $\boldsymbol{W}^{\infty}$.
4. *If* ρ *is irrational then the limit set of* $\boldsymbol{x}$ *is dense in* $\boldsymbol{W}^{\infty}$.

4.2. Limit sets for Q_1, Q_2, Q_4. For this case the dynamics of the system is much simpler. So we present the results without proofs. Let z_1, z_2 be the eigenvalues of matrix $\boldsymbol{A}$. The following theorems can be easily derived:

THEOREM 4.18. *Let* $(a, b) \in Q_1$, $\boldsymbol{x} = (x_1, x_2)^T \in \Omega$.
If $z_2 x_2 + b x_1 = 0$ *then* $\omega(\boldsymbol{x}) = \{O\}$.
If $z_2 x_2 + b x_1 \neq 0$ *then* $\omega(\boldsymbol{x}) = \{A\}$ *or* $\omega(\boldsymbol{x}) = \{C\}$.

TABLE 2

The classification of limit sets on the (a,b) parameter plane. For the definitions of sets $T, Q_1, Q_2, \ldots$ see Fig. 2

(a,b)	$\boldsymbol{x} = (x_1, x_2)^T$	$\omega(\boldsymbol{x})$
T	$\boldsymbol{x} \in \Omega$	O
Q_1	$z_2 x_2 + b x_1 = 0$ $z_2 x_2 + b x_1 \neq 0$	O $\{A\}$ or $\{C\}$
Q_2	$z_1 x_2 + b x_1 = 0$ $z_1 x_2 + b x_1 \neq 0$	O $\{B, D\}$
Q_4	$\boldsymbol{x} = 0$ $\boldsymbol{x} \neq 0$	O $\{A\}, \{C\}$,$\{B, D\}$,O_1,O_2
Q_3	$\boldsymbol{x} = 0$ $\boldsymbol{x} \neq 0$	O $\exists n_0 : \forall\, n \geq n_0 : \boldsymbol{F}^n(\boldsymbol{x}) \in \boldsymbol{K}$ where $\boldsymbol{K}$ is the boundary of an abs. convex polygon, if $\boldsymbol{F}\|_{\mathbf{K}}$ is not homeomorphic then $\omega(\boldsymbol{x})$ is periodic, if $\boldsymbol{F}\|_{\mathbf{K}}$ is homeomorphic then $\omega(\boldsymbol{x})$ is periodic or $\overline{\omega(\boldsymbol{x})} = \boldsymbol{K}$
T_1, R_1	$\boldsymbol{x} \in \Omega$	$\omega(\boldsymbol{x})$ is a point $\{\boldsymbol{y}\} \subset \overline{AC}$
T_2, R_2	$-x_2 = b x_1$ $-x_2 \neq b x_1$	O $\omega(\boldsymbol{x})$ is a period 2 orbit $\{\boldsymbol{y}, -\boldsymbol{y}\} \subset \overline{BD}$
R_3	$\boldsymbol{x} \in \overline{AC}$ $\boldsymbol{x} \notin \overline{AC}$	$\{\boldsymbol{x}\}$ $\{\boldsymbol{x}, -\boldsymbol{x}\}$
T_3	$\boldsymbol{x} = 0$ $\boldsymbol{x} \neq 0$	O if $\frac{1}{2\pi} arccos(a/2)$ is a rational number p/q then $\omega(\boldsymbol{x})$ is a periodic orbit with rotation number p/q, if $\frac{1}{2\pi} arccos(a/2)$ is irrational then $\omega(\boldsymbol{x})$ is dense on an ellipse, position of which depends on the initial value
P_1	$\boldsymbol{x} \in \overline{AC}$ $\boldsymbol{x} \notin \overline{AC}$	$\{\boldsymbol{x}\}$ $\{A\}, \{C\}$
P_2	$\boldsymbol{x} = 0$ $\boldsymbol{x} \in \overline{BD} \setminus \{O\}$ $\boldsymbol{x} \notin \overline{BD}$	O $\{\boldsymbol{x}, -\boldsymbol{x}\}$ $\{B, D\}$
P_3	$\boldsymbol{x} = 0$ $\boldsymbol{x} \in \overline{BD} \setminus \{O\}$ $\boldsymbol{x} \notin \overline{BD}$	O $\{\boldsymbol{x}, -\boldsymbol{x}\}$ $\{A\}, \{C\}$
P_4	$\boldsymbol{x} \in \overline{AC}$ $\boldsymbol{x} \notin \overline{AC}$	$\{\boldsymbol{x}\}$ $\{B, D\}$

THEOREM 4.19. *Let $(a,b) \in Q_2, \boldsymbol{x} = (x_1, x_2)^T \in \Omega$.*
If $z_1 x_2 + b x_1 = 0$ then $\omega(\boldsymbol{x}) = \{O\}$.
If $z_1 x_2 + b x_1 \neq 0$ then $\omega(\boldsymbol{x}) = \{B, D\}$.

THEOREM 4.20. *Let $(a,b) \in Q_4, \boldsymbol{x} \in \Omega$.*
$\boldsymbol{x} = O$ is unstable fixed point.
If $\boldsymbol{x} \neq O$ then the limit set of $\boldsymbol{x}$ is one of the following periodic orbits: $\{A\}$, $\{C\}$, $\{B, D\}$, $O_1 = \{(1, -\frac{a}{1-b})^T, (-\frac{a}{1-b}, 1)^T\}$, $O_2 = \{(-1, \frac{a}{1-b})^T, (\frac{a}{1-b}, -1)^T\}$. First three orbits are stable, and the last two are unstable.

In the last case every trajectory with initial point $\boldsymbol{x} \neq O$ eventually enters the invariant set $\partial\Omega$. Like in section 4.1 we can construct a map of a circle ϕ conjugate to $\boldsymbol{F}|_{\partial\Omega}$, but here a lift of ϕ is non-increasing and ϕ has degree -1.

For points (a,b) lying on borders of sets Q_1, Q_2, Q_3, Q_4, T at least one eigenvalue of the linear system lies on the unit circle. This causes some degenerations in the behaviour of the system. There exist non-isolated sets of periodic orbits. The classification of limit sets for different values of parameters and initial points is presented in Table 2 (compare also Fig. 2).

5. Arnold tongues. Now we go back to the case $(a,b) \in Q_3$. For $\boldsymbol{x} \neq O$ we have defined the rotation number of $\boldsymbol{x}$ under $\boldsymbol{F}$ denoted by $\rho_{\mathbf{F}}(\boldsymbol{x})$. We would like to investigate the layout of parameters (a,b) for which there exists an orbit with a given rotation number. The results stated in this section are not proved here. The proofs will be presented elsewhere. After [4] we use the following definition of Arnold tongue.

FIG. 6. *The rotation number as a function of parameter a.*

DEFINITION 5.1. *We say that point (a,b) belongs to the ω-Arnold tongue denoted by A_ω if there exists an orbit with rotation number $\omega \in \mathbb{R}$ for $\boldsymbol{F}$ with parameters (a,b):*

$$A_\omega = \{(a,b) : \exists \boldsymbol{x} \neq O : \rho_{\mathbf{F}}(\boldsymbol{x}) = \omega\}. \tag{21}$$

It is easy to see that $(a,b) \in A_{p/q}$ implies the existence of a periodic orbit with period q and rotation number p/q.

From Corollary 4.15 it follows that Arnold tongues with different ω are disjoint. For any $\omega \in I\!R$ the ω-Arnold tongue A_ω is closed and pathwise connected [4].

In Fig. 6 we present the rotation number as a function of parameter a. One can easily see the devil's staircase structure. Zooming-in the diagram reveals even finer structures of this type. It can be proved that if we change parameter a for a given value of b then $\rho_{\mathbf{F}}$ changes in a weakly monotonic way. It can also be proved that A_ω for irrational ω has empty interior. It follows from the fact that if $\rho_{\mathbf{F}}$ is irrational then arbitrary small change of parameter a changes the rotation number of $\boldsymbol{F}$. It is clear that all Arnold tongues have nonempty intersections with the interval T_3. Namely $A_\omega \cap T_3 = \{(2cos(2\pi\omega), -1)\}$.

FIG. 7. *Ranges of parameters (a, b) with a given rotation number (a) Global diagram (b) Fine structure of Arnold tongues, "sausage" structures.*

In Fig. 7 we present the results of computer simulations. We have calculated the rotation number for different (a, b) and shown the points where the rotation number changes its value. We discovered that for parameters above the curve l the convergence of trajectories and hence the algorithm for the calculation of the rotation number is slow. The extension of the observation interval shows the finer structure of Arnold tongues. One can notice the existence of the so-called "sausage" structures. For every rational number its corresponding Arnold tongue is composed of a finite sequence of "sausages". This pattern has apparently a self-similar infinite structure (compare Fig. 7(b)).

The structure of Arnold tongues requires more investigations. We do not know whether every Arnold tongue for rational rotation number is a finite sequence of "sausages". From Corollary 4.17 it follows that Arnold tongues for irrational rotation number cannot lie below the curve l. We know that A_ω starts with a point belonging to T_3. Do they all reach the curve l?

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